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ON THE RELATIVE INTRINSIC PSEUDO DISTANCE AND THE HYPERBOLIC IMBEDABILITY

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Dedicated to the memory of Le Van Thiem

Abstract. In this note we establidh a relation between the Kobayashi relative intrinsic pseudo distance of a holomorphi fiber bundle and the one in its base. Moreover, we prove that if $(\widetilde{Z}, \pi, Z)$ is a fiber bundle with compact hyperbolic fiber and $M \subset Z$ with $d_{M,Z}$ induces the given topology on $\overline{M}$, then M is hyperbolically imbedded in Z if and only if $\tilde{Y} = \pi^{-1}(Y)$ is hyperbolically imbedded in $\widetilde{Z}$.

1. Introduction

The relative pseudo distance was defined by Kobayashi in [2]. Let Z be a complex space and Y a relatively compact complex subspace of Z. Put

$$\mathcal{F}_{Y,Z} = \big\{ f \in \mathrm{Hol}(D, Z) | f^{-1}(Z - Y) \text{ is at most singleton} \big\}.$$

We define a pseudo distance $d_{Y,Z}$ on $\overline{Y}$ in the same way as the Kobayashi pseudo distance d_Z on Z (see [1]), but using only chains of holomorphic discs belonging to $\mathcal{F}_{Y,Z}$. Namely, writing $\ell(\alpha)$ for the length of a chain α of holomorphic discs. We set

$$d_{Y,Z}(p,q) = \inf_{\alpha} \ell(\alpha), \quad p, q \in \overline{Y},$$

where the infimum is taken over all chains α of holomorphic discs from p to q which belong to $\mathcal{F}_{Y,Z}$ (see [2], [3], [4]).

We note that $d_Z \le d_{Y,Z} \le d_Y$ and $d_{D^*,D} = d_D$, where d_D is the Poincaré distance on D, $D^* = D \setminus \{0\}$ and $d_{Y,Z}$ has the distance-decreasing property. The latter means that if (Y', Z'), $Y' \subset Z'$, is a pair of complex spaces with $\overline{Y'}$ compact, and if $Z \to Z'$ is a holomorphic map such that $f(Y) \subset Y'$, then $d_{Y',Z'}(f(p), f(q)) \le d_{Y,Z}(p,q), \ \forall\, p, q \in \overline{Y}$ (see [3]).

We say that Y is hyperbolically imbedded in Z if Y is relatively compact in Z and for every pair of distinct points p, q in $\overline{Y} \subset Z$ there exist neighbourhoods U_p and U_q of p and q in Z such that $d_Y(U_p \cap Y, U_q \cap Y) > 0$, where d_Y denotes the Kobayashi pseudo distance on Y.

In [2] Kobayashi proved the following interesting characterization of hyperbolic imbedding by using the relative intrinsic pseudo distance $d_{Y,Z}$.

Theorem 1.1. *Let Y be a relatively compact complex subspace of a complex space Z, then Y is hyperbolicaty imbedded in Z if and only if $d_{Y,Z}(p,q) > 0$ for all pairs $p, q \in \overline{Y}$, $p \neq q$.*

Using Theorem 1.1, we can prove the following result.

Theorem 1.2. *Let $(\widetilde{Z}, \pi, Z)$ be a holomorphic fiber with hyperbolic compact fiber F, where $\widetilde{Z}, Z, F$ are complex manifolds. Let M be a complex subspace of Z. Put $\widetilde{M} = \pi^{-1}(M)$. If $\widetilde{M}$ is hyperbolically imbedded in $\widetilde{Z}$ then M is hyperbolically imbedded in Z. Conversely. if $d_{M,Z}$ induces the given topology on $\overline{M}$ and M is hyperbolically imbedded in Z, then $\widetilde{M}$ is hyperbolically imbedded in $\widetilde{Z}$.*

2. Results

The following result is similar to the one in [1] for the Kobayashi pseudo-distance.

