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QUASINORMABILITY AND ASYMPTOTIC NORMABILITY OF SPACES OF ENTIRE FUNCTIONS OF BOUNDED TYPE

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Dedicated to the memory of Le Van Thiem

Abstract. It is studied under what conditions quasinormability and asymptotic normability of Frechet spaces are invariant for spaces of entire functions of bounded type on their strongly duals.

Introduction

In the period 1970-1980 Vogt has introduced and investigated "geometric" many properties of Frechet spaces. His results found important applications to some problems concerning plurisubharmonic and holomorphic functions on nuclear Frechet spaces (see for example [2], [3], [7], [9],...).

Recently the stability of certain Vogt's properties for the spaces of entire functions and of germs of holomorphic functions have been considered in [5] and [6]. In fact the problem was investigated earlier by Meise and Vogt in [7] for the nuclear case. The aim of the present paper is to study a similar problem for the quasinormability and asymptotic normability of the spaces of entire functions of bounded type on strongly dual spaces of Frechet spaces. The following two theorems are proved.

Theorem A. *Let E be a quasinormable Frechet space. Then $\mathcal{H}_b(E')$ is quasinormable if one of the following two condition holds:*

(i) *E is Hilbertisable*

(ii) *E' has an absolute basis.*

Theorem B. *Let E be an asymptotically normable Frechet space. Then $\mathcal{H}_b(E')$ so is.*

1. Preliminairies

1.1. Let E be a Frechet space, $\|\cdot\|_1 \leq \|\cdot\|_2 \leq \dots$ a fundamental system of seminorms on E defining the topology of E and $U_k = \{x \in E : \|x\|_k \leq 1\}$ for every $k \geq 1$. We say that E is

(i) Quasinormable if

(QN) $\forall p\ \exists q\ \forall \varepsilon > 0\ \exists$ a bounded set $M \subset E : U_q \subseteq M + \varepsilon U_p$.

(ii) Asymtotically normable if

(AN) $\exists p\ \forall q \geq p\ \exists k \geq q$: the semi-norms $\|\cdot\|_p$ and $\|\cdot\|_q$ define equivalent topologies on U_k.

In [8] Meise and Vogt have proved that E is quasinormable if and only if there exists a Banach space B and a nuclear Frechet - Kother space $\Lambda(A)$ such that E is a quotient space of $B\widehat{\otimes}_\pi\Lambda(A)$. Later a similar characterization for asymptotically normable Frechet spaces was given by Terzioglu and Vogt [12]. They have proved that E is asymptotically normable if and only if there exists a Banach space B and a nuclear Frechet space $\Lambda(A)$ with a continuous norm such that E is a subspace of $B\widehat{\otimes}_\pi\Lambda(A)$.

1.2. By M we denote the set of strictly increasing positive functions on $(0, \infty)$. If φ and Ψ are in M we say that Ψ dominates if for every $\lambda > 1$ there exists $C_\lambda > 0$ such that

$$\varphi(\lambda t) \leq C_\lambda \Psi(t) \quad \text{for all} \quad t \in (0, \infty). \tag{1.1}$$

Let E be a Frechet space and $\varphi \in M$. Following Vogt and Wagner [15], we say that E has the property

$$(\Omega_\varphi) \Longleftrightarrow \forall p\ \exists q\ \forall k\ \exists C > 0,\ \forall r > 0 : U_q \subseteq C\varphi(r)U_k + \frac{1}{r}U_p,$$

$$(DN_\varphi) \Longleftrightarrow \exists p\ \forall q\ \exists k,\ C > 0\ \forall r > 0 : U_q^0 \subseteq C\varphi(r)U_p^0 + \frac{1}{r}U_k^0.$$

It is known that [8] (resp. [12]) E is quasinormable (resp. asymptotically normable) if and only if $E \in (\Omega_\varphi)$ (resp. $E \in (DN_\varphi)$) for some $\varphi \in M$.

1.3. Let E and F be locally convex spaces and D an open set in E. A function $f : D \to F$ is called holomorphic if f is continuous and $u \circ f$ is Gateaux holomorphic for all $u \in F'$, the strongly dual space of F. By $\mathcal{H}(D, F)$ we denote the space of F-valued holomorphic functions on D equipped with the compact-open topology. Let $\mathcal{H}(D)$ denote $\mathcal{H}(D, C)$.

