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A SURVEY ON THE P-ADIC NEVANLINNA THEORY AND RECENT ARTICLES

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Dedicated to the memory of Le Van Thiem

Abstract. We give a brief survey of the Nevanlinna theory over non-archimedean fields and its applications in the study of p-adic hyperbolic spaces, the unique range sets for meromorphic functions.

1. Introduction

The Nevanlinna theory studies the problem *"How many times does a meromorphic fonction $f(z)$ take the value $a \in \mathbb{P}^1$?"*, in other words, *"how to measure the set $f^{-1}(a)$"?*.

The first results in this direction belong to Hadamard.

Hadamard's theorem. *Let $f(z)$ be a holomorphic function in $\mathbb{C}$. Then*

$$(\text{the number of zeros of } f \text{ in } \{|z| \leq r\}) \leq \log \max_{|z|\leq r} |f(z)| + O(1),$$

where $O(1)$ depends on f, but not on r.

This result is not yet "ideal" because of the following two deficiencies.

a) When f is a meromorphic function, we have the infinity in the right hand side of the inequality, and in this case the Hadamard theorem does not give an estimation of the number of zeros of f.

b) There are functions, for example, $f(z) = e^z$, which do not have the zeros, and in this case Hadamard's inequality becomes trivial.

For eliminating the above deficiencies, R. Nevanlinna defines the following functions.

1.1. **Counting function.** Let $a \in \mathbb{C}$. We set

$$n(a, r) = \sharp\{\text{zeros of } f(z) - a \text{ in } \{|z| < r\}, \text{ with multiplicity}\},$$

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$$N(a,r) = \int\limits_o^r \frac{n(a,t) - n(a,0)}{t} dt + n(a,0)\log r.$$

1.2. **Characteristic function.** Instead of $\log_{|z|\le r}|f(z)|$ we consider the function

$$T(r) = \frac{1}{2\pi}\int\limits_o^{2\pi} \log^+ |f(re^{i\theta})| d\theta + N(\infty, r).$$

Using these two functions one gets the following inequality:

$$N(0,r) \le T(r) + O(1).$$

This inequality is valid and non-trivial for meromorphic functions.

For eliminating the second deficiency one notices that while the function e^z does not have the zeros, it takes many values "approching to zero". Then one can "measure" this set by using

1.3. **Mean proximity function.**

$$m(a,r) = \frac{1}{2\pi}\int\limits_o^{2\pi} \log^+ \Big|\frac{1}{f(re^{i\theta}) - a}\Big| d\theta,$$

where $\log^+ = \max(0, \log)$.

It is clear that $m(a,r)$ becomes "bigger" when $f(z)$ approches to a.

There are two "Main Theorems" and defect relations which occupy a central place in the Nevanlinna theory.

1.4. **First Main theorem of Nevanlinna.** *There is a function $T(f)$ such that for any $a \in \mathbb{P}^1$ we have*

$$m(a,r) + N(a,r) = T(r) + O(1).$$

As $T(r)$ does not depend on a, one can say that a meromorphic function takes every value $a \in \mathbb{P}^1$ (and "approche to a" values) with the same frequency.

1.5. **Second Main theorem of Nevanlinna.** *For an arbitrary $q \in \mathbb{N}$ and distinct points $a_i \in \mathbb{P}^1$, $i = 1, \dots, q$,*

$$\sum_{i=1}^{q} m(a_i, r) < 2T(r) + O(\log(rT(r))),$$

where the inequality is valid beside a set of finite measure.

If we set

$$\delta(a) = \lim_{r\to\infty} \frac{m(a,r)}{T(r)}$$

then

$$\sum_{a \in \mathbb{P}^1} \delta(a) \leq 2. \tag{1}$$

We say that $\delta(a)$ is the defect value at the point a and (1) is the "defect relation". Precisely, $\delta(a) = 0$ for almost all a (except a countable set).

