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HEIGHT OF P-ADIC HOLOMORPHIC MAPS IN SEVERAL VARIABLES AND APPLICATIONS

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Dedicated to the memory of Le Van Thiem

1. Introduction

In the last years there is an increasing interest in the $p-$adic Nevanlinna theory. Hu and Yang [4], Khoai and Quang [6], and Boutabaa [1] proved $p-$adic analogues of two Main Theorems and defect relations of Nevanlinna theory. In [5], Ha Huy Khoai considered the case of several variables. He introduced the notion of height of $p-$adic holomorphic functions of several variables and proved the Poisson-Jensen formula. However, the height defined in [5] is difficult to compute. In [3], Cherry and Ye considered holomorphic maps from $\mathbb{C}_p^m$ to $\mathbb{P}^n(\mathbb{C}_p)$ and proved $p-$adic two Main Theorems. In this paper, we define the height of holomorphic maps from $\mathbb{C}_p^m$ to $\mathbb{P}^n(\mathbb{C}_p)$, which is easier to compute, and give a $p-$adic version of the two Main Theorems.

2. Height of p-adic holomorphic functions of several variables

Let p be a prime number, $\mathbb{Q}_p$ the field of p-adic numbers and $\mathbb{C}_p$ the p-adic completion of the algebraic closure of $\mathbb{Q}_p$. The absolute value in $\mathbb{Q}_p$ is normalized so that $|p| = p^{-1}$. We further use the notion $v(z)$ for the additive valuation on $\mathbb{C}_p$ which extends ord_p.

We use the notations

$b_{(m)} = (b_1, ..., b_m)$,

$D_r = \{z \in \mathbb{C}_p : |z| \leq r, r > 0\}, \quad D_{<r>} = \{z \in \mathbb{C}_p : |z| = r, r > 0\}$,

$D = \{z \in \mathbb{C}_p : |z| \leq 1\}$,

$D_{r_{(m)}} = D_{r_1} \times \cdots \times D_{r_m}$, where $r_{(m)} = (r_1, \ldots, r_m)$ for $r_i \in \mathbb{R}_+$,

$D_{<r_{(m)}>} = D_{<r_1>} \times \cdots \times D_{<r_m>}$,

$D^m = D \times \cdots \times D$ the unit polydisc in $\mathbb{C}_p^m$, $|f|_{r_{(m)}} = |f|_{(r_1,\ldots,r_m)}$,

$\gamma_i \in \mathbb{N}, \gamma = (\gamma_1, ..., \gamma_m)$, $|\gamma| = \gamma_1 + \cdots + \gamma_m$, $z^\gamma = z_1^{\gamma_1}...z_m^{\gamma_m}$, $r^\gamma = r_1^{\gamma_1}...r_m^{\gamma_m}$,

$\log = \log_p$, $\; t_i = -\log r_i$, $\; i = 1, ..., m$, $c_{(m)} + t = (c_1 + t, \ldots, c_m + t)$.

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Note that the set of $(r_1, ..., r_m) \in \mathbb{R}_+^m$ such that there exist $x_1, ..., x_m \in \mathbb{C}_p$ with $|x_i| = r_i, i = 1, ..., m$, is dense in $\mathbb{R}_+^m$. Therefore, without loss of generality one can assume that $D_{<r_{(m)}>} \neq \emptyset$.

Let f be a non-zero holomorphic function in $D_{r_{(m)}}$ represented by a convergent series

$$f = \sum_{|\gamma|=0}^{\infty} a_\gamma z^\gamma, |z_i| \leq r_i \quad \text{for } i = 1, \ldots, m.$$

We define

$$|f|_{r_{(m)}} = \max_{0 \leq |\gamma| < \infty} |a_\gamma| r^\gamma.$$

Set $\gamma t = \gamma_1 t_1 + \cdots + \gamma_m t_m$. Then we have

$$\lim_{|\gamma| \to \infty} (v(a_\gamma) + \gamma t) = +\infty.$$

Hence there exists an $(\gamma_1, \ldots, \gamma_m) \in \mathbb{N}^m$ such that $v(a_\gamma) + \gamma t$ is minimal.

