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DYNAMICS OF PERMUTABLE TRANSCENDENTAL ENTIRE FUNCTIONS

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Dedicated to the memory of Le Van Thiem

Abstract. Let f and g be two permutable transcendental entire functions. We shall prove that they have the same Julia set (i.e., $J(f) = J(g)$) if the set of the asymptotic values and critical values of f and g is bounded. This relates to a result and an open problem of Baker in the Fatou-Julia theory. In addition, for any positive integers n and m, we show that $J(f \circ g) = J(f^n \circ g^m)$.

1. Introduction and main results

Let f be a nonconstant meromorphic function. The sequence of the iterates of f is defined by setting

$$f^0 = id,\ f^1 = f,\ \dots,\ f^{n+1} = f^n(f),\ \dots.$$

Let

$$F = F(f) = \big\{z \in \mathbb{C} :\ \text{the sequence } \{f^n\} \text{ is defined and normal at } z\big\}$$

and

$$J = J(f) = \mathbb{C} - F(f).$$

These sets are called the Fatou set and the Julia set of f, respectively. Here the concept "normal" is in the sense of Montel. According to the definition, F is open (possibly empty) and J is closed.

Let f and g be two nonconstant meromorphic functions. If

$$f(g) = g(f), \tag{1}$$

then we say that f and g are permutable.

Fatou [4] proved the following result.

Theorem A. *For two given rational functions R_1 and R_2, if they are permutable, then $F(R_1) = F(R_2)$.*

The following question is natural (see Baker [1]):

Question. *For two given permutable transcendental entire functions f and g, does it follows that $F(f) = F(g)$?*

In some special cases, this question was affirmatively solved.

Theorem B (Baker [1]). *Suppose that f and g are permutable transcendental entire functions, and $f = g + c$ for some constant c. Then $F(f) = F(g)$.*

A point a is called a singular value if it is either a critical value or an asymptotic value. We denote by $\text{sing}(f^{-1})$ the set of all finite singular values of f. If the set $\text{sing}(f^{-1})$ is bounded, then we say f is of bounded type, in particular, if the set $\text{sing}(f^{-1})$ is finite, then f is called to be of finite type. We denote them by $f \in B$ and $f \in S$ respectively (cf. [2]).

Theorem C (Poon and Yang [6]). *Suppose that f and g are permutable transcendental entire functions. If both $\text{sing}(f^{-1})$ and $\text{sing}(g^{-1})$ are isolated in the finite complex plane, then $F(f) = F(g)$.*

Remark. Theorem C includes the case that f and g are of finite type. This conclusion was also proved independently by Ren and Li [7].

In this paper, by using a result due to Eremenko and Lyubich [3], we shall prove the following theorem.

Theorem 1. *Let f and g be two permutable transcendental entire functions. If f and g are of bounded type, then*

$$J(f) = J(g).$$

In the general case, we have the following statement.

Theorem 2. *If f and g are two permutable transcendental entire functions, then*

$$J(f \circ g) = J(f^n \circ g^m), \quad \forall m, n \geq 1.$$

2. Preliminaries

Let f be a transcendental entire function, $a \in \mathbb{C}$. If there exist a polynomial $p(z)$ and a nonconstant entire function $h(z)$ such that

$$f(z) = p(z)e^{h(z)} + a,$$

then we call a a big Picard exceptional value of f, and we denote the set of all such values by $PV^*(f)$. Furthermore, if $p(z) = (z - a)^k$ for some integer $k \geq 0$ then a is said to be a Fatou exceptional value of f. In particular, if $k = 0$ then a is a Picard exceptional value of f. We denote by $FV(f)$ and $PV(f)$ the Fatou exceptional values and the Picard exceptional values of f, respectively. By the Picard Theorem, each of the above three sets contains at most one point. Obviously,

$$PV(f) \subset FV(f) \subset PV^*(f).$$

For two permutable transcendental entire functions f and g, we have

$$PV(f \circ g) = PV(f) \cup PV(g).$$

In fact, if $PV(f \circ g)$ contains a point x and $x \notin PV(f)$, then there exists a point z_0 such that $f(z_0) = x$. Note that $f \circ g(z) \neq x$ for any $z \in \mathbb{C}$, thus $g(z) \neq z_0$, i.e., $z_0 \in PV(g) \subset PV^*(g)$. Since $f(g) = g(f)$, from $x \in PV(g \circ f)$ we deduce that $x \in PV^*(g)$. Thus $x, z_0 \in PV^*(g)$. Since $PV^*(g)$ contains at most one element, we have $z_0 = x$, so $x \in PV(g)$. Thus $PV(f \circ g) \subset PV(f) \cup PV(g)$. The reverse inclusion is obvious.

