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RECONSTRUCTION OF ANALYTIC FUNCTIONS ON THE UNIT DISC FROM A SEQUENCE OF MOMENTS: REGULARIZATION AND ERROR ESTIMATES

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Dedicated to the memory of Le Van Thiem

Abstract. We consider the problem of reconstructing a function in the Hardy space $H^2(U)$ on the unit disc U of the complex plane from a sequence of moments. The problem, which is ill-posed, is regularized by polynomials and error estimates are given.

We consider the problem of reconstructing a function analytic in U, the unit disc of the complex plane, from its values at a given infinite sequence of points in U. We thus deal with a moment problem. The problem of approximation of analytic functions has given rise to a huge literature. The reader is referred, e.g., to [GH, AK, CM, A] and to the momograph [G].

The present moment approach to the problem, to our knowledge, is new. The results of this paper extend those in our recent works [AT] and [TA]. We consider the problem of determining a function u in the $H^2(U)$, the Hardy space of analytic functions on U to be defined in (5) below, such that

$$u(z_n) = \mu_n, \quad n = 1, 2, ... \tag{1}$$

where (z_n) is an infinite sequence in U, (μ_n) is in l_∞, the space of all complex bounded sequences. It is easily seen that the problem is ill-posed. Let u_0 be the exact solution of (1) corresponding to the exact data $\mu^0 = (\mu_n^0) \in l_\infty$ (which is often not known exactly), i.e.,

$$u_0(z_n) = \mu_n^0, \quad n = 1, 2, ... \tag{2}$$

Let $\mu = (\mu_n) \in l_\infty$ be a known "measured" data satisfying

$$\|\mu - \mu^0\|_\infty = \sup_n |\mu_n - \mu_n^0| < \varepsilon.$$

From μ, we shall construct a polynomial that, in a sense to be specified, approximates the exact solutions u_0 (in (2)). For $m \in \mathbb{N}$, we consider the following

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system of linear equations

$$\sum_{k=0}^{m-1} a_{mk} z_n^k = \mu_n, \quad n = 1, .., m. \tag{3}$$

Put

$$P_m(z) = \sum_{0 \le k \le m/2} a_{mk} z^k. \tag{4}$$

We shall give an estimate of the error between P_n and the exact solution u_0 corresponding to the exact data $\mu^0 \in l_\infty$.

Before stating our main results, we set some notations. The Hardy space $H^2(U)$ is the class of all analytic functions F on U having the form

$$F(z) = \sum_{k=0}^{\infty} \alpha_k z^k \tag{5}$$

with

$$\|F\|_{H^2}^2 \equiv \sum_{k=0}^{\infty} |\alpha_k|^2 < \infty$$

(see, e.g., [H]). We shall denote the set $\{1, ..., m\}$ by $\overline{1, m}$.

We now state our main results.

Theorem 1. *Let $\varepsilon > 0$ and let $\alpha > 0$ be a positive number satisfying*

$$0 < \alpha < 1, \quad \frac{\sqrt{\alpha}}{(1 - \sqrt{\alpha})^4} < 1.$$

Let $(z_n) \subset U$ be such that

$$z_k \ne z_j \quad \forall k \ne j, \quad |z_n| \le \alpha \quad \forall n = 1, 2, ...$$

and let $u_0 \in H^2(U)$ be the exact solution of (2) corresponding to the exact data $\mu^0 \in l_\infty$. Let $f : [1, \infty) \longrightarrow \mathbb{R}$ be such a function that $f(\theta)$ increase to $+\infty$ as $\theta \to +\infty$ and satisfies the condition

$$f(m) \ge (m+2) m^2 D_m^2 (1+\alpha)^{2m-2},$$

where

$$D_m = \sup_{1 \le n \le m} \prod_{j \in \overline{1,m} \setminus \{n\}} |z_n - z_j|^{-1}.$$

Put

$$m(\varepsilon) = [f^{-1}(\varepsilon^{-1})].$$

Then $m(\varepsilon) \to \infty$ as $\varepsilon \downarrow 0$ and there is a function $\eta(\varepsilon)$, $0 < \varepsilon < 1$, such that $\lim_{\varepsilon \downarrow 0} \eta(\varepsilon) = 0$ and

$$\|P_{m(\varepsilon)} - u_0\|_{H^2}^2 \le \eta(\varepsilon),$$

where P_m is defined in (4).

