ACTA MATHEMATICA VIETNAMICA
Volume 27, Number 3, 2002, pp. 271-282

REPRESENTATIONS IN POLYDOMAINS

H. BEGEHR

Dedicated to the memory of Le Van Thiem

1. Introduction

Iterating Cauchy-Pompeiu operators leads to higher order Cauchy-Pompeiu representation formulas. There are two basic second order representations. One is related to the Laplacian $\partial^2/\partial z\partial\overline{z}$, the other to the Bitsadze operator $\partial^2/\partial\overline{z}^2$. These two formulas applied for different variables and iterated in proper manner lead to different representation formulas of second order for functions of several complex variables in polydomains. They are called of second order as only derivatives up to second order with respect to each variable are involved.

On one hand these representation formulas provide particular solutions to certain inhomogeneous second order overdetermined systems as inhomogeneous polyharmonic, the inhomogeneous polyanalytic and some mixed kind systems. On the other hand they even can be used to transform more general systems including lower order terms into systems of singular integral equations for some density functions. They determine a particular solution to the inhomogeneous system.

The most general second order representation is given in Theorem 3. For one group of variables the Laplace operator is involved while for the others the Bitsadze operators is. Theorem 1 and Theorem 2 present special result where only one of the two operators appears.

The representation formulas (15) and (16) in Theorem 3 correspond to different kinds of systems. While the first formula is related to

$$\begin{aligned} w_{z_\mu\overline{z_\mu}} &= f_\mu \quad \text{for } \mu\in\{\ell_1,\dots,\ell_\lambda\},\\ w_{\overline{z_\mu}\,\overline{z_\mu}} &= f_\mu \quad \text{for } \mu\in\{\ell_{\lambda+1},\dots,\ell_n\}, \end{aligned}$$

the latter is related to

$$\begin{aligned} w_{z_\mu\overline{z_\mu}} &= f_{\mu\mu} \quad \text{for} \quad \mu\in\{\ell_1,\dots,\ell_\lambda\},\\ w_{\overline{z_\mu}\,\overline{z_\mu}} &= f_{\mu\mu} \quad \text{for} \quad \mu\in\{\ell_{\lambda+1},\dots,\ell_n\},\\ w_{\overline{z_\mu}\,\overline{z_\nu}} &= f_{\mu\nu} \quad \text{for} \quad 1\le\mu<\nu\le n. \end{aligned}$$

Here $\{\ell_1,\dots,\ell_\lambda,\ell_{\lambda+1}\dots,\ell_n\}=\{1,\dots,n\}$ for some λ, $0\le\lambda\le n$, and proper compatibility conditions have to be satisfied. Dual formulas with the roles of the

Laplace and Bitsadze operators interchanged are available, too. For the origin of the Pompeiu operator see [5].

2. Basic representation formulas

The complex forms of the Gauss theorem

$$\int_D w_{\overline{z}} dxdy = \frac{1}{2i}\int_{\partial D} w dz, \quad \int_D w_z dxdy = -\frac{1}{2i}\int_{\partial D} w d\overline{z} \tag{1}$$

for $w \in C^1(D;\mathbb{C}) \cap C(\overline{D};\mathbb{C})$ with bounded smooth domains $D \subset \mathbb{C}$ lead to the Cauchy Pompeiu representations

$$w(z) = \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{\pi}\int_D w_{\overline{\zeta}}(\zeta)\frac{d\xi d\eta}{\zeta - z}, \quad z \in D, \tag{2}$$

$$w(z) = -\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\overline{\zeta}}{\overline{\zeta - z}} - \frac{1}{\pi}\int_D w_{\zeta}(\zeta)\frac{d\xi d\eta}{\overline{\zeta - z}}, \quad z \in D. \tag{2'}$$

Iterations of these formulas lead to higher order representations in terms of integral operators from a certain hierarchy, see [4]. The simplest are

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} - \frac{1}{2\pi i}\int_{\partial D} w_{\overline{\zeta}}(\zeta)\frac{\overline{\zeta - z}}{\zeta - z}d\zeta \\ &\quad + \frac{1}{\pi}\int_D w_{\overline{\zeta}\,\overline{\zeta}}(\zeta)\frac{\overline{\zeta - z}}{\zeta - z}d\xi d\eta, \quad z \in D, \end{aligned} \tag{3}$$

$$\begin{aligned} w(z) &= \frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\zeta}{\zeta - z} + \frac{1}{2\pi i}\int_{\partial D} w_{\overline{\zeta}}(\zeta)\log|\zeta - z|^2 d\overline{\zeta} \\ &\quad + \frac{1}{\pi}\int_D w_{\zeta\overline{\zeta}}(\zeta)\log|\zeta - z|^2 d\xi d\eta, \quad z \in D, \end{aligned} \tag{3'}$$

for $w \in C^2(D;\mathbb{C}) \cap C^1(\overline{D};\mathbb{C})$. As (2') is a dual representation to (2) there are dual formulas to (3) and (3') where ζ-derivatives are used instead of $\overline{\zeta}$-derivatives and vice versa. E.g. instead of (3') one has

$$\begin{aligned} w(z) &= -\frac{1}{2\pi i}\int_{\partial D} w(\zeta)\frac{d\overline{\zeta}}{\overline{\zeta - z}} - \frac{1}{2\pi i}\int_{\partial D} w_{\zeta}(\zeta)\log|\zeta - z|^2 d\zeta \\ &\quad + \frac{1}{\pi}\int_D w_{\zeta\overline{\zeta}}(\zeta)\log|\zeta - z|^2 d\zeta d\eta, \quad z \in D. \end{aligned} \tag{3''}$$

For functions of several complex variables (2) can be used to represent $w \in C^1(D^n;\mathbb{C}) \cap C(\overline{D^n};\mathbb{C})$ for polydomains $D^n = \mathop{\mathbf{X}}_{k=1}^{n} D_k \subset \mathbb{C}^n$ with bounded smooth domains $D_k \subset \mathbb{C}$, $1 \le k \le n$, as

$$\begin{aligned} w(z) = {} & \frac{1}{(2\pi i)^n} \int\limits_{\partial_0 D^n} w(\zeta) \prod_{k=1}^{n} \frac{d\zeta_k}{\zeta_k - z_k} \\ & - \sum_{k=1}^{n} \sum_{1 \le \nu_1 < \cdots < \nu_k \le n} \frac{1}{\pi^k} \int\limits_{D_{\nu_1}} \cdots \int\limits_{D_{\nu_k}} w_{\overline{\zeta_{\nu_1}} \cdots \overline{\zeta_{\nu_k}}}(\zeta) \prod_{\rho=1}^{k} \frac{d\xi_{\nu_\rho} d\eta_{\nu_\rho}}{\zeta_{\nu_\rho} - z_{\nu_\rho}}, \quad z \in D^n. \end{aligned} \tag{4}$$

Here $\partial_0 D^n := \mathop{\mathbf{X}}_{k=1}^{n} \partial D_k$ is the distinguished boundary of D so that w has to be continuous just on $D^n \cup \partial_0 D^n$ rather than on the closure $\overline{D^n}$ of D^n. Variables ζ_μ which are not integrated upon are understood to be replaced by z_μ throughout this paper.