1.6. **Why to study the p-adic Nevanlinna Theory ?** In the famous paper "De la métaphysique aux mathématiques" ([W]) A. Weil discussed the role of analogies in mathematics. For illustrating he analysed a "metaphysics" of Diophantine Geometry: the resemblance between *Algebraic Numbers* and *Algebraic Functions*. However, the striking similarity between Weil's theory of heights and Cartan's Second Main Theorem for the case of hyperplanes is pointed out by P. Vojta only after 50 years! P. Vojta observed the resemblance between *Algebraic Numbers* and *Holomorphic Functions*, and gave a "dictionary" for translating the results of Nevanlinna Theory in the one-dimensional case to Diophantine Approximations. Due to this dictionary one can regard Roth's Theorem as an analogue of the Nevanlinna Second Main Theorem. P. Vojta has also made quantitative conjectures which generalize Roth's theorem to higher dimensions. One can say that P. Vojta proposed a "new metaphysics" of Diophantine Geometry: Arithmetic Nevanlinna Theory in higher dimensions. On the other hand, in the philosophy of Hasse-Minkowski principle one hopes to have an "arithmetic result" if one had have it in p-adic cases for all prime numbers p, and in the real and complex cases. Hence one would naturally have interest to determine how the Nevanlinna Theory would look in the p-adic case.

2. Two main theorems

Let p be a prime number, Q_p the field of p-adic numbers, and C_p the p-adic completion of the algebraic closure of Q_p. The absolute value in Q_p is normalized so that $|p| = p^{-1}$. We further use the notion $v(z)$ for the additive valuation on C_p which extends ord_p.

We define the counting function in the same way as in the classical Nevanlinna theory. That is, given a meromorphic function f, we let $n(f, \infty, r)$ denote the number of poles in $\{|z| \leq r\}$, and we let

$$\begin{aligned} N(f, \infty, r) &= \int_o^r [n(f, \infty, t) - n(f, \infty, 0)] \frac{dt}{t} + n(f, \infty, 0) \log r \\ &= \sum_{|z| \leq r, z \neq 0} \max\{0, -ord_z f\} \log \frac{r}{|z|} + \max\{0, -ord_o f\} \log r, \end{aligned}$$

where $ord_z f$ denotes the order of vanishing of f at z, and negative numbers indicate poles. Counting functions for other values are defined similarly.

For the mean proximity function, note that the norms $|\ |_r$ are multiplicative on entire functions and they extend to meromorphic functions. Thus we define

$$m(f,\infty,r)=\log|f|_r,$$

and for finite a,

$$m(f,a,r)=\log\frac{1}{|f-a|_r}.$$

Note that there is no need to do any sort of "averaging" over $|z|=r$, since by the strong maximum modulus principle, for suitable generic z with $|z|=r$, we have $|f(z)|=|f|_r$. Finally, just as in the classical Nevanlinna theory, the characteristic function is given by

$$T(f,a,r)=m(f,a,r)+N(f,a,r).$$

The properties of the valuation polygon imply that

$$\log|f|_r=\sum_{|z|\le r,z\ne 0}(ord_z f)\log\frac{r}{|z|}+(ord_o f)\log r+0(1),$$

where the 0(1) term depends on the size of the first non-zero coefficient in the Laurent axpansion for f at 0. This is of course a non-Archimedean Jensen formula which can be written as

$$m(f,\infty,r)+N(f,\infty,r)=m(f,0,r)+N(f,0,r)+0(1).$$

From this formula the non-Archimedean analogue to *the Nevanlinna first Main Theorem* follows easily.

The Second Main Theorem. *Let f be a non-constant meromorphic function on $\mathbb{C}_p$, and let $a_1,a_2,\ldots,a_q$ be q distinct points on $\mathbb{C}_p\cup\{\infty\}$. Then, for all $r\ge r_o>0$,*

$$(q-2)T(f,r)-\sum_{j=1}^{q}N(f,a_j,r)-N_{Ram}(f,t)\le-\log r+0(1),$$

where

$$N_{Ram}(f,a)=N(f',0,r)+2N(f,\infty,r)-N(f',\infty,r)$$

measures the growth of the ramification of f, and the 0(1) term depends only on the a_j, the function f, and the number r_o.