Besides, if $u_0' \in H^2(U)$ then

$$\|P_{m(\varepsilon)} - u_0\|_{H^2}^2 \le \varepsilon + \frac{m(\varepsilon)+2}{(1-\sqrt{\alpha})^2}\left(\frac{\sqrt{\alpha}}{(1-\sqrt{\alpha})^4}\right)^{m(\varepsilon)} \|u_0\|_{H^2}^2 + \frac{4\|u_0'\|_{H^2}^2}{m^2(\varepsilon)}.$$

Theorem 2. *Let the assumptions of Theorem 1 hold. If, in addition, there exists a C^1-function $\psi : [1,\infty) \longrightarrow U$ such that*

$$|\psi'(x)| \ge \beta x^{-2} \quad \forall x \in [1,\infty)$$

for some $\beta > 0$, and $\psi(n) = z_n$, then there exists an $\varepsilon_0 > 0$ such that

$$\|P_{m(\varepsilon)} - u_0\|_{H^2}^2 \le (1 + \|u_0\|_{H^2}^2 + 4\|u_0'\|_{H^2}^2)\left(\ln\frac{1}{\varepsilon}\right)^{-1}$$

for $0 < \varepsilon < \varepsilon_0$, where $m(\varepsilon) = [f^{-1}(\varepsilon^{-1})]$ and

$$f(\theta) = 4(\theta+2)\theta^2(1+\alpha)^{2\theta-2}\left(\frac{2\theta}{\beta}\right)^{2\theta-2} \qquad \forall\,\theta > 0.$$

Remark. As an example of a function ψ with the properties described in the above theorem, we can consider $\psi(x) = 1/x$, $x \ge 1$. In this case one has $z_n = 1/n$.

Proof of Theorem 1. Since $u_0 \in H^2(U)$, u_0 can be represented by a series

$$u_0(z) = \sum_{k=0}^{\infty} a_k z^k, \tag{6}$$

where

$$\sum_{k=0}^{\infty} |a_k|^2 < \infty.$$

From (4) it follows that

$$P_m(z) - u_0(z) = \sum_{0\le k\le m/2} c_{mk} z^k - \sum_{k>m/2} a_k z^k,$$

where $c_{mk} = a_{mk} - a_k$ (a_{mk} is defined in (3)). Hence

$$\|P_m - u_0\|_{H^2}^2 = \sum_{0\le k\le m/2} |c_{mk}|^2 + \sum_{k>m/2} |a_k|^2. \tag{7}$$

The latter equality shows that an estimate of the first sum in the right hand side of (7) is essential.

The remainder of our proof is divided into three steps. In Step 1 we shall represent the numbers c_{mk} as the coefficients of Lagrange polynomials. In Step 2 we shall estimate c_{mk}. Finally, in Step 3 we shall complete our proof.

Step 1. Representation of $c_{mk} \equiv a_{mk} - a_k$ as a coefficient of a Lagrange polynomial.

Recall that

$$u_0(z_n) = \mu_n^0, \quad n = 1, 2, ...$$

Hence, (6) gives

$$\sum_{k=0}^{m-1} a_k z_n^k = \mu_n^0 - \sum_{k=m}^{\infty} a_k z_n^k. \tag{8}$$

Substracting (8) from (3) gives

$$\sum_{k=0}^{m-1} c_{mk} z_n^k = \varphi_{mn}, \quad n = 1, ..., m, \tag{9}$$

where

$$\varphi_{mn} = \varepsilon_n + \sum_{k=m}^{\infty} a_k z_n^k,$$
$$\varepsilon_n = \mu_n - \mu_n^0, \quad n = 1, ..., m.$$

We wish to estimate $|c_{mk}|$. For this purpose, we shall give an explicit form for (c_{mk}). Put

$$Q_m(z) = \sum_{k=0}^{m-1} c_{mk} z^k. \tag{10}$$

Condition (9) implies that

$$Q_m(z_n) = \varphi_{mn}, \quad n = 1, .., m.$$

Hence, using Lagrange's interpolation (cf, e.g., [G], p. 61) we get

$$Q_m(z) = \sum_{n=1}^{m} \varphi_{mn} \prod_{j \in \overline{1,m} \setminus \{n\}} \left(\frac{z - z_j}{z_n - z_j} \right).$$

Since

$$\varphi_{mn} = \varepsilon_n + \sum_{k=m}^{\infty} a_k z_n^k = \varepsilon_n + \sum_{l=0}^{\infty} a_{m+l} z_{m+l}^k,$$

we have

$$Q_m(z) = Q_{1m}(z) + Q_{2m}(z), \tag{11}$$

where

$$Q_{1m}(z) = \sum_{n=1}^{m} \varepsilon_n \prod_{j \in \overline{1,m} \setminus \{n\}} \left(\frac{z - z_j}{z_n - z_j} \right),$$
$$Q_{2m}(z) = \sum_{l=0}^{\infty} a_{m+l} \sum_{n=1}^{m} z_n^{m+l} \prod_{j \in \overline{1,m} \setminus \{n\}} \left(\frac{z - z_j}{z_n - z_j} \right).$$