In (4) only first order derivatives with respect to each variables z_k, $1 \le k \le n$, are involved as the higher order derivatives occurring only in mixed form. In [3] the iteration of (4) with its dual formula is used to develop a formula where second order derivatives with respect to each variable occur in order to solve the inhomogeneous pluriharmonic system, see also [1, 2].

Here (3) and (3') will be iterated for creasing "second order" representation formulas for functions in several complex variables.

But also first order and second order representations can be combined leading to representations for solutions to mixed systems. There are many combinations possible.

3. Iteration of (3)

Let $w \in C^2(D^2; \mathbb{C}) \cap C^1(\overline{D}^2; \mathbb{C})$ for a bounded smooth bidomain $D^2 = D_1 \times D_2 \subset \mathbb{C}^2$. By $w \in C^1(D^2; \mathbb{C})$ we mean a complex-valued function with continuous derivatives w_{z_1}, w_{z_2}, $w_{z_1 z_2}$, $w_{\overline{z_1}}$, $w_{\overline{z_2}}$, $w_{\overline{z_1 z_2}}$, $w_{z_1 \overline{z_2}}$, $w_{\overline{z_1} z_2}$ in D^2, etc.

Applying (3) for D_1 to ω, and for D_2 to w and to $w_{\overline{z_1}}$ and inserting the last two into the first gives

$$\begin{aligned} w(z) = {} & \frac{1}{(2\pi i)^2} \int\limits_{\partial D_1} \int\limits_{\partial D_2} w(\zeta_1, \zeta_2) \frac{d\zeta_1}{\zeta_1 - z_1} \frac{d\zeta_2}{\zeta_2 - z_2} \\ & - \frac{1}{(2\pi i)^2} \int\limits_{\partial D_1} \int\limits_{\partial D_2} w_{\overline{\zeta_2}}(\zeta_1, \zeta_2) \frac{d\zeta_1}{\zeta_1 - z_1} \frac{\overline{\zeta_2 - z_2}}{\zeta_2 - z_2} d\zeta_2 \\ & + \frac{1}{2\pi i} \frac{1}{\pi} \int\limits_{\partial D_1} \int\limits_{D_2} w_{\overline{\zeta_1}\, \overline{\zeta_2}}(\zeta_1, \zeta_2) \frac{d\zeta_1}{\zeta_1 - z_1} \frac{\overline{\zeta_2 - z_2}}{\zeta_2 - z_2} d\xi_2 d\eta_2 \\ & - \frac{1}{(2\pi i)^2} \int\limits_{\partial D_1} \int\limits_{\partial D_2} w_{\overline{\zeta_1}}(\zeta_1, \zeta_2) \frac{\overline{\zeta_1 - z_1}}{\zeta_1 - z_1} d\zeta_1 \frac{d\zeta_2}{\zeta_2 - z_2} \end{aligned}$$

$$+\frac{1}{(2\pi i)^2}\int\limits_{\partial D_1}\int\limits_{\partial D_2} w_{\overline{\zeta_1}\,\overline{\zeta_2}}\frac{\overline{\zeta_1-z_1}}{\zeta_1-z_1}d\zeta_1\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}d\zeta_2$$
$$-\frac{1}{2\pi i}\frac{1}{\pi}\int\limits_{\partial D_1}\int\limits_{D_2} w_{\overline{\zeta_1}\,\overline{\zeta_2}}\frac{\overline{\zeta_1-z_1}}{\zeta_1-z_1}d\zeta_1\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}d\zeta_2 d\eta_2$$
$$+\frac{1}{\pi}\int\limits_{D_1} w_{\overline{\zeta_1}\,\overline{\zeta_1}}(\zeta_1,z_2)\frac{\overline{\zeta_1-z_1}}{\zeta_1-z_1}d\xi_1 d\eta_1.$$

Using (3) once more for D_1 to $w_{\overline{z_2}\overline{z_2}}$ shows

$$\begin{aligned} w(z_1,z_2) =&\frac{1}{(2\pi i)^2}\int\limits_{\partial D_1}\int\limits_{\partial D_2}\Big\{\frac{w(\zeta_1,\zeta_2)}{(\zeta_1-z_1)(\zeta_2-z_2)}-\frac{w_{\overline{\zeta_1}}(\zeta_1,\zeta_2)}{\zeta_2-z_2}\frac{\overline{\zeta_1-z_1}}{\zeta_1-z_1}\\ &-\frac{w_{\overline{\zeta_2}}(\zeta_1,\zeta_2)}{\zeta_1-z_1}\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}+w_{\overline{\zeta_1}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\frac{\overline{\zeta_1-z_1}}{\zeta_1-z_1}\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}\Big\}d\zeta_1 d\zeta_2\\ &+\frac{1}{\pi}\int\limits_{D_1} w_{\overline{\zeta_1}\,\overline{\zeta_2}}(\zeta_1,z_2)\frac{\overline{\zeta_1-z_1}}{\zeta_1-z_1}d\zeta_1 d\eta_1+\frac{1}{\pi}\int\limits_{D_2} w_{\overline{\zeta_2}\,\overline{\zeta_2}}(z_1,\zeta_2)\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}d\xi_2 d\eta_2\\ &-\frac{1}{\pi^2}\int\limits_{D_1}\int\limits_{D_2} w_{\overline{\zeta_1}\,\overline{\zeta_1}\,\overline{\zeta_2}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\frac{\overline{\zeta_1-z_1}}{\zeta_1-z_1}\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}d\xi_1 d\eta_1 d\xi_2 d\eta_2, \end{aligned}\tag{5}$$

$(z_1,z_2)\in D_1,D_2$.