Theorem 2.1. *Let Z be a complex space and $\pi : \widetilde{Z} \to Z$ be a convering space of Z. Let Y be a complex subspace of Z and $\widetilde{Y} = \pi^{-1}(Y)$. If $p, q \in \overline{Y}$, $\tilde{p} \in \pi^{-1}(p)$, then*

$$d_{Y,Z}(p,q) = \inf_{\tilde{q}\in\pi^{-1}(q)} d_{\widetilde{Y},\widetilde{Z}}(\tilde{p},\tilde{q}).$$

Proof. Since π is holomorphic and $\pi(\widetilde{Y}) \subset Y$, we have

$$d_{Y,Z}(p,q) = d_{Y,Z}(\pi(\tilde{p}), \pi(\tilde{q})) \leq d_{\widetilde{Y},\widetilde{Z}}(\tilde{p},\tilde{q}), \quad \forall\, \tilde{p} \in \pi^{-1}(p),\ \forall\, \tilde{q} \in \pi^{-1}(q).$$

Hence

$$d_{Y,Z}(p,q) \leq \inf_{\tilde{q}\in\pi^{-1}(q)} d_{\widetilde{Y},\widetilde{Z}}(\tilde{p},\tilde{q}).$$

We will prove the reverse inequality.

If $d_{Y,Z}(p,q) = \infty$, i.e. there is not any chain α of holomorphic discs belonging to $\mathcal{F}_{Y,Z}$ from p to q, then there is not any chain $\tilde{\alpha}$ of holomorphic discs belonging to $\mathcal{F}_{\widetilde{Y},\widetilde{Z}}$ from $\tilde{p}$ to $\tilde{q}$. So $d_{\widetilde{Y},\widetilde{Z}}(\tilde{p},\tilde{q}) = \infty$, for every $\tilde{q} \in \pi^{-1}(q)$. In this case we have

$$d_{Y,Z}(p,q) = \inf_{\tilde{q}\in\pi^{-1}(q)} d_{\widetilde{Y},\widetilde{Z}}(\tilde{p},\tilde{q}) = \infty.$$

If $d_{Y,Z}(p,q) < \infty$ and the strict inequality

$$d_{Y,Z}(p,q) < \inf_{\tilde{q}\in\pi^{-1}(q)} d_{\widetilde{Y},\widetilde{Z}}(\tilde{p},\tilde{q})$$

holds, then there is a positive number $\varepsilon > 0$ such that

$$d_{Y,Z}(p,q) + \varepsilon + \inf_{\tilde{q}\in\pi^{-1}(q)} d_{\widetilde{Y},\widetilde{Z}}(\tilde{p},\tilde{q}). \tag{2.1}$$

Let $b_1, \ldots, b_k \in D$ and $\{f_i\} \subset \mathcal{F}_{Y,Z}$ be such that $f_1(0) = p$, $f_i(b_i) = f_{i+1}(0)$, $i = 1, \ldots, k-1$, $f_k(b_k) = q$, and

$$\sum_{i=1}^{k} d_D(0, b_i) < d_{Y,Z}(p, q) + \varepsilon.$$

Then we can lift $f_1, \ldots, f_k$ to holomorphic mappings $\tilde{f}_1, \ldots, \tilde{f}_k \in \mathcal{F}_{\widetilde{Y},\widetilde{Z}}$ in such a way that $\tilde{p} = \tilde{f}_1(0)$, $\tilde{f}_i(b_i) = \tilde{f}_{i+1}(0)$ for $i = 1, \ldots, k-1$, $\pi \circ \tilde{f}_i = f_i$ for $i = 1, \ldots, k$. If we set $\tilde{q} = \tilde{f}_k(b_k)$, then $\pi(\tilde{q}) = q$ and $d_{\widetilde{Y},\widetilde{Z}}(\tilde{p}, \tilde{q}) \leq \sum_{i=1}^{k} d_D(0, b_i)$. Hence $d_{\widetilde{Y},\widetilde{Z}}(\tilde{p}, \tilde{q}) \leq d_{Y,Z}(p, q) + \varepsilon$, which contradicts the inequality (2.1).

Thus $d_{Y,Z}(p, q) = \inf_{\tilde{q} \in \pi^{-1}(q)} d_{\widetilde{Y},\widetilde{Z}}(\tilde{p}, \tilde{q})$. □

The following theorem corresponding to Theorem A in [6] for the Kobayashi pseudo distance.

Theorem 2.2. *Let $(\widetilde{Z}, \pi, Z)$ be a holomorphic fiber bundle with hyperbolic fiber F, where $\widetilde{Z}, Z, F$ are complex manifolds. Let Y be a complex subspace of Z and $\widetilde{Y} = \pi^{-1}(Y)$, $p, q \in \overline{Y}$, $\tilde{p} \in \pi^{-1}(p)$, $\tilde{q} \in \pi^{-1}(q)$. Then*

$$d_{Y,Z}(p, q) = \inf_{\tilde{q} \in \pi^{-1}(q)} d_{\widetilde{Y},\widetilde{Z}}(\tilde{p}, \tilde{q}).$$

Proof. Since π is holomorphic and $\pi(\widetilde{Y}) \subset Y$, we have

$$d_{Y,Z}(p, q) \leq d_{\widetilde{Y},\widetilde{Z}}(\tilde{p}, \tilde{q}), \quad \forall\, \tilde{p} \in \pi^{-1}(p),\ \forall \tilde{q} \in \pi^{-1}(q).$$

Hence

$$d_{Y,Z}(p, q) \leq \inf_{\tilde{q} \in \pi^{-1}(q)} d_{\widetilde{Y},\widetilde{Z}}(\tilde{p}, \tilde{q}), \tag{2.2}$$

We now prove the reverse inequality.