Now, if we denote by c_{1mk}, c_{2mk} $(k = 0, ..., m-1)$ the coefficients of z^k in the expansion of the polynomials Q_{1m}, Q_{2m} respectively, then

$$Q_{jm} = \sum_{k=0}^{m-1} c_{jmk} z^k, \quad j = 1, 2. \tag{12}$$

From (10), (11), (12), one has

$$c_{mk} = c_{1mk} + c_{2mk}. \tag{13}$$

Step 2. Estimates of c_{1mk}, c_{2mk}, c_{mk}.

i) Estimation of c_{1mk}.

We claim that

$$|c_{1mk}| \le m\varepsilon D_m (1+\alpha)^{m-1}, \quad 0 \le k \le m-1. \tag{14}$$

Let us introduce some notations. For $j = 1, 2, .., m$, put

$$\begin{aligned} \hat{z}_1 &= (z_2, ..., z_m), \\ \hat{z}_j &= (z_1, ..., z_{j-1}, z_{j+1}, ..., z_m), \quad 2 \le j \le m-1, \\ \hat{z}_m &= (z_1, ..., z_{m-1}). \end{aligned}$$

Setting $t = (t_1, ..., t_{m-1})$, we define

$$\begin{aligned} \sigma_0(t) &= 1, \\ \sigma_1(t) &= t_1 + t_2 + ... + t_{m-1}, \\ \sigma_2(t) &= \sum_{1 \le j_1 < j_2 \le m-1} t_{j_1} t_{j_2}, \\ \sigma_3(t) &= \sum_{1 \le j_1 < j_2 < j_3 \le m-1} t_{j_1} t_{j_2} t_{j_3}, \\ &... \quad ... \\ \sigma_{m-1}(t) &= t_1 ... t_{m-1}. \end{aligned}$$

Then we put

$$s_{mn} = \prod_{j \in \overline{1,m} \setminus \{n\}} (z_n - z_j).$$

Using these notations, we can rewrite the representation of Q_{1m} as follows

$$\begin{aligned} Q_{1m}(z) &= \sum_{n=1}^{m} \varepsilon_n s_{mn}^{-1} \sum_{k=0}^{m-1} z^k (-1)^{m-k-1} \sigma_{m-k-1}(\hat{z}_n) \\ &= \sum_{k=0}^{m-1} z^k \left(\sum_{n=1}^{m} \varepsilon_n s_{mn}^{-1} (-1)^{m-k-1} \sigma_{m-k-1}(\hat{z}_n) \right). \end{aligned}$$

This gives

$$c_{1mk} = \sum_{n=1}^{m} \varepsilon_n s_{mn}^{-1} (-1)^{m-k-1} \sigma_{m-k-1}(\hat{z}_n), \quad 0 \le k \le m-1.$$

Since

$$|s_{mn}|^{-1} \le \sup_{1\le n\le m} \prod_{j\in\overline{1,m}\setminus\{n\}} |z_n - z_j|^{-1}$$
$$= D_m \quad (m = 1, 2, ..., \ 1 \le n \le m),$$

we have

$$|c_{1mk}| \le \varepsilon D_m \sum_{n=1}^{m} |\sigma_{m-k-1}(\hat{z}_n)|. \tag{15}$$

On the other hand, since $|z_n| < \alpha$ for all $n = 1, 2, ...$, one has

$$|\sigma_{m-k-1}(\hat{z}_n)| \le \alpha^{m-k-1} \left(\sum_{1\le j_1<j_2<..<j_{m-k-1}\le m-1} 1 \right).$$

Note that the latter sum is the number of $(m-k-1)$-element subsets of the set $\{1, 2, .., m-1\}$. Hence

$$\sum_{1\le j_1<j_2<..<j_{m-k-1}\le m-1} 1 = C_{m-1}^{m-k-1} \tag{16}$$

where $C_m^k = m!/k!(m-k)!$.

From (15) and (16) we deduce that

$$|c_{1mk}| \le \varepsilon D_m \sum_{n=1}^{m} \alpha^{m-k-1} C_{m-1}^{m-k-1}$$
$$= \varepsilon m D_m \alpha^{m-k-1} C_{m-1}^{m-k-1}$$
$$\le \varepsilon m D_m (1+\alpha)^{m-1}.$$

ii) Estimation of $|c_{2mk}|$.