In order to eliminate the boundary values of $w_{\overline{\zeta_1}\,\overline{\zeta_2}}$ one can apply the Gauss theorem to deduce that

$$\begin{aligned} &\frac{1}{(2\pi i)^2}\int\limits_{\partial D_1}\int\limits_{\partial D_2} w_{\overline{\zeta_1}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\frac{\overline{\zeta_1-z_1}}{\zeta_1-z_1}\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}d\zeta_1 d\zeta_2\\ &=\frac{1}{\pi^2}\int\limits_{D_1}\int\limits_{D_2}\frac{\partial^2}{\partial\overline{\zeta_1}\partial\overline{\zeta_2}}\Big[w_{\overline{\zeta_1}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\frac{\overline{\zeta_1-z_1}}{\zeta_1-z_1}\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}\Big]d\xi_1 d\eta_1 d\xi_2 d\eta_2\\ &=\frac{1}{\pi^2}\int\limits_{D_1}\int\limits_{D_2}\Big\{w_{\overline{\zeta_1}\,\overline{\zeta_1}\,\overline{\zeta_2}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\frac{\overline{\zeta_1-z_1}}{\zeta_1-z_1}\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}\\ &\quad+w_{\overline{\zeta_1}\,\overline{\zeta_2}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\frac{\overline{\zeta_1-z_1}}{\zeta_1-z_1}\frac{1}{\zeta_2-z_2}+w_{\overline{\zeta_1}\,\overline{\zeta_2}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\frac{1}{\zeta_1-z_1}\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}\\ &\quad+\frac{w_{\overline{\zeta_1}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)}{(\zeta_1-z_1)(\zeta_2-z_2)}\Big\}d\xi_1 d\eta_1 d\xi_2 d\eta_2. \end{aligned}$$

Hence, for $(z_1,z_2)\in D_1\times D_2$,

$$
\begin{aligned}
w(z) = & \frac{1}{(2\pi i)^2} \int\limits_{\partial D_1} \int\limits_{\partial D_2} \Big\{ \frac{w(\zeta_1,\zeta_2)}{(\zeta_1 - z_2)(\zeta_2 - z_2)} \\
& - \frac{w_{\overline{\zeta_1}}(\zeta_1,\zeta_2)}{\zeta_2 - z_2} \frac{\overline{\zeta_1 - z_1}}{\zeta_1 - z_1} - \frac{w_{\overline{\zeta_2}}(\zeta_1,\zeta_2)}{\zeta_1 - z_1} \frac{\overline{\zeta_2 - z_2}}{\zeta_2 - z_2} \Big\} d\zeta_1 d\zeta_2 \\
& + \frac{1}{\pi} \int\limits_{D_1} w_{\overline{\zeta_1}\,\overline{\zeta_2}}(\zeta_1, z_2) \frac{\overline{\zeta_1 - z_1}}{\zeta_1 - z_1} d\xi_1 d\eta_1 + \frac{1}{\pi} \int\limits_{D_2} w_{\overline{\zeta_2}\,\overline{\zeta_2}}(z_1, \zeta_2) \frac{\overline{\zeta_2 - z_2}}{\zeta_2 - z_2} d\xi_2 d\eta_2 \\
& + \frac{1}{\pi^2} \int\limits_{D_1} \int\limits_{D_2} w_{\overline{\zeta_1}\,\overline{\zeta_2}}(\zeta_1,\zeta_2) \frac{d\xi_1 d\eta_2}{\zeta_1 - z_1} \frac{d\xi_2 d\eta_2}{\zeta_2 - z_2} \\
& + \frac{1}{\pi^2} \int\limits_{D_1} \int\limits_{D_2} \Big\{ w_{\overline{\zeta_1}\,\overline{\zeta_1}\,\overline{\zeta_2}}(\zeta_1,\zeta_2) \frac{\overline{\zeta_1 - z_1}}{\zeta_1 - z_1} \frac{1}{\zeta_2 - z_2} \\
& + w_{\overline{\zeta_1}\,\overline{\zeta_2}\,\overline{\zeta_2}}(\zeta_1,\zeta_2) \frac{1}{\zeta_1 - z_1} \frac{\overline{\zeta_2 - z_2}}{\zeta_2 - z_2} \Big\} d\xi_1 d\eta_1 d\xi_2 d\eta_2.
\end{aligned} \tag{6}
$$

Generalizing (5) and (6) to more than two variables gives the following result.

Theorem 1. *Let* $w \in C^2(D^n;\mathbb{C}) \cap C^1(\overline{D}^n,\mathbb{C})$ *for a bounded smooth polydomain* $D^n := \mathop{\mathbf{X}}_{k=1}^{n} D_k \subset \mathbb{C}^n$*. Then for* $z \in D^n$

$$
\begin{aligned}
w(z) = & \sum_{k=0}^{n} (-1)^k \sum_{1 \le \nu_1 < \cdots < \nu_k \le n} \frac{1}{(2\pi i)^n} \int\limits_{\partial_0 D^n} w_{\overline{\zeta_{\nu_1}}\,\overline{\zeta_{\nu_2}} \ldots \overline{\zeta_{\nu_k}}}(\zeta) \\
& \times \prod_{\rho=1}^{k} \overline{(\zeta_{\nu_\rho} - z_{\nu_\rho})} \prod_{\mu=1}^{n} \frac{d\zeta_\mu}{\zeta_\mu - z_\mu} \\
& + \sum_{k=1}^{n} (-1)^{k+1} \sum_{1 \le \nu_1 < \cdots < \nu_k \le n} \frac{1}{\pi^k} \int\limits_{D_{\nu_1}} \cdots \int\limits_{D_{\nu_k}} w_{\overline{\zeta_{\nu_1}}\,\overline{\zeta_{\nu_1}} \ldots \overline{\zeta_{\nu_k}}\,\overline{\zeta_{\nu_k}}}(\zeta) \\
& \times \prod_{\rho=1}^{k} \frac{\overline{\zeta_{\nu_\rho} - z_{\nu_\rho}}}{\zeta_{\nu_\rho} - z_{\nu_\rho}} d\xi_{\nu_\rho} d\eta_{\nu_\rho},
\end{aligned} \tag{7}
$$