An entire function $f : E \to F$ is said to be of bounded type if f is bounded on every bounded set in E. We denote by $\mathcal{H}_b(E, F)$ the space of F-valued entire functions of bounded type E equipped with the topology of uniform convergence on all bounded subsets of E. Write $\mathcal{H}_b(E)$ for $H_b(E, C)$. It is known [10] that if E is a bornologicall (DF)-space, then $\mathcal{H}_b(E)$ is a Frechet space.

For more details concerning holomorphic functions on locally convex spaces the readers may consult [1].

2. Proof of Theorem A

The proof requires some lemmas.

Lemma 2.1. *Let E be a quasinormale Frechet space. Then so is E''.*

Proof. Let W be a neighbourhood of $0 \in E''$. Take a neighbourhood U of $0 \in E$ such that $U^{00} \subset W$. Since E is quasinormable we can find a neighbourhood V of $0 \in E$ such that

$\forall \varepsilon > 0$ there exists a bounded set M in $E : V \subset M + \varepsilon U$.

Since M^{00} is $\sigma(E'', E')$-compact, the bipolar theorem implies that

$$V^{00} \subset Cl_{\sigma(E'',E')}(M + \varepsilon U) \subset M^{00} + \varepsilon U^{00} \subset M^{00} + \varepsilon W.$$

By an appropriate modification of Vogt [13] we get the following.

Proposition 2.1. *Let*

$$0 \to E \to F \to \ell^2(I)\widehat{\otimes}_\pi \Lambda(A) \to 0$$

be an exact sequence of Frechet-Hilbertisbale spaces. Assume that $E \in (\Omega_\varphi)$ for some $\varphi \in M$, while $\Lambda(A) \in (DN_\Psi)$ for some Ψ which dominates φ. Then the sequence split.

Proposition 2.2. *Let E be a quasinormable Frechet-Hilbertisable space. Then there exist an index set I and a nuclear Frechet-Köthe space $\Lambda(A)$ such that E' is a subspace of $[\ell^2(I)\widehat{\otimes}_\pi \Lambda(A)]'$.*

Proof. Let $\{\|\cdot\|_k\}$ be a fundamental system of Hilbert seminorms of E satisfying

$$\forall k \geq 1 \ \forall \varepsilon > 0 \ \exists \text{ a bounded set } M \subset E : U_{k+1} \subset M + \varepsilon U_k$$

(i) Let us consider the canonical resolution of Palamodov [11]

$$0 \longrightarrow E \xrightarrow{e} \prod_{k\geq 1} E_k \xrightarrow{q} \prod_{k \geq 1} E_k \longrightarrow 0,$$

where

$$\begin{aligned} q &: \{x_k\} \longrightarrow (\rho_{k+1,k} x_{k+1} - x_k), \\ e &: x \longrightarrow (w_k x); \end{aligned}$$

here $\rho_{k+1,k} : E_{k+1} \to E_k$ and $w : E \to E_k$ are canonical maps and E_k are Hilbert spaces associated to $\|\cdot\|_k$.

We shall prove that every bounded set in $\prod_{k\geq 1} E_k$ is the image of a bounded set in $\prod_{k\geq 1} E_k$ under the maps q. Indeed, by virtue of [11] it is enough to check that for any index set S the space $\ell^\infty(S, E)$ is dense in $\ell^\infty(S, E_{k+1})$ with respect to the norm of $\ell^\infty(S, E_k)$.

Given $\sigma \in \ell^\infty(S, E_{k+1})$ and $\varepsilon > 0$. Choose a bounded set M in E satisfying

$$U_{k+1} \subset M + \frac{\varepsilon}{\|\sigma\|_{k+1}} U_k.$$

Since $\left\{\dfrac{\sigma(s)}{\|\sigma\|_{k+1}} : s \in S\right\} \subset U_{k+1}$, we deduce that there exists $\beta \in \ell^\infty(S, E)$ such that

$$\Big\| \frac{\sigma(s)}{\|\sigma\|_{k+1}} - \beta(s) \Big\|_k < \frac{\varepsilon}{\|\sigma\|_{k+1}} \quad \text{for} \quad s \in S.$$

Thus for $\gamma = \|\sigma\|_{k+1}\beta \in \ell^\infty(S, E)$ we have

$$\|\sigma - \gamma\|_k < \varepsilon.$$

(ii) Following [8], we put

$$F = \Big\{ x = \{x_k\} \in \prod_{k\geq 1} E_k : \|x\|^2 = \sum_{k \geq 1} \|x_k\|^2 < \infty \Big\}.$$