We claim that

$$|c_{2mk}| \le \sup_n |a_n| \frac{(\sqrt{\alpha})^{m-k}}{(1-\sqrt{\alpha})^{2m+1}}, \quad 0 \le k \le m-1. \tag{17}$$

Put $Z_s = (z_1, .., z_s)$. For $\beta = (\beta_1, ..., \beta_s)$ ($\beta_i = 0, 1, 2, ..., 1 \le i \le s$), we define

$$Z_s^\beta = z_1^{\beta_1} ... z_s^{\beta_s}.$$

For each $s > 1$, we put

$$R_p(Z_s, z) = \sum_{n=1}^{s} z_n^p \prod_{j\in\overline{1,s}\setminus\{n\}} \left(\frac{z - z_j}{z_n - z_j} \right)$$

and

$$R_p(Z_1, z) = z_1^p.$$

From the definition of Q_{2m} we have

$$Q_{2m}(z) = \sum_{l=0}^{\infty} a_{m+l} R_{m+l}(Z_m, z). \tag{18}$$

To continue the proof of (17), we need the following lemma (which will be proved later on).

Lemma 1. *One has*

$$R_p(Z_s, z) = \sum_{k=0}^{s-1} c_{pk}(Z_s) z^k, \quad 1 \le s \le p,$$

where

$$c_{pk}(Z_s) = \sum_{|\beta|=p-k} C_{\beta,k,p} Z_s^{\beta}, \quad 0 \le k \le s-1, \tag{19}$$

($\beta = (\beta_1, .., \beta_s)$, $|\beta| = \beta_1 + ... + \beta_s$, $Z_s^{\beta} = z_1^{\beta_1} ... z_s^{\beta_s}$), and

$$|C_{\beta,k,p}| \le C_p^s. \tag{20}$$

We now return to the proof of (17). By Lemma 1, from (18) we get

$$\begin{aligned} Q_{2m}(z) &= \sum_{l=0}^{\infty} a_{m+l} R_{m+l}(Z_m, z) \\ &= \sum_{l=0}^{\infty} a_{m+l} \sum_{k=0}^{m-1} c_{m+l,k}(Z_m) z^k \\ &= \sum_{k=0}^{m-1} \left(\sum_{l=0}^{\infty} a_{m+l} c_{m+l,k}(Z_m) \right) z^k. \end{aligned}$$

Hence

$$c_{2mk} = \sum_{l=0}^{\infty} a_{m+l} c_{m+l,k}(Z_m).$$

This implies in view of (19) and (20) that

$$\begin{aligned} |c_{2mk}| &\le \sup_n |a_n| \sum_{l=0}^{\infty} \left(\sum_{|\beta|=m+l-k} |C_{\beta,k,m+l} Z_m^{\beta}| \right) \\ &\le \sup_n |a_n| \sum_{l=0}^{\infty} C_{m+l}^m \sum_{|\beta|=m+l-k} |Z_m^{\beta}|. \end{aligned} \tag{21}$$

On the other hand, one has

$$|Z_m^{\beta}| = |z_1|^{\beta_1} ... |z_m|^{\beta_m} \le \alpha^{\beta_1} ... \alpha^{\beta_m} = \alpha^{|\beta|}.$$

Hence (21) implies

$$|c_{2mk}| \le \sup |a_n| \sum_{l=0}^{\infty} C_{m+l}^{m} \alpha^{m+l-k} \left(\sum_{|\beta|=m+l-k} 1 \right). \tag{22}$$

We shall calculate the quantity

$$\sum_{|\beta|=m+l-k} 1, \quad \beta = (\beta_1, ..., \beta_m).$$

For all z in U and $(\theta_n) \subset \overline{U}$, one has the expansion

$$\frac{1}{1-\theta_n z} = 1 + \theta_n z + \theta_n^2 z^2 + ..., \quad \forall n = 1, 2, ...$$

Hence

$$\prod_{n=1}^{m} \frac{1}{1-\theta_n z} = \sum_{p=0}^{\infty} \left(\sum_{|\beta|=p} \theta_1^{\beta_1} ... \theta_m^{\beta_m} \right) z^p.$$

In particular, one has for $\theta_n = 1 \ \ (n = 1, 2, .., m)$ the equality

$$\frac{1}{(1-z)^m} = \sum_{p=0}^{\infty} \left(\sum_{|\beta|=p} 1 \right) z^p. \tag{23}$$

On the other hand, one has the Taylor's expansion (cf, e.g., [T])

$$\frac{1}{(1-z)^m} = \sum_{p=0}^{\infty} C_{m-1+p}^{m-1} z^p. \tag{24}$$

Combining (23) with (24) gives

$$\sum_{|\beta|=p} 1 = C_{m-1+p}^{m-1}.$$

For $p = m + l - k$, the latter equality implies

$$\sum_{|\beta|=m+l-k} 1 = C_{2m+l-k-1}^{m-1}.$$

So we can write (22) as follows

$$|c_{2mk}| \le \sup_n |a_n| \sum_{l=0}^{\infty} C_{m+l}^{m} C_{2m+l-k-1}^{m-1} \alpha^{m+l-k}.$$