and

$$
\begin{aligned}
w(z) = & \sum_{k=0}^{n-1} (-1)^k \sum_{1 \le \nu_1 < \cdots < \nu_k \le n} \frac{1}{(2\pi i)^n} \int\limits_{\partial_0 D^n} w_{\overline{\zeta_{\nu_1}} \ldots \overline{\zeta_{\nu_k}}}(\zeta) \prod_{\rho=1}^{k} \overline{(\zeta_{\nu_\rho} - z_{\nu_\rho})} \prod_{\mu=1}^{n} \frac{d\zeta_\mu}{\zeta_\mu - z_\mu} \\
& + \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1 \le \nu_1 < \cdots < \nu_k \le n} \frac{1}{\pi^k} \int\limits_{D_{\nu_1}} \cdots \int\limits_{D_{\nu_k}} w_{\overline{\zeta_{\nu_1}}\,\overline{\zeta_{\nu_1}} \ldots \overline{z_{\nu_k}}\,\overline{\zeta_{\nu_k}}} \prod_{\rho=1}^{k} \frac{\overline{\zeta_{\nu_\rho} - z_{\nu_\rho}}}{\zeta_{\nu_\rho} - z_{\nu_\rho}} d\xi_{\nu_\rho} d\eta_{\nu_\rho}
\end{aligned}
$$

$$
+\sum_{k=0}^{n-1} \sum_{\substack{1\le\nu_1<\cdots<\nu_k\le n\\ 1\le\nu_{k+1}<\cdots<\nu_n\le n\\ \{\nu_1,\dots,\nu_n\}=\{1,\dots,n\}}} \frac{(-1)^n}{\pi^n}\int_{D^n} w_{\overline{\zeta_{\nu_1}}\,\overline{\zeta_{\nu_1}}\dots\overline{\zeta_{\nu_k}}\,\overline{\zeta_{\nu_k}}\,\overline{\zeta_{\nu_{k+1}}}\dots\overline{\zeta_{\nu_n}}}(\zeta)
$$

$$
\times\prod_{\rho=1}^{k}\overline{(\zeta_{\nu_\rho}-z_{\nu_\rho})}\prod_{\mu=1}^{n}\frac{d\xi_\mu d\eta_\mu}{\zeta_\mu-z_\mu}\,. \tag{8}
$$

Here the variables which are not integrated upon are through to be z's rather than ζ's. Moreover, $\partial_0 D^n := \mathop{\mathbf{X}}\limits_{k=1}^{n}\partial D_k$ is the distinguished boundary of D^n.

The proof is by introduction and follows the above argumentation in the case $n=2$.

4. Other iteration

In an analogous way one can proceed with (3').

Theorem 2. *Any $w\in C^2(D^n;\mathbb{C})\cap C^1(\overline{D}^n;\mathbb{C})$ can in D^n be represented as*

$$
w(z)=\sum_{k=0}^{n}\sum_{\substack{1\le\nu_1<\cdots<\nu_k\le n\\ 1\le\nu_{k+1}<\cdots<\nu_n\le n\\ \{\nu_1,\dots,\nu_n\}=\{1,\dots,n\}}}\frac{1}{(2\pi i)^n}\int_{\partial_0 D^n} w_{\overline{\zeta_{\nu_1}}\dots\overline{\zeta_{\nu_k}}}(\zeta)
$$

$$
\times\prod_{\rho=1}^{k}\log|\zeta_{\nu_\rho}-z_{\nu_\rho}|^2 d\overline{\zeta_{\nu_\rho}}\prod_{\rho=k+1}^{n}\frac{d\zeta_{\nu_\rho}}{\zeta_{\nu_\rho}-z_{\nu_\rho}}
$$

$$
+\sum_{k=1}^{n}(-1)^{k+1}\sum_{1\le\nu_1<\cdots<\nu_k\le n}\frac{1}{\pi^k}\int_{D_{\nu_1}}\cdots\int_{D_{\nu_k}} w_{\zeta_{\nu_1}\overline{\zeta_{\nu_1}}\dots\zeta_{\nu_k}\overline{\zeta_{\nu_k}}}(\zeta)
$$

$$
\times\prod_{\rho=1}^{k}\log|\zeta_{\nu_\rho}-z_{\nu_\rho}|^2 d\xi_{\nu_\rho}d\eta_{\nu_\rho} \tag{9}
$$

and

$$
w(z)=\sum_{k=0}^{n-1}\sum_{\substack{1<\nu_1<\cdots<\nu_k\le n\\ 1<\nu_{k+1}<\cdots<\nu_n\le n}}\frac{1}{(2\pi i)^n}\int_{\partial_0 D^n} w_{\overline{\zeta_{\nu_1}}\dots\overline{\zeta_{\nu_k}}}(\zeta)
$$

$$
\times\prod_{\rho=1}^{k}\log|\zeta_{\nu\rho}-z_{\nu\rho}|^2 d\zeta_{\nu_\rho}\prod_{\rho=k+1}^{n}\frac{d\zeta_{\nu_\rho}}{\zeta_{\nu_\rho}-z_{\nu_\rho}}
$$

$$+\sum_{k=1}^{n-1}(-1)^{k+1}\sum_{1\le\nu_1<\cdots<\nu_k\le n}\frac{1}{\pi^k}\int\limits_{D_{\nu_1}}\cdots\int\limits_{D_{\nu_k}} w_{\zeta_{\nu_1}\overline{\zeta_{\nu_1}}\ldots\zeta_{\nu_k}\overline{\zeta_{\nu_k}}}(\zeta)$$

$$\times\prod_{\rho=1}^{k}\log|\zeta_{\nu_\rho}-z_{\nu_\rho}|^2 d\xi_{\nu_\rho}d\eta_{\nu_\rho}$$

$$+\sum_{k=0}^{n-1}\sum_{\substack{1\le\nu_1<\cdots<\nu_k\le n\\ 1\le\nu_{k+1}<\cdots<\nu_n\le n}}\frac{(-1)^n}{\pi^n}\int\limits_{D^n} w_{\zeta_{\nu_1}\overline{\zeta_{\nu_1}}\ldots\zeta_{\nu_k}\overline{\zeta_{\nu_k}}\,\overline{\zeta_{\nu_{k+1}}}\ldots\overline{\zeta_{\nu_n}}}(\zeta)$$

$$\times\prod_{\rho=1}^{k}\log|\zeta_{\nu_\rho}-z_{\nu_\rho}|^2\prod_{\rho=k+1}^{n}\frac{1}{\zeta_{\nu_\rho}-z_{\nu_\rho}}\prod_{\mu=1}^{n}d\xi_\mu d\eta_\mu. \tag{10}$$