Corollary. *Let f and $a_1,a_2,\ldots,a_q$ be as in the preceding theorem. Then for all $r\ge r_o$*

$$(q-2)T(f,r)\le\sum_{j=1}^{q}N_1(f,a_j,r)-\log r+0(1),$$

where $N_1(f,a_j,r)$ denotes a modified counting function in that each point where $f=a$ is counted only with multiplicity 1, and again 0(1) term depends on the a_j,f,r_o.

3. The height function

In the p-adic case we can use the so-called "the height function". Note that the Newton polygon gives expression to one of the most basic diffrences between p-adic analytic functions and complex analytic functions. Namely, the modulus of a p-adic analytic function depends only on the modulus of the argument, except for a discrete set of values of the modulus of argument. This fact often makes it easier to prove the p-adic analogues of classical results. Now we give the definition of the height function.

Let $f(z)$ be an analytic function on $\mathbb{C}_p$, which is represented by a convergent power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n.$$

For each n we draw the graph Γ_n which depicts $v(a_n z^n)$ as a function of $v(z) = t$. This graph is a straight line with slope n. Since we have $\lim_{n\to\infty}\{v(a_n)+nt\} = \infty$ for all t, it follows that for every t there exists an n for which $v(a_n) + nt$ is minimal. Let $h(f,t)$ denote the boundary of the intersection of all of the half-planes lying under the lines Γ_n. Then in any finite segment $[r,s]$, there are only finitely many Γ_n which appear in $h(f,t)$. Thus $h(f,t)$ is a polygonal line. This line is what we call *the height* of the function $f(z)$. The points t at which $h(f,t)$ has vertices are called the critical points of $f(z)$. A finite segment contains only finitely many critical points. If t is a critical point, then $v(a_n) + nt$ attains its minimum at least at two values of n. If $v(z) = t$ is not a critical point, then $|f(z)| = p^{-h(f,t)}$.

The height of a function $f(z)$ gives complete information about the number of zeros of $f(z)$. Namely, f has zeros when $v(z) = t_i$ (a critical point) and the number of zeros of f such that $v(z) = t_i$ is equal to the difference $n_{i+1} - n_i$ between slopes of $h(f,t)$ at t_{i-0} and t_{i+0}

For a meromorphic function $f = \dfrac{\phi}{\psi}$, the height of f is defined by $h(f,t) = h(\phi,t) - h(\psi,t)$. We also use the notation

$$h^+(f,t) = -h(f,t).$$

Theorem 3.1. *Let f be a meromorphic function and let $a_1, a_2, \dots, a_q$ be q distinct points in $\mathbb{C}_p \cup \{\infty\}$. Then for t sufficiently small,*

$$(q-2)h^+(f,t) \le \sum_{j=1}^{q} N_1(f,a_j,t) + t + 0(1),$$

where $N_1(f,a,t)$ denotes a modified function in that each point where $f = a$ is counted only with multiplicity 1, and the 0(1) is a bounded value as $t \longrightarrow -\infty$.

The height function is applied to the interpolation problem (see [K1]). Let $u = \{u_1, u_2, \dots\}$ be a sequence of points in $\mathbb{C}_p$. In what follows we shall consider only sequences u for which the numbers of points u_i satisfying $v(u_i) \geq t$ is finite for every t. We shall always assume that $v(u_i) \geq v(u_{i+1}),\ (i = 1, 2, \dots)$.

Definition 3.1. The sequence $u = \{u_i\}$ is called an *interpolating sequence* of f if the sequence of interpolating polynomials for f on u converges to f.