This yields

$$\begin{aligned} |c_{2mk}| &\le \sup_n |a_n| \sum_{l=0}^{\infty} C_{m+l}^{m} (\sqrt{\alpha})^{m+l-k} C_{2m+l-k-1}^{m-1} \sqrt{\alpha}^{m+l-k} \\ &\le \sup_n |a_n| (\sqrt{\alpha})^{m-k} \left(\sum_{l=0}^{\infty} C_{m+l}^{m} (\sqrt{\alpha})^{l} \right) \left(\sum_{l=0}^{\infty} C_{2m+l-k-1}^{m-1} (\sqrt{\alpha})^{m+l-k} \right). \end{aligned}$$

Using Taylor's expansion (24), one has

$$\begin{aligned}|c_{2mk}| &\leq \sup_n |a_n|(\sqrt{\alpha})^{m-k}\frac{1}{(1-\sqrt{\alpha})^{m+1}}\cdot\frac{1}{(1-\sqrt{\alpha})^{m}}\\ &\leq \sup_n |a_n|\frac{(\sqrt{\alpha})^{m-k}}{(1-\sqrt{\alpha})^{2m+1}}\cdot\end{aligned}$$

Inequality (17) has been established.

iii) Estimation of c_{mk}.

One has $c_{mk} = c_{1mk} + c_{2mk}$. Hence, using (14) and (17) we obtain

$$\begin{aligned}|c_{mk}| &\leq |c_{1mk}| + |c_{2mk}|\\ &\leq \varepsilon m D_m(1+\alpha)^{m-1} + \sup_n |a_n|\frac{(\sqrt{\alpha})^{m-k}}{(1-\sqrt{\alpha})^{2m+1}}\cdot\end{aligned}$$

For $0 \leq k \leq m/2$, the latter inequality implies that

$$|c_{mk}| \leq \varepsilon m D_m(1+\alpha)^{m-1} + \frac{\sup_n |a_n|}{1-\sqrt{\alpha}}\left(\frac{\sqrt[4]{\alpha}}{(1-\sqrt{\alpha})^2}\right)^m. \tag{25}$$

Step 3. Estimation of $\|P_{m(\varepsilon)} - u_0\|_{H^2}$.

From (7), it follows that

$$\|P_{m(\varepsilon)} - u_0\|^2_{H^2} = \sum_{0\leq k\leq m(\varepsilon)/2} |c_{mk}|^2 + \sum_{k>m(\varepsilon)/2} |a_k|^2. \tag{26}$$

By (25), we can apply the inequality

$$(a+b)^2 \leq 2(a^2+b^2), \quad \forall a, b \in \mathbf{R},$$

to get for $0 \leq k \leq m/2$ the following

$$|c_{mk}|^2 \leq 2\left(\varepsilon^2 m^2 D_m^2(1+\alpha)^{2m-2} + \frac{\sup_n |a_n|^2}{(1-\sqrt{\alpha})^2}\left(\frac{\sqrt{\alpha}}{(1-\sqrt{\alpha})^4}\right)^m\right). \tag{27}$$

Combining (26) with (27) gives

$$\begin{aligned}\|P_{m(\varepsilon)} - u_0\|^2_{H^2} \leq& 2(1+m(\varepsilon)/2)\varepsilon^2 m^2(\varepsilon) D^2_{m(\varepsilon)}(1+\alpha)^{2m(\varepsilon)-2} +\\ &+ 2(1+m(\varepsilon)/2)\frac{\sup\limits_n |a_n|^2}{(1-\sqrt{\alpha})^2}\left(\frac{\sqrt{\alpha}}{(1-\sqrt{\alpha})^4}\right)^{m(\varepsilon)}\\ &+ \sum_{k>m(\varepsilon)/2} |a_k|^2.\end{aligned} \tag{28}$$

From the definition of $m(\varepsilon)$, one sees that $m(\varepsilon) \to +\infty$ as $\varepsilon \downarrow 0$ and

$$\varepsilon^{-1} \geq f(m(\varepsilon)) \geq (m(\varepsilon)+2)m^2(\varepsilon)D^2_{m(\varepsilon)}(1+\alpha)^{2m(\varepsilon)-2}.$$