Let (3") applied for D_2 be inserted into (3') used for D_1. Then for $(z_1,z_2)\in D_1\times D_2$ analogously as before one gets

$$\begin{aligned}
w(z_1,z_2) = &-\frac{1}{(2\pi i)^2}\int\limits_{\partial D_1}\int\limits_{\partial D_2}\Big\{\frac{w(\zeta_1,\zeta_2)}{(\zeta_1-z_1)\overline{(\zeta_2-z_2)}}d\zeta_1 d\overline{\zeta_2}\\
&+\frac{w_{\overline{\zeta_1}}(\zeta_1,\zeta_2)}{\overline{\zeta_2-z_2}}\log|\zeta_1-z_1|^2 d\overline{\zeta_1}d\overline{\zeta_2}+\frac{w_{\zeta_2}(\zeta_1,\zeta_2)}{\zeta_1-z_1}\log|\zeta_2-z_2|^2 d\zeta_1 d\zeta_2\\
&+w_{\overline{\zeta_1}\zeta_2}(\zeta_1,\zeta_2)\log|\zeta_1-z_1|^2\log|\zeta_2-z_2|^2 d\overline{\zeta_1}d\zeta_2\Big\}\\
&+\frac{1}{\pi}\int\limits_{D_1} w_{\zeta_1\overline{\zeta_1}}(\zeta_1,z_2)\log|\zeta_1-z_1|^2 d\xi_1 d\eta_1\\
&+\frac{1}{\pi}\int\limits_{D_2} w_{\zeta_2\overline{\zeta_2}}(z_1,\zeta_2)\log|\zeta_2-z_2|^2 d\xi_2 d\eta_2\\
&-\frac{1}{\pi^2}\int\limits_{D_1}\int\limits_{D_2} w_{\zeta_1\overline{\zeta_1}\zeta_2\overline{\zeta_2}}(\zeta_1,\zeta_2)\log|\zeta_1-z_1|^2\log|\zeta_2-z_2|^2 d\xi_1 d\eta_1 d\xi_2 d\eta_2.
\end{aligned}\tag{11}$$

This can be reformulated as

$$\begin{aligned}
w(z_1,z_2) = &-\frac{1}{(2\pi i)^2}\int\limits_{\partial D_1}\int\limits_{\partial D_2}\Big\{\frac{w(\zeta_1,\zeta_2)}{(\zeta_1-z_1)\overline{(\zeta_2-z_2)}}d\zeta_1 d\overline{\zeta_2}\\
&+\frac{w_{\overline{\zeta_1}}(\zeta_1,\zeta_2)}{\overline{\zeta_2-z_2}}\log|\zeta_1-z_1|^2 d\overline{\zeta_1}d\overline{\zeta_2}+\frac{w_{\zeta_2}(\zeta_1,\zeta_2)}{\zeta_1-z_1}log|\zeta_2-z_2|^2 d\zeta_1 d\zeta_2\Big\}\\
&+\frac{1}{\pi}\int\limits_{D_1} w_{\zeta_1\overline{\zeta_1}}(\zeta_1,z_2)\log|\zeta_1-z_1|^2 d\xi_1 d\eta_1
\end{aligned}$$

$$+\frac{1}{\pi}\int\limits_{D_2} w_{\zeta_2\overline{\zeta_2}}(z_1,\zeta_2)\log|\zeta_2-z_2|^2 d\xi_2 d\eta_2$$

$$+\frac{1}{\pi^2}\int\limits_{D_1}\int\limits_{D_2}\Big\{\frac{w_{\overline{\zeta_1}\zeta_2}(\zeta_1,\zeta_2)}{(\zeta_1-z_1)\overline{(\zeta_2-z_2)}}+\frac{w_{\zeta_1\overline{\zeta_1}\zeta_2}(\zeta_1,\zeta_2)}{\overline{\zeta_1-z_2}}\log|\zeta_1-z_1|^2$$

$$+\frac{w_{\overline{\zeta_1}\zeta_2\overline{\zeta_2}}(\zeta_1,\zeta_2)}{\zeta_1-z_1}\log|\zeta_2-z_2|^2\Big\}d\xi_1 d\eta_1 d\xi_2 d\eta_2, \tag{12}$$

Generalizations of (11) and (12) are not yet available.

5. Iteration of (3) with (3')

Inserting (3') for D_2 into (3) applied for D_1 gives for $(z_1,z_2)\in D_1\times D_2$

$$w(z_1,z_2)=\frac{1}{(2\pi i)^2}\int\limits_{\partial D_1}\int\limits_{\partial D_2} w(\zeta_1,\zeta_2)\frac{d\zeta_1}{\zeta_1-z_1}\frac{d\zeta_2}{\zeta_2-z_2}$$

$$-\frac{1}{(2\pi i)^2}\int\limits_{\partial D_1}\int\limits_{\partial D_2} w_{\overline{\zeta_2}}(\zeta_1,\zeta_2)\frac{d\zeta_1}{\zeta_1-z_1}\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}d\zeta_2$$

$$+\frac{1}{2\pi i}\frac{1}{\pi}\int\limits_{\partial D_1}\int\limits_{D_2} w_{\overline{\zeta_2}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\frac{d\zeta_1}{\zeta_1-z_1}\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}d\xi_2 d\eta_2$$

$$+\frac{1}{(2\pi i)^2}\int\limits_{\partial D_1}\int\limits_{\partial D_2} w_{\overline{\zeta_1}}(\zeta_1,\zeta_2)\log|\zeta_1-z_1|^2 d\overline{\zeta_1}\frac{d\zeta_2}{\zeta_2-z_2}$$

$$-\frac{1}{(2\pi i)^2}\int\limits_{\partial D_1}\int\limits_{\partial D_2} w_{\overline{\zeta_1}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\log|\zeta_1-z_1|^2 d\overline{\zeta_1}\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}d\zeta_2$$

$$+\frac{1}{2\pi i}\frac{1}{\pi}\int\limits_{\partial D_1}\int\limits_{D_2} w_{\overline{\zeta_1}\,\overline{\zeta_2}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\log|\zeta_1-z_1|^2 d\overline{\zeta_1}\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}d\xi_2 d\eta_2$$

$$+\frac{1}{\pi}\int\limits_{D_1} w_{\zeta_1\overline{\zeta_1}}(\zeta_1,z_2)\log|\zeta_1-z_1|^2 d\xi_1 d\eta_1$$