Lemma 1. ([8]) *Let f be a transcendental entire function. Then*

$$f^{-1}(F(f)) = F(f) = f(F(f)) \cup \{PV(f) \cap F(f)\},$$
$$f^{-1}(J(f)) = J(f) = f(J(f)) \cup \{PV(f) \cap J(f)\}.$$

Lemma 2. (Baker [1]) *Let f and g be two permutable transcendental entire functions. Then $g(J(f)) \subset J(f)$ and $f(J(g)) \subset J(g)$.*

We define

$$I(f) = \{a : a \in \mathbb{C},\ f^n(a) \to \infty \quad \text{as} \quad n \to \infty\}.$$

Lemma 3. ([3]) *If $f \in B$ be a transcendental entire function, then $J(f) = \overline{I(f)}$.*

Lemma 4. (cf. [5]) *Let f be a transcendental entire function, $b \in \mathbb{C} \backslash FV(f)$. We have*

$$J(f) \subset \overline{\left(\bigcup_{n=0}^{\infty} f^{-n}(b) \right)}.$$

Furthermore, if $b \in J(f) \backslash FV(f)$ then

$$J(f) = \overline{\left(\bigcup_{n=0}^{\infty} f^{-n}(b) \right)}.$$

Lemma 5. *Let f_1 and f_2 be two permutable transcendental entire functions. Then*

$$F(f_1 \circ f_2) \subset F(f_1) \cap F(f_2). \tag{2}$$

Proof. Since f_1 and f_2 are permutable, we have

$$f_1 \circ f_1(f_2) = f_1(f_2) \circ f_1, \quad f_2 \circ f_1(f_2) = f_1(f_2) \circ f_2.$$

It follows from Lemma 2 that

$$f_1(J(f_1(f_2))), f_2(J(f_1(f_2))) \subset J(f_1(f_2)). \tag{3}$$

This and Lemma 1 imply that

$$\begin{aligned} J(f_1(f_2)) &= f_1 \circ f_2(J(f_1(f_2))) \cup \{PV(f_1(f_2)) \cap J(f_1(f_2))\} \\ &= f_2 \circ f_1(J(f_1(f_2))) \cup \{PV(f_1(f_2)) \cap J(f_1(f_2))\} \\ &\subset f_2(J(f_1(f_2))) \cup \{PV(f_1(f_2)) \cap J(f_1(f_2))\} \\ &\subset J(f_1(f_2)). \end{aligned}$$

Thus

$$f_2(J(f_1(f_2))) \cup \{PV(f_1(f_2)) \cap J(f_1(f_2))\} = J(f_1(f_2)). \tag{4}$$

Similarly we have

$$f_1(J(f_1(f_2))) \cup \{PV(f_1(f_2)) \cap J(f_1(f_2))\} = J(f_1(f_2)). \tag{5}$$

Next we shall prove that

$$f_2^{-1}(J(f_1(f_2))) = J(f_1(f_2)) \tag{6}$$

and

$$f_1^{-1}(J(f_1(f_2))) = J(f_1(f_2)). \tag{7}$$

In fact, for any $a \in f_2^{-1}(J(f_1(f_2)))$, i.e., $f_2(a) \in J(f_1(f_2))$, from (3) we deduce that $f_1(f_2(a)) \in J(f_1(f_2))$. Applying Lemma 1 to the function $f_1(f_2)$ we obtain $a \in J(f_1(f_2))$. Hence

$$f_2^{-1}(J(f_1(f_2))) \subset J(f_1(f_2)).$$

The converse follows from (4). Thus (6) holds.

The equation (7) can be proved similarly. It follows from (4)-(7) that

$$f_2^{-1}(F(f_1(f_2))) = F(f_1(f_2)) = f_2(F(f_1(f_2))) \cup \{PV(f_2) \cap F(f_1(f_2))\}. \tag{8}$$

In fact, if $b \in F(f_1(f_2)) \backslash PV(f_2)$, then there exists $c \in \mathbb{C}$ such that $f_2(c) = b$. From (4) we see that $c \in F(f_1(f_2))$, and so $b \in f_2(F(f_1(f_2)))$. Thus

$$F(f_1(f_2)) \subset f_2(F(f_1(f_2))) \cup \{PV(f_2) \cap F(f_1(f_2))\}.$$

All other relations can be proved similarly.