If $(p, q) \in \overline{Y} \times \overline{Y}$ is such that $d_{Y,Z}(p, q) = +\infty$, i.e. there is not any chain α of holomorphic discs belonging to $\mathcal{F}_{Y,Z}$ from p to q, then there is not any chain $\tilde{\alpha}$ of holomorphic discs belonging to $\mathcal{F}_{\widetilde{Y},\widetilde{Z}}$ from $\tilde{p}$ to $\tilde{q}$. So $d_{\widetilde{Y},\widetilde{Z}}(\tilde{p}, \tilde{q}) = +\infty$ for every $q \in \pi^{-1}(q)$. In this case we have

$$d_{Y,Z}(p, q) = \inf_{\tilde{q} \in \pi^{-1}(q)} d_{\widetilde{Y},\widetilde{Z}}(\tilde{p}, \tilde{q}) = \infty.$$

Suppose that $d_{Y,Z}(p, q) < +\infty$. We want to prove that

$$d_{Y,Z}(p, q) \geq \inf_{\tilde{q} \in \pi^{-1}(q)} d_{\widetilde{Y},\widetilde{Z}}(\tilde{p}, \tilde{q}).$$

Take arbitrary points $p, q \in \overline{Y} \subset Z$ and $\tilde{p} \in \pi^{-1}(p)$. Let $\alpha = \{f_1 \ldots, f_k\}$ be a chain of holomorphic discs from p to q. Suppose that $b_1, \ldots, b_k \in D$ and $f_1, \ldots, f_k \in \mathcal{F}_{Y,Z}$ are such that $f_1(0) = p, \ldots,\ f_i(b_i) = f_{i+1}(0)$, $i = 1, \ldots, k-1$, $f_k(b_k) = q$.

Consider the pull-back diagram

$$\begin{array}{ccc} D \times_Z \widetilde{Z} & \xrightarrow{\sigma_1} & \widetilde{Z} \\ \eta_1 \Big\downarrow & & \Big\downarrow \pi \\ D & \xrightarrow{f_1} & Z \end{array}$$

There is an equivalence $\Phi_1 : D \times F \to D \times_Z \widetilde{Z}$ of holomorphic fiber bundles over Z (see [5]). Thus, there exists $c_1 \in F$ such that $\sigma_1 \circ \Phi_1(0, c_1) = \tilde{p}$.

We define a holomorphic map $\varphi_1 : D \to \widetilde{Z}$ by $\varphi_1(z) = \sigma_1 \circ \Phi_1(z, c_1)$ for every $z \in D$. We have

$$\pi \circ \varphi_1(z) = \pi \circ \sigma_1 \circ \Phi_1(z, c_1) = f_1 \circ \eta_1 \circ \Phi_1(z, c_1) = f_1(z),$$

hence φ_1 is a lift of f_1. Since $f_1 \in \mathcal{F}_{Y,Z}$, $\varphi_1 \in \mathcal{F}_{\widetilde{Y},\widetilde{Z}}$.

Consider the pull-back diagram

$$\begin{array}{ccc} D \times_Z \widetilde{Z} & \xrightarrow{\sigma_2} & \widetilde{Z} \\ \eta_2 \Big\downarrow & & \Big\downarrow \pi \\ D & \xrightarrow{f_2} & Z \end{array}$$