For each k we let F_k be the topological complement of E_k in F, i.e. $F = E_k \oplus F_k$. By taking the direct sum of the above canonical resolution with the exact sequence

$$0 \longrightarrow 0 \longrightarrow \prod_{k=1}^{\infty} F_k \xrightarrow{id} \prod_{k=1}^{\infty} F_k \longrightarrow 0$$

we obtain the exact sequence

$$0 \longrightarrow E \longrightarrow F^N \xrightarrow{\tilde{q}} F^N \longrightarrow 0.$$

It is easy to check that every bounded set in F^N is the image under $\tilde{q}$ of a bounded set in F^N. By Proposition 2.1 and the same argument as in [8], we deduce that E is a quotient space of $\ell^2(I)\widehat{\otimes}_\pi \Lambda(A)$ for some index set I and a nuclear Frechet-Köthe space $\Lambda(A)$ such that every bounded set in E is the image of a bounded set in $\ell^2(I)\widehat{\otimes}_\pi \Lambda(A)$. It follows that E' is a subspace of $[\ell^2(I)\widehat{\otimes}_\pi \Lambda(A)]' \cong \ell^2(I)\widehat{\otimes}_\pi \Lambda(A)$. □

Lemma 2.2. *Let B be a Banach space and E a nuclear Frechet space. Then $\mathcal{H}_b(B\widehat{\otimes}_\pi E')$ is quasinormable.*

Proof. It is easy to see that the topology of $\mathcal{H}_b(B\widehat{\otimes}_\pi E')$ is defined by the system of seminorms

$$|||f|||_p = \sup\big\{p^n |P_n f(\omega)| : \omega \in A_p, n \geq 0\big\}, \tag{1}$$

where

$$f(\omega) = \sum_{n \geq 0} P_n f(\omega), \quad P_n f = \frac{1}{2\pi i} \int\limits_{|\lambda| = p} \frac{f(\lambda\omega)}{\lambda^{n+1}} d\lambda$$

is the Taylor expansion of $f \in \mathcal{H}_b(B\widehat{\otimes}_\pi E')$ at $0 \in B\widehat{\otimes}_\pi E'$ and $\{A_p\}$ is an exhaustion sequence of bounded sets in $B\widehat{\otimes}_\pi E'$.

Let $p \geq 1$. Choose $q > p+1$ such that the canonical map

$$\omega_{pq}; F_q \longrightarrow F_p$$

is nuclear. Now we claim that for every $\varepsilon > 0$ there exists a bounded set $M \subset \mathcal{H}_b(B\widehat{\otimes}_\pi E')$ such that $W_q \subset M + 2\varepsilon W_p$, where for each $k \geq 1$ we put

$$W_k = \big\{ f \in \mathcal{H}_b(B\widehat{\otimes}_\pi E') : |||f|||_k \leq 1 \big\}.$$

Given $\varepsilon > 0$, we select $n_\varepsilon > 0$ satisfying

$$\Big(\frac{p}{p+1}\Big)^n < \varepsilon \quad \forall n > n_\varepsilon.$$

It follows that

$$\begin{aligned} \Big|\Big|\Big| \sum_{n > n_\varepsilon} P_n f \Big|\Big|\Big|_p &\leq \sup \Big\{ \Big(\frac{p}{p+1}\Big)^n q^n |P_n f(\omega)| : \omega \in A_q \Big\} \\ &\leq \varepsilon \quad \forall f \in W_q. \end{aligned}$$

This implies that

$$\sum_{n > n_\varepsilon} P_n f \in \varepsilon W_p, \quad \forall f \in W_q. \tag{2}$$

Since for each $n \geq 0$, $\mathcal{P}((^n(B\widehat{\otimes}_\pi E'))$, the space of continuous homogeneous polynomials of degee n on $B\widehat{\otimes}_\pi E'$, is isomorphic to a complemented subspace of

$$((B\widehat{\otimes}_\pi E')\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi (B\widehat{\otimes}_\pi E'))' \cong (B\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi B)'\widehat{\otimes}_\pi (E\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E)$$

and the map

$$w_{pq}\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi w_{pq} : E_q\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_q \to E_p\widehat{\otimes}_\pi \cdots \widehat{\otimes}_\pi E_p$$

is compact, we can find for each $n \geq 1$ a bounded set $M_n \subset \mathcal{P}(^n(B\widehat{\otimes}_\pi E'))$ such that

$$W_q \cap \sum_{0 \leq n \leq n_\varepsilon} \mathcal{P}(^n(B\widehat{\otimes}_\pi E')) \subset \sum_{0 \leq n \leq n_\varepsilon} M_n + \varepsilon W_p. \tag{3}$$

combining (2) and (3) we have

$$W_q \subset \sum_{0 \leq n \leq n_\varepsilon} M_n + 2\varepsilon W_p.$$

Now we are able to prove Theorem A.