$$=\frac{1}{(2\pi i)^2}\int\limits_{\partial D_1}\int\limits_{\partial D_2}\Big\{\frac{w(\zeta_1,\zeta_2)}{(\zeta_1-z_1)(\zeta_2-z_2)}d\zeta_1 d\zeta_2$$

$$+\frac{w_{\overline{\zeta_1}}(\zeta_1,\zeta_2)}{\zeta_2-z_2}\log|\zeta_1-z_1|^2 d\overline{\zeta_1}d\zeta_2-\frac{w_{\overline{\zeta_2}}(\zeta_1,\zeta_2)}{\zeta_1-z_1}\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}d\zeta_1 d\zeta_2$$

$$
\begin{aligned}
&- w_{\overline{\zeta_1}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\log|\zeta_1-z_1|^2\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}d\overline{\zeta_1}d\zeta_2\Big\}\\
&+\frac{1}{\pi}\int\limits_{D_1} w_{\zeta_1\overline{\zeta_1}}(\zeta_1,z_2)\log|\zeta_1-z_1|^2 d\zeta_1 d\eta_1+\frac{1}{\pi}\int\limits_{D_2} w_{\overline{\zeta_2}\,\overline{\zeta_2}}(z_1,\zeta_2)\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}d\xi_1 d\eta_2\\
&-\frac{1}{\pi^2}\int\limits_{D_1}\int\limits_{D_2} w_{\zeta_1\overline{\zeta_1}\,\overline{\zeta_2}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\log|\zeta_1-z_1|^2\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}d\xi_1 d\eta_1 d\xi_2 d\eta_2,
\end{aligned}
\tag{13}
$$

where again (3') is used to get rid of the mixed integrals. The Gauss theorem

$$
\begin{aligned}
&\frac{1}{(2\pi i)^2}\int\limits_{\partial D_1}\int\limits_{\partial D_2} w_{\overline{\zeta_1}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\log|\zeta_1-z_1|^2\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}d\overline{\zeta_1}d\zeta_2\\
&=-\frac{1}{\pi^2}\int\limits_{D_1}\int\limits_{D_2}\frac{\partial^2}{\partial\zeta_1\partial\overline{\zeta_2}}\Big[w_{\overline{\zeta_1}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\log|\zeta_1-z_1|^2\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}\Big]d\xi_1 d\eta_1 d\xi_2 d\eta_2\\
&=-\frac{1}{\pi^2}\int\limits_{D_1}\int\limits_{D_2}\Big\{w_{\zeta_1\overline{\zeta_1}\,\overline{\zeta_2}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\log|\zeta_1-z_1|^2\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}+w_{\zeta_1\overline{\zeta_1}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\frac{\log|\zeta_1-z_1|^2}{\zeta_2-z_2}\\
&\quad+w_{\overline{\zeta_1}\,\overline{\zeta_2}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\frac{1}{\zeta_1-z_1}\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}+w_{\overline{\zeta_1}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\frac{1}{(\zeta_1-z_1)(\zeta_2-z_2)}\Big\}d\xi_1 d\eta_1 d\xi_2 d\eta_2
\end{aligned}
$$

gives then for $(z_1,z_2)\in D_1\times D_2$

$$
\begin{aligned}
w(z)=&\frac{1}{(2\pi i)^2}\int\limits_{\partial D_1}\int\limits_{\partial D_2}\Big(w(\zeta_1,\zeta_2)\frac{d\zeta_1}{\zeta_1-z_2}\frac{d\zeta_2}{\zeta_2-z_2}+\frac{w_{\overline{\zeta_1}}(\zeta_1,\zeta_2)}{\zeta_2-z_2}\\
&\times\log|\zeta_1-z_1|^2 d\overline{\zeta_1}d\zeta_2-\frac{w_{\overline{\zeta_2}}(\zeta_1,\zeta_2)}{\zeta_1-z_1}\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}d\zeta_1 d\zeta_2\Big\}\\
&+\frac{1}{\pi}\int\limits_{D_1} w_{\zeta_1\overline{\zeta_1}}(\zeta_1,z_2)\log|\zeta_1-z_1|^2 d\xi_1 d\eta_1+\frac{1}{\pi}\int\limits_{D_2} w_{\overline{\zeta_2}\,\overline{\zeta_2}}(z_1,\zeta_2)\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}d\xi_2 d\eta_2\\
&+\frac{1}{\pi^2}\int\limits_{D_1}\int\limits_{D_2} w_{\overline{\zeta_1}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\frac{d\xi_1 d\eta_1}{\zeta_1-z_1}\frac{d\xi_2 d\eta_2}{\zeta_2-z_2}+\\
&+\frac{1}{\pi^2}\int\limits_{D_1}\int\limits_{D_2}\Big\{w_{\zeta_1\overline{\zeta_1}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\frac{\log|\zeta_1-z_1|^2}{\zeta_2-z_2}\\
&+w_{\zeta_1\overline{\zeta_2}\,\overline{\zeta_2}}(\zeta_1,\zeta_2)\frac{1}{\zeta_1-z_1}\frac{\overline{\zeta_2-z_2}}{\zeta_2-z_2}\Big\}d\xi_1 d\eta_1 d\xi_2 d\eta_2.
\end{aligned}
\tag{14}
$$

These representations can be generalized by induction.

Theorem 3. *If $w \in C^2(D^n;\mathbb{C}) \cap C^1(\overline{D}^n;\mathbb{C})$ then for $z \in D^n$ and any grouping $\{\ell_1,\dots,\ell_\lambda\}$, $\{\ell_{\lambda+1},\dots,\ell_n\}$, $0 \le \lambda \le n$, with $\{\ell_1,\dots,\ell_n\} = \{1,\dots,n\}$, $1 \le \ell_1 < \cdots < l_\lambda \le n$, $1 \le \ell_{\lambda+1} < \cdots < \ell_n \le n$,*