For every sequence u we define a holomorphic function Φ_u as follows. Let

$$N_u(t) = \#\{u_i | v(u_i) \geq t\}.$$

Write the sequence u in the form

$$u = \{u_1, u_2, \dots, u_{n_1}, u_{n_1+1}, \dots, u_{n_2}, \dots\},$$

where

$$v(u_i) = t_k \ \text{ for } n_{k-1} < i \leq n_k$$

(we take $u_o = 0$), and

$$\lim_{k \to \infty} t_k = -\infty.$$

We choose a sequence a_k with the property

$$v(a_o) = -n_1 t_1,\ v(a_{k+1}) = v(a_k) + (n_k - n_{k+1}) t_{k+1},\ (k = 1, 2, \dots).$$

We set

$$\Phi_u(z) = 1 + \sum_{k=1}^{\infty} a_k z^{n_k}.$$

Then the series converges for $z \in \mathbb{C}_p$ and determines an analytic function $\Phi_u(z)$ on $\mathbb{C}_p$, for which the number of zeros in each region $\{z | v(z) > t\}$ is equal to $N_u(t)$, and

$$h(\Phi_u, t) = \int_{\infty}^{t} N_u(t) dt.$$

Theorem 3.2. *The sequence $u = \{u_i\}$ is an interpolating sequence of the function $f(z)$ if and only if*

$$\lim_{t \to \infty} \{h(f, t) - h(\Phi_u, t)\} = \infty.$$

Remark 3.1. This is the first interpolation theorem for p-adic analytic functions not necessarily bounded. A similar theorem for analytic functions in the unit disc implies that the p-adic L-functions associated to modular forms are uniquelly defined by the values on Dirichlet characters (see [K2]).

Remark 3.2. We can use the interpolation theorem to recover a p-adic meromorphic function if we know the preimages (with multiplicity) of two points (see [K3]).

For *high dimensions*, as well as in the complex case, instead of the study the preimage of a point, we should consider the preimage of a divisor of codimension one. The reason is that in the p-adic case there exist also Fatou-Bieberbach domains (see [S]).

Now let $f = (f_1, \dots, f_{n+1}) : C_p \to P^n(C_p)$ be a p-adic holomorphic curve, where the functions f_j have no common zeros.

Definition 3.2. The height of the holomorphic curve f is defined by

$$h(f,t) = \min_{1 \le j \le n+1} h(f_j, t),$$

where $h(f_j, t)$ is the height of p-adic holomorphic function on $\mathbb{C}_p$.

Notes that the height of a curve is well defined up to a bounded value.

The following theorem is a p-adic version of the Second Main Theorem in the case of holomorphic curves.

Theorem 3.3. ([KT) *Let $H_1, \dots, H_q$ be q hyperplanes in general position, and let f be a non-degenerate holomorphic curve in $P^n(C_p)$. Then we have*

$$(q-n-1)h^+(f,t) \le \sum_{j=1}^{q} N_n(f.H_j, t) + \frac{n(n+1)}{2}.t + 0(1),$$

where $0(1)$ *is bounded when* $t \to -\infty$ *and* $h^+(h,t) = -h(f,t)$.

Cherry and Ye [CY] extend the theorem to several variables. Moreover, they considered the case of degenerate curves by using the so-called Nochka's weights ([N]). Recenly, Hu and Yang [HY] obtain similar results for moving targets.

For the case of hypersurfaces we have the following result

Theorem 3.4. ([KA1]) *Let $H_1, \dots, H_q$ be hypersurfaces of degree d in $\mathbb{P}^n(\mathbb{C}_p)$ in general position. Let f be a non-degenerate holomorphic curve. Then*

$$(q-n)h^+(f,t) \le \sum_{j=1}^{q} \frac{N(f \circ H_j, t}{d} + 0(1),$$

where $0(1)$ *is bounded when* $t \to -\infty$.

This is a p-adic version of Eremenko-Sodin's theorem ([ES]).

A holomorphic curve f is called *k-non-degenerate* if the image of f is contained in a linear subspace of dimension k and is not contained in any linear subspace of dimension $k-1$.

We conclude this section by the following conjecture.