(i) Assume that E is a quasinormable Frechet-Hilbertisable space. By Proposition 2.2 we can find an index set I and a nuclear Frechet-Köthe space $\Lambda(A)$ such that E' is a subspace of $\ell^2(I)\widehat{\otimes}_\pi \Lambda'(A)$. Since $\Lambda'(A)$ is nuclear we infer that $\ell^2(I)\widehat{\otimes}_\pi \Lambda'(A)$ has a fundamental system of Hilbert semi-norms. Combining this with the fact that every entire function of bounded type on a (DF)-space can be factoried through a Banach space [4] we can assert that the restriction map

$$R : \mathcal{H}_b(\ell^2(I)\widehat{\otimes}_\pi \Lambda'(A) \to \mathcal{H}_b(E')$$

is surjective and, hence it is an open mapping by [10]. Consequently, we deduce from Lemma 2.2 that $\mathcal{H}_b(E')$ is quasinormable.

(ii) Assume that E' has an absolute basis $\{u_j\}$ with the sequence of coefficient functionals $\{e_j\} \subset E''$. For each $k \geq 1$, put

$$E'(k) = \Big\{u \in E' : \|u\|_k = \sum_{j\geq 1} |e_j(u)| \big\|u_j\big\|_k^* < \infty\Big\},$$

where

$$\big\|u\big\|_k^* = \sup\Big\{|u(x)| : x \in U_k,\ u \in E'\Big\}.$$

Since E is quasinormable (hence E' is bornological) and since $\{u_j\}$ is an absolute basis of E', it is easy to check by [10] that $E' \cong \lim\operatorname{ind}_k E'(k)$ and $\mathcal{P}(^nE') \cong \lim\operatorname{proj}\mathcal{P}(^nE'(k))$ for $n \geq 0$.

a) First we show that every bounded set in $\mathcal{P}(^n(E'(k+1))$ can be approximated in $\mathcal{P}(^nE'(k))$ by a bounded set in $\mathcal{P}(^nE')$. We consider only the case $n = 2$, because the others cases can be proved similarly.

Let M be a bounded set in $\mathcal{P}(^2E'(k+1)) \cong \mathcal{L}(E'(k+1))$, $E''(k+1)$ and $\varepsilon > 0$. Since $E''(k+1) \cong \ell^\infty$, every $f \in \mathcal{P}(^2E'(k+1))$ can be considered as a sequence $\{f_n\} \subset E''(k+1)$ with

$$\begin{aligned}\|f\|_{k+1,k+1} &:= \sup\Big\{|f(u,v)| : (u,v) \in A_{k+1} \times A_{k+1}\Big\}\\ &= \sup\Big\{|f_n(u)| : u \in A_{k+1}, n \geq 1\Big\},\end{aligned}$$

where

$$A_{k+1} = \Big\{u \in E'(k+1) : |||u|||_{k+1} \leq 1\Big\}.$$

By Lemma 2.1, E'' is quasinormable. Hence, without loss of generality we may assume that

$$\begin{gathered}\forall p \geq 1\ \forall \varepsilon > 0\ \exists \text{ a bounded set } B \text{ in } E'' :\\ A^0_{p+1} \subseteq B + \delta A^0_p \quad \text{and} \quad 2A_p \subseteq A_{p+1} \text{ for } p \geq 1.\end{gathered} \tag{4}$$

Applying (4) to the bounded set $\big\{f_n : f \in M, n \geq 1\big\} \subset E''(k+1)$ we can find a bounded set $\{g_{f,n} : f \in M, n \geq 1\}$ such that

$$|f_n(u) - g_{f,n}(u)| < \frac{\varepsilon}{2^2} \tag{5}$$

for $u \in A_k$ and $n \geq 1$.