Hence (28) implies

$$\begin{aligned}\|P_{m(\varepsilon)} - u_0\|^2_{H^2} \leq & \varepsilon + 2(1 + m(\varepsilon)/2)\frac{\sup_n |a_n|^2}{(1-\sqrt{\alpha})^2}\left(\frac{\sqrt{\alpha}}{(1-\sqrt{\alpha})^4}\right)^{m(\varepsilon)} \\ & + \sum_{k>m(\varepsilon)/2} |a_k|^2 \equiv \eta(\varepsilon).\end{aligned} \tag{29}$$

Now, if $u_0' \in H^2(U)$ then

$$\sum_{k=0}^{\infty} k^2 |a_k|^2 = \|u_0'\|^2_{H^2} < \infty.$$

Hence

$$\sum_{k>m(\varepsilon)/2} |a_k|^2 \leq \frac{4}{m^2(\varepsilon)} \sum_{k>m(\varepsilon)/2} k^2 |a_k|^2 \leq \frac{4\|u_0'\|^2_{H^2}}{m^2(\varepsilon)}. \tag{30}$$

From (29) and (30) we get

$$\|P_{m(\varepsilon)} - u_0\|^2_{H^2} \leq \varepsilon + (2 + m(\varepsilon))\frac{\|u_0\|^2_{H^2}}{(1-\sqrt{\alpha})^2}\left(\frac{\sqrt{\alpha}}{(1-\sqrt{\alpha})^4}\right)^{m(\varepsilon)} + \frac{4\|u_0'\|^2_{H^2}}{m^2(\varepsilon)}.$$

This completes the proof of Theorem 1 provided that Lemma 1 is proved. □

Proof of Lemma 1. We shall prove the lemma by induction in p. For $p = 1$, we have $s = 1$. From the definition of $R_p(Z_s, z)$ one has $R_1(Z_1, z) = z_1$, i.e., the lemma is true for $p = 1$.

To proved by induction we suppose that the statement holds for $p = r$. We shall prove that the statement also holds for $p = r + 1$. We first claim that

(31)

$$R_{r+1}(Z_s, z) = z_s(R_r(Z_s, z) - R_r(Z_{s-1}, z)) + zR_r(Z_{s-1}, z), \quad 2 \leq s \leq r + 1.$$

In fact, one has

$$\begin{aligned}R_{r+1}(Z_s, z) - z_s R_r(Z_s, z) &= \sum_{n=1}^{s-1} (z_n^{r+1} - z_s z_n^r) \prod_{j \in \overline{1,s}\setminus\{n\}} \left(\frac{z - z_j}{z_n - z_j}\right) \\ &= (z - z_s) R_r(Z_{s-1}, z)\end{aligned}$$

From this it follows that (31) holds.

To continue the proof, we first consider the case $p = r + 1$, $s = 1$. Since $R_{r+1}(Z_1, z) = z_1^{r+1}$, Lemma 1 holds for $p = r + 1$, $s = 1$. For $2 \leq s \leq r$, one has from the induction hypothesis that

$$R_r(Z_s,z)=\sum_{k=0}^{s-1}\left(\sum_{|\beta|=r-k}C_{\beta,k,r}Z_s^{\beta}\right)z^k, \tag{32}$$

$$R_r(Z_{s-1},z)=\sum_{k=0}^{s-2}\left(\sum_{|\tilde\beta|=r-k}C_{\tilde\beta,k,r}Z_s^{\tilde\beta}\right)z^k, \tag{33}$$

where $\beta=(\beta_1,...,\beta_s)$, $\tilde\beta=(\tilde\beta_1,...,\tilde\beta_{s-1})$, $2\le s\le r$, and

$$|C_{\beta,k,r}|\le C_r^s \qquad (0\le k\le s-1), \tag{34}$$

$$|C_{\tilde\beta,k,r}|\le C_r^{s-1} \quad (0\le k\le s-2). \tag{35}$$

But we have

$$\beta=(\tilde\beta,\beta_s),\quad Z_s^{\beta}=Z_{s-1}^{\tilde\beta}z_s^{\beta_s},\quad Z_{s-1}^{\tilde\beta}=Z_s^{(\tilde\beta,0)}.$$

Hence (32) can be written as

(36)

$$R_r(Z_s,z)=\sum_{k=0}^{s-1}\left(\sum_{|\tilde\beta|=r-k}C_{\beta,k,r}Z_s^{(\tilde\beta,0)}\right)z^k+\sum_{k=0}^{s-1}\left(\sum_{|\tilde\beta|<r-k}C_{\beta,k,r}Z_s^{(\tilde\beta,r-k-|\tilde\beta|)}\right)z^k.$$