$$
\begin{aligned}
w(z) = &\sum_{k=0}^{n} \sum_{1\le\nu_1<\cdots<\nu_k\le n} \frac{1}{(2\pi i)^n} \int_{\partial_0 D^n} w_{\overline{\zeta_{\nu_1}}\cdots\overline{\zeta_{\nu_k}}}(\zeta) \prod_{\nu_\rho\in\{\ell_1,\dots,\ell_\lambda\}} \log|\zeta_{\nu_\rho}-z_{\nu_\rho}|^2 d\overline{\zeta_{\nu_\rho}} \\
&\times \prod_{\nu_\rho\in\{\ell_{\lambda+1},\dots,\ell_n\}} \frac{\overline{z_{\nu_\rho}-\zeta_{\nu_\rho}}}{\zeta_{\nu_\rho}-z_{\nu_\rho}} d\zeta_{\nu_\rho} \prod_{\mu\notin\{\nu_1,\dots,\nu_k\}} \frac{d\zeta_\mu}{\zeta_\mu - z_\mu} \\
&+ \sum_{k=1}^{n} (-1)^{k+1} \sum_{1\le\nu_1<\cdots<\nu_k\le n} \frac{1}{\pi^k} \int_{D_{\nu_1}} \cdots \int_{D_{\nu_k}} \prod_{\nu_\rho\in\{\ell_1,\dots,\ell_\lambda\}} \frac{\partial^2}{\partial\zeta_{\nu_\rho}\partial\overline{\zeta_{\nu_\rho}}} \\
&\times \prod_{\nu_\rho\in\{\ell_{\lambda+1},\dots,\ell_n\}} \frac{\partial^2}{\partial\overline{\zeta_{\nu_\rho}}^2} w(\zeta) \prod_{\nu_\rho\in\{\ell_1,\dots,\ell_\lambda\}} \log|\zeta_{\nu_\rho}-z_{\nu_\rho}|^2 \\
&\times \prod_{\nu_\rho\in\{\ell_{\lambda_1},\dots,\ell_n\}} \frac{\overline{\zeta_{\nu_\rho}-z_{\nu_\rho}}}{\zeta_{\nu_\rho}-z_{\nu_\rho}} \prod_{\rho=1}^{k} d\xi_{\nu_\rho} d\eta_{\nu_\rho}
\end{aligned}
\tag{15}
$$

and

$$
\begin{aligned}
w(z) = &\sum_{k=0}^{n-1} \sum_{1\le\nu_1<\cdots<\nu_k\le n} \frac{1}{(2\pi i)^n} \int_{\partial_0 D^n} w_{\overline{\zeta_1}\cdots\overline{\zeta_{\nu_k}}}(\zeta) \prod_{\nu_\rho\in\{\ell_1,\dots,\ell_\lambda\}} \log|\zeta_{\nu_\rho}-z_{\nu_\rho}|^2 d\overline{\zeta_{\nu_\rho}} \\
&\times \prod_{\nu_\rho\in\{\ell_{\lambda+1},\dots,\ell_n\}} \frac{\overline{z_{\nu_\rho}-\zeta_{\nu_\rho}}}{\zeta_{\nu\rho}-z_{\nu_\rho}} d\zeta_{\nu_\rho} \prod_{\mu\notin\{\nu_1,\dots,\nu_k\}} \frac{d\zeta_\mu}{\zeta_\mu - z_\mu} \\
&+ \sum_{k=1}^{n-1} (-1)^{k+1} \sum_{1\le\nu_1<\cdots<\nu_k\le n} \frac{1}{\pi^k} \int_{D_{\nu_1}} \cdots \int_{D_{\nu_k}} \prod_{\nu_\rho\in\{\ell_1,\dots,\ell_\lambda\}} \frac{\partial^2}{\partial\zeta_{\nu_\rho}\partial\overline{\zeta_{\nu_\rho}}} \\
&\times \prod_{\nu_\rho\in(\ell_{\lambda+1},\dots,\ell_n\}} \frac{\partial^2}{\partial\overline{\zeta_{\nu_\rho}}^2} w(\zeta) \prod_{\nu_\rho\in\{\ell_1,\dots,\ell_\lambda\}} \log|\zeta_{\nu_\rho}-z_{\nu_\rho}|^2 \\
&\times \prod_{\nu_\rho\in\{\ell_{\lambda+1},\dots,\ell_n\}} \frac{\overline{\zeta_{\nu_\rho}-z_{\nu_\rho}}}{\zeta_{\nu_\rho}-z_{\nu_\rho}} \prod_{\rho=1}^{k} d\xi_{\nu_\rho} d\eta_{\nu_\rho} \\
&+ \sum_{\rho=0}^{\lambda-1} \sum_{\sigma=\lambda}^{n-1} \sum_{\substack{1\le\nu_1<\cdots<\nu_\rho\le\lambda \\ 1\le\nu_{\rho+1}<\cdots<\nu_\lambda\le\lambda}} \sum_{\substack{\lambda+1\le\nu_{\lambda+1}<\cdots<\nu_\sigma\le n \\ \lambda+1\le\nu_{\sigma+1}<\cdots<\nu_n\le n}} \frac{(-1)^n}{\pi^n} \int_{D^n} \\
&\times \prod_{\tau=1}^{p} \frac{\partial^2}{\partial\zeta_{\ell_{\nu_\tau}}\partial\overline{\zeta_{\ell_{\nu_\tau}}}} \prod_{i=\lambda+1}^{\sigma} \frac{\partial^2}{\partial\overline{\zeta_{\ell_{\nu_i}}}^2} \prod_{\mu\notin\{\ell_1,\dots,\ell_{\nu_\rho},\ell_{\nu_{\lambda+1}},\dots,\ell_{\nu_\sigma}\}} \frac{\partial}{\partial\overline{\zeta_\mu}} w(\zeta)
\end{aligned}
$$

$$\times \prod_{\tau=1}^{\rho} \log|\zeta_{\ell_{\nu_\tau}} - z_{\ell_{\nu_\tau}}|^2 \times \prod_{i=\lambda+1}^{\sigma} \frac{\overline{\zeta_{\ell_{\nu_i}} - z_{\ell_{\nu_i}}}}{\zeta_{\ell_{\nu_i}} - z_{\ell_{\nu_i}}}$$

$$\times \prod_{\mu \notin \{\ell_{\nu_1},\dots,\ell_{\nu_\rho},\ell_{\nu_{\lambda+1}},\dots,\ell_{\nu_\sigma}\}} \frac{1}{\zeta_\mu - z_\mu} \prod_{k=1}^{n} d\xi_k d\eta_k. \tag{16}$$

These formulas are second order representations in some sense as they express the functions w through its derivatives up to second order with respect to each single variable. They can be used to solve overdetermined second order systems. E.g. the Bitsadze system

$$w_{\overline{z_k}\,\overline{z_\ell}} = f_{k\ell}, \quad 1 \le k,\ \ell \le n, \quad \text{in } D^n,$$

satisfying $f_{k\ell} = f_{\ell k}$ and the compatibility conditions

$$f_{k\ell\overline{z_j}} = f_{kj\overline{z_\ell}}, \quad 1 \le k,\ \ell,\ j \le n$$

can be solved via (8). Let on $\partial_0 D^n$

$$w_{\overline{z_{\nu_1}}\dots\overline{z_{\nu_k}}}(z) = \varphi_{\nu_1\dots\nu_k}(z), \quad 1 \le \nu_1 \dots < \nu_k \le n, \quad 0 \le k \le n-1.$$