From (5) we see that the form

$$g_f(u) = \{g_{f,n}(u)\}, \quad u \in E'(k+2) \tag{6}$$

defines $g_f \in \mathcal{L}(E'(k+2), E''(k))$ such that

$$\big\|f - g\big\|_{k+1,k} < \frac{\varepsilon}{2^2} \quad \text{for} \quad f \in M.$$

Choose a bounded set $\{x_{f,j} : f \in M, j \geq 1\}$ in $E''(k+2)$ such that

$$\Big\|g_f\Big(\frac{u_j}{\big\|u_j\big\|^*_{k+2}}\Big) - x_{f,j}\Big\|_{k-1} < \frac{\varepsilon}{2^2} \quad \text{for} \quad j \geq 1,\ f \in M.$$

Then, for each $f \in M$, the form

$$h_f^{(1)}(u) = \sum_{j \geq 1} e_j(u) \cdot \left\|u_j\right\|_{k+2}^{*} x_{f,j} \quad \text{for} \quad u \in E'(k+2)$$

defines a bounded set $\left\{h_f^{(1)} : d \in M\right\}$ in $\mathcal{L}(E'(k+2), E''(k+2))$ such that

$$\left\|h_f^{(1)} - h_f^{(2)}\right\|_{k+2,k} < \frac{\varepsilon}{2^2} \quad \text{for} \quad f \in M.$$

Continuing this process we get for each $p \geq 1$ a bounded set

$$\left\{h_f^{(p)} : f \in M\right\} \subset \mathcal{L}(E'(k+p+1), E''(k+p+1))$$

such that

$$\left\|h_f^{(p+1)} - h_f^{(p)}\right\|_{k+p+1,k+p-1} < \frac{\varepsilon}{2^{p+1}} \quad \text{for} \quad f \in M. \tag{7}$$

From (7) it follows that for each $p \geq 1$ and each $f \in M$ the series

$$h_f^{(p)} + \sum_{q \geq p} \left[h_f^{(q+1)} - h_f^{(q)}\right]$$

converges to $\hat{h}_f^{(p)}$ in $\mathcal{L}(E'(k+p+1), E''(k+p))$ and the sequence $\left\{\hat{h}_f^{(p)}\right\}_{p \geq 1}$ defines

$$\hat{h}_f \in \mathcal{L}(E', E'') \cong \mathcal{P}(^2E')$$

such that

$$\left\|\hat{h}_f - f\right\|_{k+1,k} < \varepsilon \quad \text{for} \quad f \in M.$$

Moreover, the set $\left\{\hat{h}_f : f \in M\right\}$ is bounded in $\mathcal{P}(^2E')$.

b) To prove that $\mathcal{H}_b(E')$ is quasinormable it is enough to check that

$$\forall k \geq 1 \ \forall \varepsilon > 0 \ \exists \text{ a bounded set } M \subset \mathcal{H}_b(E') : W_{k+1} \subseteq M + \varepsilon W_k, \tag{8}$$

where

$$W_p = \left\{f \in \mathcal{H}_p(E') : \|f\|_p \leq 1\right\} \quad \text{for} \quad p \geq 1.$$

Given $k \geq 1$ and $\varepsilon > 0$, we choose N such that

$$\left(\frac{k}{k+1}\right)^n < \varepsilon \quad \text{for} \quad n > N. \tag{9}$$

For each $f \in W_{k+1}$, we consider the Taylor expansion of f at $0 \in E'$

$$f(u) = \sum_{n \geq 0} P_n f(u), \quad P_n f(u) = \frac{1}{2\pi i} \int\limits_{|\lambda| = p} \frac{f(\lambda u)}{\lambda^{n+1}} d\lambda. \tag{10}$$

From (9) and (10) we have

$$\sum_{n > N} P_n f \in \varepsilon W_k.$$

By a) we can find a bounded set M in $\sum\limits_{0\le n\le N}\mathcal{P}({}^nE')$ such that

$$\sum_{0\le n\le N} W^n_{k+1}\subseteq M+\varepsilon\sum_{0\le n\le N} W^k_n\subseteq M+\varepsilon W_k.$$

Hence

$$W_{k+1}\subseteq M+2\varepsilon W_k,$$

and (8) is proved.

3. Proof of Theorem B

Let E be an asymptotically normable Frechet space. By the theorem of Terzioglu-Vogt [12] we can find a Banach space B and an asymptotically normable nuclear Frechet space F such that E is a subspace of $B\widehat{\otimes}_\pi F$. It follows from the Hahn-Banach theorem that $\mathcal{H}_b(E')$ is a subspace of $\mathcal{H}_b(B'\widehat{\otimes}_\pi F')$. Hence, it remains to check $\mathcal{H}_b(B'\widehat{\otimes}_\pi F')$ is asymptotically normable.