For convenience, in the following calculations we put $C_{\tilde\beta,s-1,r}=0$. Substituting (33), (36) into (31), one has for $2\le s\le r$ the following

$$\begin{aligned}R_{r+1}(Z_s,z)&=\sum_{k=0}^{s-1}\left(\sum_{|\tilde\beta|=r-k}(C_{\beta,k,r}-C_{\tilde\beta,k,r})z_sZ_{s-1}^{\tilde\beta}\right)z^k\\&\quad+\sum_{k=0}^{s-1}\left(\sum_{|\tilde\beta|<r-k}C_{\beta,k,r}Z_s^{(\tilde\beta,r-k-|\tilde\beta|+1)}\right)z^k+\sum_{k=0}^{s-2}\left(\sum_{|\tilde\beta|=r-k}C_{\tilde\beta,k,r}Z_{s-1}^{\tilde\beta}\right)z^{k+1}\\&=\sum_{k=0}^{s-1}c_{r+1,k}(Z_s)z^k,\end{aligned}\tag{37}$$

where

$$\begin{aligned}c_{r+1,k}(Z_s)&=\sum_{|\tilde\beta|=r-k}(C_{\beta,k,r}-C_{\tilde\beta,k,r})z_sZ_{s-1}^{\tilde\beta}+\sum_{|\tilde\beta|<r-k}C_{\beta,k,r}Z_s^{(\tilde\beta,r-k-|\tilde\beta|+1)}\\&\quad+\sum_{|\tilde\beta|=r-k+1}C_{\tilde\beta,k-1,r}Z_{s-1}^{\tilde\beta}\quad(1\le k\le s-1),\end{aligned}\tag{38}$$

$$c_{r+1,0}(Z_s)=\sum_{|\tilde\beta|=r}(C_{\beta,k,r}-C_{\tilde\beta,k,r})z_sZ_{s-1}^{\tilde\beta}+\sum_{|\tilde\beta|<r}C_{\beta,0,r}Z_s^{(\tilde\beta,r-|\tilde\beta|+1)}. \tag{39}$$

Let us check (20) for $p = r+1,\ 2 \le s \le r$. From (38), one obtains for $1 \le k \le s-1$ the equality

$$c_{r+1,k}(Z_s) = \sum_{|\beta|=r-k+1} C_{\beta,k,r+1} Z_s^{\beta}.$$

where $\beta = (\tilde{\beta}, \beta_s)$ and

$$C_{\beta,k,r+1} = \begin{cases} C_{\beta,k,r} - C_{\tilde{\beta},k,r} & \text{for } |\tilde{\beta}| = r-k, \\ C_{\beta,k,r} & \text{for } |\tilde{\beta}| < r-k, \\ C_{\tilde{\beta},k,r} & \text{for } |\tilde{\beta}| = r-k+1. \end{cases}$$

Then we can use (34), (35) and the identity

$$C_r^{s-1} + C_r^s = C_{r+1}^s$$

to deduce that

$$|C_{\beta,k,r+1}| \le C_{r+1}^k \quad (1 \le k \le s-1).$$

Similarly, for $k = 0$, we can use the representation (39) to get

$$|C_{\beta,0,r+1}| \le C_{r+1}^0.$$

Thus the lemma holds for $p = r+1$ and $2 \le s \le r$. In the case where $p = r+1$, $s = r+1$, we can replace (32) by the following equality

$$R_r(Z_{r+1}, z) = z^r. \tag{40}$$

(In fact, the degree of the polynomial $R_r(Z_{r+1}, .)$ is r and $R_r(Z_{r+1}, z_n) = z_n^r$ for $n = 1, 2, .., r+1$. Hence (40) holds). Using the same arguments as for the case $2 \le s \le r$ we shall get (19), (20) for $p = r+1$, $s = r+1$. As mentioned earlier, Lemma 1 holds for $p = r+1$, $s = 1$; hence the above arguments show that Lemma 1 holds for $p = r+1$, $1 \le s \le r+1$. The proof of Lemma 1 is complete. $\square$

Proof of Theorem 2. Put

$$\psi_1(x) = \psi(1/x), \quad \forall x \in (0, 1].$$

One has

$$|\psi_1'(x)| = \left| \frac{1}{x^2} \psi(1/x) \right| \ge \beta \quad \forall x \in (0, 1],$$

and

$$\lim_{x \to 0^+} \psi_1(x) = 0, \quad z_n = \psi_1(1/n).$$

Using the mean value theorem, one has

$$|z_n - z_m| = |\psi_1(1/n) - \psi_1(1/m)| \ge \beta \left| \frac{1}{n} - \frac{1}{m} \right| .$$