Then in D^n

$$w(z) = \sum_{k=0}^{n-1}(-1)^k \sum_{1\le\nu_1<\dots<\nu_k\le n} \frac{1}{(2\pi i)^n} \int_{\partial_0 D^n} \varphi_{\nu_1\dots\nu_k}(\zeta) \prod_{\rho=1}^{k} \overline{(\zeta_{\nu_\rho} - z_{\nu_\rho})} \prod_{\mu=1}^{n} \frac{d\zeta_\mu}{\zeta_\mu - z_\mu}$$

$$+ \sum_{k=1}^{n-1}(-1)^{k+1} \sum_{1\le\nu_1<\dots<\nu_k\le n} \frac{1}{\pi^k} \int_{D_{\nu_1}} \cdots \int_{D_{\nu_k}} f_{\nu_1\nu_1\overline{\zeta_{\nu_2}}\,\overline{\zeta_{\nu_2}}\dots\overline{\zeta_{\nu_k}}\,\overline{\zeta_{\nu_k}}}(\zeta)$$

$$\times \prod_{\rho=1}^{k} \frac{\overline{\zeta_{\nu_\rho} - z_{\nu_\rho}}}{\zeta_{\nu_\rho} - z_{\nu_\rho}} d\xi_{\nu_\rho} d\eta_{\nu_\rho}$$

$$+ \sum_{k=0}^{n-1} \sum_{\substack{1\le\nu_1<\dots<\nu_k\le n \\ 1\le\nu_{k+1}<\dots<\nu_n\le n}} \frac{(-1)^n}{\pi^n} \int_{D^n} f_{\nu_1\nu_{k+1}\overline{\zeta_{\nu_1}}\,\overline{\zeta_{\nu_2}}\,\overline{\zeta_{\nu_2}}\dots\overline{\zeta_{\nu_k}}\,\overline{\zeta_{\nu_k}}\,\overline{\zeta_{\nu_{k+2}}}\dots\overline{\zeta_{\nu_n}}}(\zeta)$$

$$\times \prod_{\rho=1}^{k} \overline{(\zeta_{\nu_\rho} - z_{\nu_\rho})} \prod_{\mu=1}^{n} \frac{d\xi_\mu d\eta_\mu}{\zeta_\mu - z_\mu}\,. \tag{17}$$

It is easily checked that the boundary integrals in (17) form a solution to the homogeneous system

$$w_{\overline{z_k}\,\overline{z_\ell}} = 0,$$

i.e. are pluriholomorphic functions while the area integrals altogether form a particular solution to the inhomogeneous Bitsadze system. Similar remarks hold for (10) and (16). It should be mentioned that (17) does not give a solution to some boundary value problem. Only if w is the solution to the above problem it can be represented by (17). To solve this problem solvability conditions have to be found first and then the solution has to be determined. For examples see [3].

Similarly, from (7) a solution to the system

$$w_{\overline{z_k}\,\overline{z_\ell}} = f_k, \quad f_{k\,\overline{z_\ell}\,\overline{z_\ell}} = f_{\ell\,\overline{z_k}\,\overline{z_k}}, \quad 1 \le k,\ \ell \le n, \quad \text{in } D$$

with

$w_{\overline{z_{\nu_1}}\,\overline{z_{\nu_2}}\ldots\overline{z_{\nu_k}}}(z) = \varphi_{\nu_1\nu_2\ldots\nu_k}(a), \quad 1 \le \nu_1 < \nu_2 < \cdots < \nu_k \le n,\ 0 \le k \le n,$ on $\partial_0 D^n$

is seen to be given as

$$\begin{aligned} w(z) = &\sum_{k=0}^{n}(-1)^k \sum_{1\le\nu_1<\cdots<\nu_k\le n} \frac{1}{(2\pi i)^n} \int_{\partial_0 D^n} \varphi_{\nu_1\nu_2\ldots\nu_k}(\zeta) \prod_{\rho=1}^{k} \overline{(\zeta_{\nu_\rho} - z_{\nu_\rho})} \prod_{\mu=1}^{n} \frac{d\zeta_\mu}{\zeta_\mu - z_\mu} \\ &+ \sum_{k=1}^{n}(-1)^{k+1} \sum_{1\le\nu_1<\cdots<\nu_k\le n} \frac{1}{\pi^k} \int_{D_{\nu_1}} \cdots \int_{D_{\nu_k}} f_{\nu_1\overline{\zeta}_{\nu_2}\,\overline{\zeta}_{\nu_2}\ldots\overline{\zeta_{\nu_k}}\,\overline{\zeta_{\nu_k}}}(\zeta) \\ &\times \prod_{\rho=1}^{k} \frac{\overline{\zeta_{\nu_\rho} - z_{\nu_\rho}}}{\zeta_{\nu_\rho} - z_{\nu_\rho}} d\xi_{\nu_\rho} d\eta_{\nu_\rho}. \end{aligned}$$

Similar remarks can be made for (9) and (15). Moreover, higher order representation formulas are available in an analogous way starting from higher order Cauchy Pompeiu representations with higher order operators from the hierarchy in [4].

References

[1] H. Begehr, *Complex analytic methods for partial differential equations*, ZAMM **76** (1996), Suppl. 2, 21-24.

[2] H. Begehr, *Boundary value problems in* $\mathbb{C}$ *and* $\mathbb{C}^n$, Acta Math. Vietnam. **22** (1997), 407-425.

[3] H. Begehr and A. Dzhuraev, *An Introduction to Several Complex Variables and Partial Differential Equations*, Addison Wesley Longman, Harlow, 1997.

[4] H. Begehr and G. N. Hile, *A hierarchy of integral operator*, Rocky Mountain J. Math. **27** (1997), 669-706.

[5] M. Mitrea and F. Sabac, *Pompeiu's integral representation formula. History and mathematics*, Revue Romaine Pure Appl. **43** (1998), 211-226.

Freie Universität Berlin
I. Mathematisches Institut
Arnimallee 3, D - 14195 Berlin
Germany

E-mail address: begehr@math.fu-berlin.de