Conjecture 3.1. *Let $H_1, \dots, H_q$ be hypersurfaces of degree d_j, $j = 1, \dots, q$ in $\mathbb{P}^n(\mathbb{C}_p)$ in general position. Let f be a k-non-degenerate holomorphic curve. Let s be an integer $\ge k$, or $s = \infty$. Then*

$$(q-2n+k-1)h^+(f,t) \le \sum_{j=1}^{q} \frac{N_s(H_j \circ f, t)}{d_j} + 0(1).$$

Remark 3.3. In the complex case the above conjecture corresponds to the following cases:

1. Nevanlinna's Second Main Theorem: $n = 1$, $k = 1$, $d_j = 1$, $s = \infty$.
2. Cartan Theorem: $\forall n$, $k = n$, $d_j = 1$, $s = n$.
3. Nochka Theorem (Cartan's conjecture): $\forall n$, $\forall k \leq n$, $s = k$, $d = 1$.
4. Eremenko-Sodin's theorem: $\forall n, k = n$, $\forall d_j$, $s = \infty$.

4. Defect relation and Borel's Lemmas

Let H be a hyperplane of $\mathbb{P}^n(\mathbb{C}_p)$ such that the image of f is not contained in H. We say that f ramifies at least d ($d > 0$) over H if for all $z \in f^{-1}H$ the degree of the pull-back divisor f^*H, $deg_z f^*H \geq d$. In case $f^{-1}H = \emptyset$ we set $d = \infty$.

Theorem 4.1. *Let $H_1, \ldots, H_q$ be q hyperplanes in general position. Assume f is linearly non-degenerate and ramifies at least d_j over H_j. Then*

$$\sum_{j=1}^{q}(1 - \frac{n}{d_j}) < n + 1.$$

Remark 4.1. In the complex case we have a similar inequality, but with the sign $\leq$. The reason is that in the p-adic case, the error term in Second Main Theorem is simpler that the complex one. This is important for applications.

From Theorem 4.1 one can deduce the following p-adic version of Borel's Lemma.

Theorem 4.2. (p-adic Borel's Lemma [Q]). *Let $f_1, f_2, \ldots, f_n$ $(n \geq 3)$ be p-adic holomorphic functions without common zeros on $\mathbb{C}_p$ such that $f_1 + f_2 + ... + f_n = 0$. Then the functions $f_1, \ldots, f_{n-1}$ are linearly dependent if for $j = 1, \ldots, n$ every zero of f_j is of multiplicity at least d_j and the following condition holds:*

$$\sum_{j=1}^{n} \frac{1}{d_j} \leq \frac{1}{n-2}.$$

By using the defect relation one can prove some generalizations of Borel's lemma.

Let

$$M_j = z_1^{\alpha_{j,1}} \ldots z_{n+1}^{\alpha_{j,n+1}}, \quad 1 \leq j \leq s,$$

be distinct monomials of degree l with non-negative exponents. Let X be a hypersurface of degree dl of $\mathbb{P}^n(\mathbb{C}_p)$ defined by

$$X: \quad c_1 M_1^d + \ldots c_s M_s^d = 0,$$

where $c_j \in \mathbb{C}_p^*$ are non-zero constants.

Theorem 4.3. (p-adic analogue of Masuda-Noguchi's Theorem [M-N]). *Let $f = (f_1, .., f_{n+1}) : C_p \longrightarrow X$ be a non-constant holomorphic curve such that any $f_j \not\equiv 0$. Assume that*

$$d \geq s(s-2).$$

Then there is a decomposition of indices $\{1, 2, ..., s\} = \cup I_\gamma$ such that

(i) *Every I_γ contains at least 2 indices;*

(ii) *The ratio of $M_j^d \circ f(z)$ and $M_j^k \circ f(z)$ is constant for $j, k \in I_\gamma$;*

(iii) $\sum_{j \in I_\gamma} c_j M_j^d \circ f(z) \equiv 0$ *for all γ.*

Corollary 4.1. *For $d \geq 3$ there is no solutions of the following equation in the set of p-adic non-constant holomorphic functions having no common zeros:*

$$x^d + y^d = z^d.$$

5. P-ADIC HYPERBOLIC SPACES

Recall that a complex space is said to be hyperbolic if every holomorphic curve in it is a constant curve. In the complex case, the Borel Lemma is often used to establish the hyperbolicity of a complex space. In what follows we show some applications of p-adic Borel's Lemma in the study of p-adic hyperbolic hypersurfaces.