By a reasoning similar to the above, we can show that there exist an equivalence $\Phi_2 : D \times F \to D \times_Z \widetilde{Z}$ of holomorphic fiber bundles over Z and a point $c_2 \in F$ such that $\sigma_2 \circ \Phi_2(0, c_2) = \varphi_1(b_1)$. We define a holomorphic map $\varphi_2 : D \to \widetilde{Z}$ by setting $\varphi_2(z) = \sigma_2 \circ \Phi_2(z, c_2)$ for $z \in D$. The map φ_2 is a lift of f_2, hence $\varphi_2 \in \mathcal{F}_{\widetilde{Y},\widetilde{Z}}$. Continuing this process we find maps $\varphi_1, \ldots, \varphi_k \in \mathcal{F}_{\widetilde{Y},\widetilde{Z}}$ such that φ_i is a lift of f_i and $\varphi_1(0) = \tilde{p}$, $\varphi_i(b_i) = \varphi_{i+1}(0)$, $\tilde{q} = \varphi_k(b_k) \in \pi^{-1}(q)$. Thus

$$d_{\widetilde{Y},\widetilde{Z}}(\tilde{p}, \tilde{q}) \le \sum_{i=1}^{k} d_D(0, b_i) = \ell(\alpha).$$

Hence $\inf_{\tilde{q} \in \pi^{-1}(q)} d_{\widetilde{Y},\widetilde{Z})}(\tilde{p}, \tilde{q}) \le d_{Y,Z}(p, q)$ for all $\tilde{p} \in \pi^{-1}(p)$. ☐

Theorem 2.3. *Let $(\widetilde{Z}, \pi, Z)$ be a holomorphic fiber bundle with hyperbolic compact fiber F, where $\widetilde{Z}, Z, F$ are complex manifolds. Let M be a complex subspace of Z. Put $\widetilde{M} = \pi^{-1}(M)$. Then we have*

a) *If $\widetilde{M}$ is hyperbolically imbedded in $\widetilde{Z}$ then M is hyperbolically imbedded in Z*

b) *If M is hyperbolically imbedded in Z and $d_{M,Z}$ induces the given topology on $\overline{M}$, then $\widetilde{M}$ is hyperbolically imbedded in $\widetilde{Z}$.*

Proof. Obviously, $\widetilde{M}$ is relatively compact in $\widetilde{Z}$ if and only if M is relatively compact in Z.

a) Assume that $\widetilde{M}$ is hyperbolically imbedded in $\widetilde{Z}$. According to Theorem 1.1, we need only to prove that $d_{M,Z}(p,q) > 0$ for all $p, q \in \overline{M}$, $p \neq q$. By Theorem 2.2, we have

$$d_{M,Z}(p,q) = \inf_{\tilde{q}\in\pi^{-1}(q)} d_{\widetilde{M},\widetilde{Z}}(\tilde{p},\tilde{q}) \quad \text{for all } \tilde{p} \in \pi^{-1}(p).$$

Since $p \neq q$, we have $\tilde{p} \neq \tilde{q}$. It follows that $d_{\widetilde{M},\widetilde{Z}}(\tilde{p},\tilde{q}) > 0$ for all $\tilde{q} \in \pi^{-1}(q)$, $\tilde{p} \in \pi^{-1}(p)$. Since F is compact and $d_{\widetilde{M},\widetilde{Z}}$ is lower semicontinuous on $\overline{\widetilde{M}} \times \overline{\widetilde{M}}$ (see [3]). we have

$$\inf_{\tilde{q}\in\pi^{-1}(q)} d_{\widetilde{M},\widetilde{Z}}(\tilde{p},\tilde{q}) = \min_{\tilde{q}\in\pi^{-1}(q)} d_{\widetilde{M},\widetilde{Z}}(\tilde{p},\tilde{q}) > 0 \quad \text{for all } \tilde{p} \in \pi^{-1}(p).$$

Hence $d_{M,Z}(p,q) > 0$ for all $p, q \in \overline{M}$, $p \neq q$.

b) Suppose that M is hyperbolically imbedded in Z. Let $\tilde{p}$, $\tilde{q} \in \overline{\widetilde{M}}$, $\tilde{p} \neq \tilde{q}$. We have to prove that $d_{\widetilde{M},\widetilde{Z}}(\tilde{p},\tilde{q}) > 0$. Since M is hyperbolically imbedded in Z, if $\pi(\tilde{p}) \neq \pi(\tilde{q})$, then $d_{\widetilde{M},\widetilde{Z}}(\tilde{p},\tilde{q}) \geq d_{M,Z}(\pi(\tilde{p}),\pi(\tilde{q})) > 0$.

Suppose that $\pi(\tilde{p}) = \pi(\tilde{q}) = p$. We choose a neighbourhood U of p in Z such that $\pi^{-1}(U) = U \times F$. Let $B_s = \{x \in \overline{M}; |d_{M,Z}(x,p) < s\}$, $D_r = \{z \in C \big| \, |z| < r\}$.