(i) Let $\big\{\|\cdot\|_k\big\}_{k=1}^\infty$ be a fundamental system of seminorms of F. Since F is asymptotically normable, we have

$$\text{(AN)}\qquad \exists p\ \forall q\ \exists k;\ \|\cdot\|_p\sim\|\cdot\|_q\quad\text{on}\quad U_k,$$

i.e., the topologies on U_k which are defined by $\|\cdot\|_p$ and $\|\cdot\|_q$ coincide. First we check that for p, q and k as in (AN) we have

$$\pi^n_p\sim\pi^n_q\quad\text{on}\quad W^n_k\quad\text{for}\quad n\ge 1,$$

where W^n_k denotes the unit ball in

$$\underbrace{(B\widehat{\otimes}_\pi F)\widehat{\otimes}_\pi\cdots\widehat{\otimes}_\pi(B\widehat{\otimes}_\pi F)}_{n}$$

of the semi-norm π^n_k induced by $\|\cdot\|_k$. For simplicity, we consider only the case $n=2$.

Let $\{f_n\}\subset W^2_k$ with $\pi^2_p(f_n)\to 0$ as $n\to\infty$. Since

$$(B\widehat{\otimes}_\pi F)\widehat{\otimes}_\pi(B\widehat{\otimes}_\pi F)\cong F\widehat{\otimes}_\pi(B\widehat{\otimes}_\pi B\widehat{\otimes}_\pi F)\cong\mathcal{L}(F',B\widehat{\otimes}_\pi B\widehat{\otimes}_\pi F)$$

the sequence $\{f_n\}$ can be considered as a sequence

$$\big\{\hat f_n\big\}\subset\mathcal{L}(F',B\widehat{\otimes}_\pi B\widehat{\otimes}_\pi F)$$

for which

$$\sup\Big\{\big|(w\circ\hat f_n)(u)\big| : u\in U^0_k, w\in(V\otimes V\otimes U_k)^0, n\ge 1\Big\}\le 1,$$

and

$$\varepsilon^2_p(\hat f_n):=\sup\Big\{\big|(w\circ\hat f_n)(u)\big| : u\in U^0_k, w\in(V\otimes V\otimes U_k)^0\Big\}\to 0$$

Assume that $\varepsilon^2_q(f_n)\not\to 0$. Then for each $n\ge 1$ there exists $w_n\in(V\otimes V\otimes U_q)^0$ such that

$$\sup\Big\{\big|(w_n\circ\hat f_n)(u) : u\in U^0_q\Big\}\not\to 0\quad\text{as } n\to\infty.$$

This is impossible, because $\{w_n \circ \hat{f}_n\} \subset U_k^0$ and $\|w_n \circ f\|_p \to 0$ as $n \to \infty$.

(ii) For each $k \geq 1$, put

$$W_k \Big\{ f \in \mathcal{H}_b(B' \widehat{\otimes}_\pi F') : |||f|||_k \leq 1 \Big\}$$

where

$$|||f|||_k = \sup \Big\{ |f(w)| : w \in \operatorname{conv}(V^0 \otimes U_k^0) \Big\}.$$

It remains to check that

$$||| \cdot |||_p \sim ||| \cdot |||_q \quad \text{on} \quad W_k$$

for p, q, k as in (AN). Assume that $\{f_n\} \subset W_k$, $|||f_n|||_p \to 0$ as $n \to \infty$. Write

$$f_n(w) = \sum_{j \geq 0} P_j f_n(w), \quad P_j f_n(w) = \frac{1}{2\pi i} \int\limits_{|\lambda| = p} \frac{f_n(\lambda w)}{\lambda^{j+1}} d\lambda.$$

Take $0 < \delta < 1$ such that

$$\operatorname{conv}(V \otimes U_q^0) \subset \delta \ \operatorname{conv}\ (V \otimes U_k^0).$$

For each $\varepsilon > 0$, choose j_0 for which

$$\sum_{j > j_0} |||P_j f_n|||_q \leq \sum_{j > j_0} \delta^j < \varepsilon.$$

Then by (i) we have

$$\sum_{0 \leq j \leq j_0} |||P_j f_n|||_q < \varepsilon$$

for n sufficiently large.

Hence $|||f_n|||_q \to 0$ as $n \to \infty$.

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