Although the set of hyperbolic hypersurfaces of degree d large enough with respect to n is conjectured to be Zariski dense ([Ko]), it is not easy to construct explicit examples of hyperbolic hypersurfaces.

The first example of smooth hyperbolic surfaces of even degree $d \geq 50$ was given by R. Brody and M. Green ([BG]). Now we show how to use p-adic Borel's Lemmas to construct explicit examples of p-adic hyperbolic hypersurfaces.

Let X be a hypersurface defined as above, and let $d \geq s(s-2)$. Suppose that X is not hyperbolic, and let

$$f = (f_1, ..., f_{n+1}) : \mathbb{C}_p \longrightarrow X$$

be a nonconstant holomorphic curve in X. We are going to show that $\{c_j\}$ belongs to an algebraic subset of $(\mathbb{C}_p^*)^s$. We can assume that $f_j \not\equiv 0$ for every j. By Theorem 4.3, there is a decomposition of indices $\{1, \ldots, s\} = \cup I_\xi$ such that

(i) every I_ξ contains at least 2 indices,

(ii) the ratio of $M_j^d \circ f(z)$ and $M_k^d \circ f(z)$ is constant for $j, k \in I_\xi$,

(iii) $\sum_{j \in I_\xi} c_j M_j^d \circ f(z) \equiv 0$ for all ξ.

Now for a decomposition of $\{1, \ldots, s\}$ as above, we set $b_{jk} = M_j^d \circ f(z) \,/\, M_k^d \circ f(z)$. Then the linear system of equations

$$AY = B\,,$$

where A is the matrix $\{\alpha_{j\ell} - \alpha_{k\ell}\}$, $Y = \begin{pmatrix} y_0 \\ \vdots \\ y_n \end{pmatrix}$, $B = \{\log b_{jk}\}$, has the solution $\{\log f_0, ..., \log f_n\}$. Thus, the matrix A satisfies certain conditions on the rank. On the other hand, by condition (iii) there exist $(A_0, ..., A_n) \in \mathbb{P}^n$ such that $(c_i) \in (\mathbb{C}_p^*)^s$

$$\sum_{i \in I_\xi} c_i \, A_o^{\alpha_{io}} \cdots A_n^{\alpha_{in}} = 0.$$

Hence, $(c_i) \in (\mathbb{C}_p^*)^s$ belongs to the projection $\Sigma \subset (\mathbb{C}_p^*)^s$ of an algebraic subset in $(\mathbb{C}_p^*)^s \times \mathbb{P}^n$. If we take $(c_i) \notin \Sigma$, then we have a hyperbolic hypersurface.

Examples 5.1. Let $N = 4n - 3$, $k = N(N-2) = 16(n-1)^2$. Then for generic linear functions $H_j(z_0, \ldots, z_n) \in \mathbb{C}_p^{n+1}$ $(1 \leq j \leq n)$ the hypersurface

$$X : \sum_{j=1}^{N} H_j^k = 0$$

is hyperbolic. This is p-adic version of a recent result of Siu and Yeung ([SY], 1997).

Proof. By p-adic Borel's lemma, if $f : \mathbb{C}_p \longrightarrow X$ is a non-constant holomorphic curve then $\mathrm{Im} f \subset \cap_\xi X_\xi$, where

$$X_\xi : \sum_{j \in I_\xi} H_j^k = 0.$$

The genericity of $\{H_j\}$ implies $\cap X_\xi = \emptyset$.

For the case of surfaces in $\mathbb{P}^3$ we can use the following method. Take at first a surface $X \subset \mathbb{P}^3$ such that every holomorphic curve in X is degenerate. This means that the image of a holomorphic map $f : \mathbb{C}_p \longrightarrow X$ from $\mathbb{C}_p$ into X is contained in a proper algebraic subset of X. If one could prove that the image $f(\mathbb{C}_p)$ is contained in a curve of genus at least 1, then f is a constant map (Bercovich's theorem).