Similarly we have

$$f_1^{-1}(F(f_1(f_2))) = F(f_1(f_2)) = f_1(F(f_1(f_2))) \cup \{PV(f_1) \cap F(f_1(f_2))\}. \tag{9}$$

It follows from (8) and (9) that, for any positive integer k,

$$F(f_1(f_2)) = f_2^k(F(f_1(f_2))) \cup \Big\{ \bigcup_{j=0}^{k} f_2^j(PV(f_2)) \cap F(f_1(f_2)) \Big\} \tag{10}$$

and

$$F(f_1(f_2)) = f_1^k(F(f_1(f_2))) \cup \Big\{ \bigcup_{j=0}^{k} f_1^j(PV(f_2)) \cap F(f_1(f_2)) \Big\}. \tag{11}$$

In fact, to prove (10) we need only to show that

$$F(f_1(f_2)) \subset f_2^k(F(f_1(f_2))) \cup \Big\{ \bigcup_{j=0}^{k} f_2^j(PV(f_2)) \cap F(f_1(f_2)) \Big\}. \tag{12}$$

The reverse inclussion follows from (8). Let

$$a \in F(f_1(f_2)) \backslash \bigcup_{j=0}^{k} f_2^j(PV(f_2)).$$

Then by (8), $a \in f_2(F(f_1(f_2)))$. Thus there exists a point $y_1 \in F(f_1(f_2))$ such that $a = f_2(y_1)$. Since

$$y_1 \in F(f_1(f_2)) \backslash \bigcup_{j=0}^{k-1} f_2^j(F(f_1(f_2)),$$

there exists a point $y_2 \in F(f_1(f_2))$ such that $y_1 = f_2(y_2)$, hence $a = f_2^2(y_2)$. By induction, there exists a point $y_k \in F(f_1(f_2))$ such that $a = f_2^k(F(f_1(f_2)))$, hence (12) holds. This proves (10). The proof of (11) is the same.

Combining (10), (11) and Montel's theorem, $\{f_2^k\}$ and $\{f_1^k\}$ are normal in $F(f_1(f_2))$. We thus get (2).

Lemma 6. (cf. [5]) *Let f be a transcendental entire function, $n \geq 1$. Then we have $F(f) = F(f^n)$.*

3. Proof of Theorem 1

At first we prove that

$$g^{-1}[I(f)] \subset I(f). \tag{13}$$

Let $a \in g^{-1}[I(f)]$, that is, $g(a) \in I(f)$. Then $f^n(g(a)) \longrightarrow \infty$ as $n \to \infty$. Note that $f^n(g) = g(f^n)$ for any positive integer n. We thus have $g(f^n(a)) \longrightarrow \infty$ as $n \to \infty$. This implies that $f^n(a) \longrightarrow \infty$ as $n \to \infty$. Therefore $a \in I(f)$ and (13) holds.

We now take a point $a \in I(f)$ such that a is not a Fatou exceptional value of f. From (13) it follows that

$$g^{-n}(a) \subset I(f), \quad \forall n \geq 1,$$

consequently

$$\bigcup_{n=1}^{\infty} g^{-n}(a) \subset I(f).$$

By Lemmas 3 and 4,

$$J(g) \subset \overline{\bigcup_{n=1}^{\infty} g^{-n}(a)} \subset \overline{I(f)} = J(f).$$

Similarly we can get $J(f) \subset J(g)$. Thus $J(g) = J(f)$.

The proof of Theorem 1 is complete. □

4. Proof of Theorem 2

For two given positive integers n and m, we shall prove that

$$F(f \circ g) = F(f^n \circ g^m). \tag{14}$$

Let $t > \max\{n, m\}$. From Lemma 6 we get

$$F(f \circ g) = F((f \circ g)^t). \tag{15}$$

Now by (1),

$$(f \circ g)^t = (f^{t-n} \circ g^{t-m}) \circ (f^n \circ g^m) = (f^n \circ g^m) \circ (f^{t-n} \circ g^{t-m}).$$

Applying Lemma 5 to $f_1 = f^n \circ g^m$ and $f_2 = f^{t-n} \circ g^{t-m}$ we get

$$F((f \circ g)^t) \subset F(f^n \circ g^m). \tag{16}$$

Similarly, by (1) we have

$$f^n \circ g^m = (f \circ g) \circ (f^{n-1} \circ g^{m-1}) = (f^{n-1} \circ g^{m-1}) \circ (f \circ g).$$

Applying Lemma 5 to $f_1 = f \circ g$ and $f_2 = f^{n-1} \circ g^{m-1}$ we obtain

$$F(f^n \circ g^m) \subset F(f \circ g).$$

Combining this with (15) and (16) we get (14).

The proof is complete. □

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