Since $d_{M,Z}$ induces the given topology on $\overline{M}$, we can choose $s > 0$ and $r > 0$ in such a way that $B_{2s} \subset U$ and $d_D(z,0) < s$ for $z \in D_r$, where D denotes the unit disc in C. Thus, if $f : D \to \widetilde{Z}$ is holomorphic and $f(0) \in \pi^{-1}(B_s)$, then $f(D_r) \subset U \times F$. Indeed, we have $d_{\widetilde{M},\widetilde{Z}}(f(0), f(z)) \leq d_D(0,z) < s$, for all $z \in D_r$. Hence

$$d_{M,Z}(\pi f(0), \pi f(z)) < s.$$

Since $\pi f(0) \in B_s$, it follows that $\pi f(z) \in B_{2s} \subset U$. So we get $f(z) \in \pi^{-1}(U) = U \times F$.

Choose $c > 0$ such that $d_D(0,z) \geq c d_{D_r}(0,z)$ for all $z \in D_{r/2}$. Let $\alpha = \{f_i\} \subset \mathcal{F}_{\widetilde{M},\widetilde{Z}}$ be a holomorphic chain from $\tilde{p}$ to $\tilde{q}$ such that $f_1(0) = \tilde{p}$, $f_1(b_1) = f_2(0), \ldots, f_k(b_k) = \tilde{q}$, where $b_i \in D$ $(i = 1, \ldots, k)$. By imserting extra terms in this chain if necessary, we may assume that $b_i \in D_{r/2}$ for all i. We set $\tilde{p}_0 = \tilde{p}$, $\tilde{p}_1 = f_1(b_1), \ldots, \tilde{p}_k = f_k(b_k) = \tilde{q}$.

We consider two cases.

Case 1. At least one of the $\tilde{p}_i$'s is not contained in $\pi^{-1}(B_s)$. Then we get

$$\begin{aligned}\sum_{i=1}^{k} d_D(0,b_i) &= \sum_{i=1}^{k} d_{D^*,D}(0,b_i) \\ &\geq \sum_{i=1}^{k} d_{\widetilde{M},\widetilde{Z}}(f_i(0), f_i(b)) = \sum_{i=1}^{k} d_{\widetilde{M},\widetilde{Z}}(\tilde{p}_{i-1}.\tilde{p}_i)\end{aligned}$$

$$\geq \sum_{i=1}^{k} d_{M,Z}(\pi(\tilde{p}_{i-1}), \pi(\tilde{p}_i))$$
$$\geq s.$$

Case 2. All the $\tilde{p}_i$'s are in $\pi^{-1}(B_s)$. Then

$$\begin{aligned}\sum_{i=1}^{k} d_D(0, b_i) &\geq c \sum_{i=1}^{k} d_{D_r}(0, b_i)\\ &\geq c \sum_{i=1}^{k} d_{U\times F}(\tilde{p}_{i-1}, \tilde{p}_i)\\ &\geq c \sum_{i=1}^{k} d_F(\varphi(\tilde{p}_{i-1}), \varphi(\tilde{p}_i))\\ &\geq c d_F(\tilde{p}, \tilde{q}) > 0.\end{aligned}$$

where $\varphi : U \times F \to F$ is the projection and $d_F(\tilde{p}, \tilde{q}) > 0$ because F is hyperbolic. This implies that

$$d_{\widetilde{M},\widetilde{Z}}(\tilde{p}, \tilde{q}) = \inf_{\alpha} \sum_{i=1}^{k} d_D(0, b_i) \geq \min[s, c d_F(\tilde{p}, \tilde{q})] > 0.$$

The proof is complete. □

References

[1] S. Kobayashi, *Hyperbolic Manifolds and Holomorphic Mappings*, Dekker, New York, 1970.

[2] S. Kobayashi, *Relative intrinsic distance and hyperbolic imbedding*, Symposia Mathematica, Proceedings of "Recent Advances in Differential Geometry", Pisa, 1993.

[3] S. Kobayashi, *Hyperbolic Complex Spaces*, Springer Verlag, Berlin-Heidelberg, 1998.

[4] J. E. Joseph and M. H. Kwack, *Hyperbolic imbedding and spaces of continuous extensions of holomorphic maps*, J. Geom. Anal. **4** (1994), 361-378.

[5] H. L. Royden, *Holomorphic fiber bundles with hyperbolic fiber*, Proc. Amer. Math. Soc. **43** (1974), 311-312.

[6] D. D. Thai and N. L. Huong, *A note on the Kobayashi pseudodistance and the tautness of holomorphic fiber bundles*, Annales Polonici Mathematici LVIII, 1 (1993), 1-5.

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