Example 5.2. Let X be a surface in $\mathbb{P}^3(\mathbb{C}_p)$ defined by the equation

$$X: \; z_1^d + z_2^d + z_3^d + z_4^d + c z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} z_4^{\alpha_4} = 0,$$

where $c \neq 0$, $\sum_{i=1}^{4} \alpha_i = d$, and if there is an exponent $\alpha_i = 0$, then the others must be $\neq 1$. Then X is hyperbolic if $d \geq 24$.

Example 5.3. Let X be a curve in $\mathbb{P}^2(\mathbb{C}_p)$ defined by the equation:

$$X: \; z_1^d + z_2^d + z_3^d + c z_1^{\alpha_1} z_2^{\alpha_2} z_3^{\alpha_3} = 0,$$

where $d \geq 24$, $d > \alpha_i \geq 0$, $\sum \alpha_i = d$. Then the complement of X is p-adic hyperbolic in $\mathbb{P}^2(\mathbb{C}_p)$.

6. Unique range set of meromorphic functions

For a non-constant meromorphic function f on C and a set $S \subset C \cup \{\infty\}$ we define

$$E_f(S) = \cup_{a \in S}\{(m, z) | f(z) = a \text{ with multiplicity } m\},$$

and

$$\bar{E}_f(S) = \cup_{a \in S}\{z | f(z) = a \text{ ignoring multiplicities}\}.$$

A set $S \subset C \cup \{\infty\}$ is call *an unique range set for meromorphic functions* (URSM) if for any pair of non-constant meromorphic functions f and g on C, the condition $E_f(S) = E_g(S)$ implies $f = g$. A set $S \subset C \cup \{\infty\}$ is called *an unique range set for entire functions* (URSE) if for any pair of non-constant entire functions f and g on C, the condition $E_f(S) = E_g(S)$ implies $f = g$. The classical theorems of Nevanlinna show that $f = g$ if $\bar{E}_f(a_j) = \bar{E}_g(a_j)$ for distinct values $a_1, \ldots, a_5$, and that f is a Möbius transformation of g if $E_f(a_j) = E_g(a_j)$ for distinct values $a_1, \ldots, a_4$. Gross and Yang show that the set

$$S = \{z \in C | z + e^z = 0\}$$

is an URSE. Recently, URSE and also URSM with finitely many elements have been found by Yi ([Y1], [Y2]), Li and Yang ([LY1], [LY2]), Mues and Reinders [MR], Frank and Reinders [FR]. Li and Yang introduced the notation

$$\lambda_M = \inf\{\#S | S \text{ is a URSM}\},$$
$$\lambda_E = \inf\{\#S | S \text{ is a URSE}\},$$

where $\#S$ is the cardinality of the set S. The best lower and upper bounds known so far are

$$5 \leq \lambda_E \leq 7, \quad 6 \leq \lambda_M \leq 11.$$

For p-adic meromorphic or entire function f on C_p, similarly we can define $E_f(S)$ and $\bar{E}_f(S)$ for a set $S \subset C_p \cup \{\infty\}$ and introduce the notation λ_M and λ_E. By using p-adic Nevanlinna theory and theory of singularities we can prove the following theorems:

Theorem 6.1. *Let P be a generic polynomial of degree at least 5. Let f and g be p-adic meromorphic functions such that $P(f) = CP(g)$ with a constant C. Then $f \equiv g$.*

Theorem 6.2. *Let $S = \{a_1, a_2, a_3, a_4\}$ be a generic set of 4 points in $\mathbb{C}_p$. Then for p-adic meromorphic functions f and g, the conditions $E_f(S) = E_g(S)$ and $E_f(\infty) = E_g(\infty)$ imply $f \equiv g$.*

For the proof, see [K7].

Conjecture 6.1. *A generic set of 5 points in $\mathbb{C}_p \cup \{\infty\}$ is a URS for p-adic meromorphic functions.*

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