Hence,

$$
\begin{aligned}
|s_{mn}| &\equiv \prod_{j\in\overline{1,m}\setminus\{n\}} |(z_n - z_j)| \\
&\geq \beta^{m-1}\left|\left(\frac{1}{n}-1\right)\left(\frac{1}{n}-\frac{1}{2}\right)\cdots\left(\frac{1}{n}-\frac{1}{n-1}\right)\left(\frac{1}{n}-\frac{1}{n+1}\right)\cdots\left(\frac{1}{n}-\frac{1}{m}\right)\right| \\
&= \frac{\beta^{m-1}(n-1)!(m-n)!}{n^{m-2}m!}, \qquad (1\leq n\leq m).
\end{aligned}
$$

This implies that

$$|s_{mn}|^{-1} \leq \frac{n^{m-2}m!}{\beta^{m-1}(n-1)!(m-n)!}\,.$$

Hence

$$
\begin{aligned}
D_m = \sup_{1\leq n\leq m} |s_{mn}|^{-1} &\leq \frac{m!}{\beta^{m-1}} \sup_{1\leq n\leq m} \frac{n^{m-2}}{(n-1)!(m-n)!} \\
&= \frac{1}{\beta^{m-1}} \sup_{1\leq n\leq m} n^{m-1} C_m^n.
\end{aligned}
$$

But, one has

$$C_m^n \leq (1+1)^m = 2^m.$$

Therefore

$$D_m \leq 2\frac{m^{m-1}2^{m-1}}{\beta^{m-1}} = 2\left(\frac{2m}{\beta}\right)^{m-1}.$$

Put

$$f(\theta) = 4(\theta+2)\theta^2(1+\alpha)^{2\theta-2}\left(\frac{2\theta}{\beta}\right)^{2\theta-2}.$$

We can verify that f is increasing for $\theta > \beta e/2$. By Theorem 1, one has

(41)

$$\|P_{m(\varepsilon)} - u_0\|_{H^2}^2 \leq \varepsilon + 2(1+m(\varepsilon)/2)\frac{\|u_0\|_{H^2}^2}{(1-\sqrt{\alpha})^2}\left(\frac{\sqrt{\alpha}}{(1-\sqrt{\alpha})^4}\right)^{m(\varepsilon)} + \frac{4\|u_0'\|_{H^2}^2}{m^2(\varepsilon)}$$

where $m(\varepsilon) = [f^{-1}(\varepsilon^{-1})]$. We shall estimate the latter quantity.

There exists an $\varepsilon_0' > 0$ such that $m(\varepsilon) > \beta e/2$ for $0 < \varepsilon < \varepsilon_0'$. In this case, since f is increasing on $(\beta e/2, +\infty)$, one has

$$\varepsilon^{-1} \leq f(m(\varepsilon)+1)$$

or, equivalently,

$$4(\theta_\varepsilon+2)\theta_\varepsilon^2(1+\alpha)^{2\theta_\varepsilon-2}\left(\frac{2\theta_\varepsilon}{\beta}\right)^{2\theta_\varepsilon-2} \geq \frac{1}{\varepsilon}\,, \quad \theta_\varepsilon = m(\varepsilon)+1.$$

So we have

$$2(\theta_\varepsilon-1)\ln\left(\frac{2(1+\alpha)\theta_\varepsilon}{\beta}\right) + \ln(4(\theta_\varepsilon+2)\theta_\varepsilon^2) \geq \ln\left(\frac{1}{\varepsilon}\right). \tag{42}$$

For $\varepsilon \to 0$, one has $\theta_\varepsilon \to +\infty$ and

$$(\theta_\varepsilon - 1)^2 \geq LHS \quad \text{of (42)} \;\geq \ln\left(\frac{1}{\varepsilon}\right)\cdot$$

Hence

$$m(\varepsilon) = \theta_\varepsilon - 1 \geq \left(\ln\frac{1}{\varepsilon}\right)^{1/2}. \tag{43}$$

By (43), we can find an ε_0 such that $0 < \varepsilon_0 < \varepsilon_0'$ and

$$\varepsilon \leq (m(\varepsilon))^{-2} \leq \left(\ln\frac{1}{\varepsilon}\right)^{-1}, \tag{44}$$

$$\frac{(2+m(\varepsilon))}{(1-\sqrt{\alpha})^2}\left(\frac{\sqrt{\alpha}}{(1-\sqrt{\alpha})^4}\right)^{m(\varepsilon)} \leq \left(\ln\frac{1}{\varepsilon}\right)^{-1} \tag{45}$$

for $0 < \varepsilon < \varepsilon_0$.

Substituting (43)-(45) into (41) gives

$$\|P_{m(\varepsilon)} - u_0\|^2_{H^2} \leq (1 + \|u_0\|^2_{H^2} + 4\|u_0'\|^2_{H^2})\left(\ln\frac{1}{\varepsilon}\right)^{-1}.$$

This completes the proof of Theorem 2. $\square$

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