Definition 2.1. *The height* of the holomorphic function $f(z_{(m)})$ is defined by

$$H_f(t_{(m)}) = \min_{0 \leq |\gamma| < \infty} (v(a_\gamma) + \gamma t).$$

We also use the notation

$$H_f^+(t_{(m)}) = -H_f(t_{(m)}).$$

Set

$$\begin{aligned} I_f(t_{(m)}) &= \Big\{(\gamma_1, \ldots, \gamma_m) \in \mathbb{N}^m : v(a_\gamma) + \gamma t = H_f(t_{(m)})\Big\}, \\ rn_f^+(t_{(m)}) &= \min\big\{|\gamma| : \gamma \in I_f(t_{(m)})\big\}, \\ n_f^-(t_{(m)}) &= \max\big\{|\gamma| : \gamma \in I_f(t_{(m)})\big\}, \\ n_f(0,0) &= \min\big\{|\gamma| : a_\gamma \neq 0\big\}. \end{aligned}$$

Theorem 2.1. *Let $f(z)$ be a holomorphic function on D_r. Assume that f is not identically zero. Then there exist a polynomial*

$$g(z) = b_0 + b_1 z + \cdots + b_v z^v, \; deg\ g = n_f^-(t), t = -\log_p r,$$

and a holomorphic function $h = 1 + \sum_{n=1}^{\infty} c_n z^n$ on D_r such that

1) *$f(z) = g(z)h(z)$,*
2) *$f(z)$ just has $n_f^-(t)$ zeros in D_r,*
3) *$n_f^-(t) - n_f^+(t)$ is equal to the number of zeros of f at $v(z) = t$,*
4) *h has no zeros in D_r*

For the proof, see the Weierstrass Preparation Theorem [4].

The set of z in $\mathbb{C}_p$ with $|z| \leq 1$ forms a closed subring of $\mathbb{C}_p$. We denote this subring by $\mathcal{O}$ (called the ring of integers of $\mathbb{C}_p$), and the set of z with $|z| < 1$

forms a maximal ideal I in $\mathcal{O}$. We denote the field $\mathcal{O}/I$, which is called the residue class field, by $\widehat{\mathbb{C}_p}$. Note that since $\mathbb{C}_p$ is algebraically closed, so is $\widehat{\mathbb{C}_p}$, and in particular $\widehat{\mathbb{C}_p}$ cannot be a finite field. Given an element w in $\mathcal{O}$, we denote its equivalence class in $\widehat{\mathbb{C}_p}$ by $\widehat{w}$.

Let $f = \sum_{|\gamma|=0}^{\infty} a_\gamma z^\gamma$ be a non-zero entire function on $\mathbb{C}_p^m$. Choose $y = y_{(m)}$ such that

$$|y| = \max\{|\gamma| : |a_\gamma| = |f|_{(1,\dots,1)}\}.$$

Define $\widehat{f}$ by

$$\widehat{f} = \sum_{|\gamma|=0}^{\infty} \frac{\widehat{a_\gamma}}{a_y} z^\gamma .$$

Since f is entire, $\left|\frac{a_\gamma}{a_y}\right| < 1$ for all but finitely many γ, and thus $\widehat{f}$ is a polynomial in $m-$variables with coefficients in $\widehat{\mathbb{C}_p}$. Since

$$\left|\frac{a_y}{a_y}\right| = 1,$$

$\widehat{f}$ is not the zero polynomial.

Lemma 2.1. *Let*

$$f_s(z_{(m)}) = \sum_{|\gamma|=0}^{\infty} a_\gamma^s z^\gamma, \quad s = 1, \dots, q,$$

be q non-zero entire functions on $\mathbb{C}_p^m$. Then for any $D_{r_{(m)}}$ in $\mathbb{C}_p^m$ ($D_{<r_{(m)}>} \neq \emptyset$) there exists $u = u_{(m)} \in D_{r_{(m)}}$ such that

$$\left|f_s(u_{(m)})\right| = \left|f_s\right|_{r_{(m)}}, \quad s = 1, \dots, q.$$

Proof. We first prove that if $r_{(m)} = (1, \dots, 1)$, then there exists $w = w_{(m)} \in D^m$ such that

$$|f_s(w)| = \max_{0 \le |\gamma| < \infty} |a_\gamma^s|, \quad s = 1, \dots, q. \tag{2.1}$$

For each $s = 1, \dots, q$, choose $y_s = (y_1^s, \dots, y_m^s)$ such that

$$|y_s| = \max\left\{|\gamma| : |a_\gamma^s| = |f|_{(1,\dots,1)}\right\}.$$

Set

$$\mathcal{M} = \{\widehat{f}_s, s = 1, \dots, q\}.$$

Since $\widehat{f}_s$ is not the zero polynomial, so is $\prod_{s=1}^{q} \hat{f}_s$.

Let $w = w_{(m)} \in D^m$ be such that $\widehat{w}$ is not a solution of $\prod_{s=1}^{q} \hat{f}_s$. Set

$$\frac{f_s(w)}{a_{y_s}} = b_s, \quad s = 1, \dots, q.$$

We have

$$\widehat{b_s} = \widehat{f_s}(\widehat{w}).$$

Since $\widehat{w}$ is not a solution of all $\widehat{f_s}$,

$$b_s \notin I.$$

Thus

$$\Big|\frac{f_s(w)}{a_{y_s}}\Big| = 1.$$

Hence $|f_s(w)| = |a_{y_s}|$.

Now let $x_1, \dots, x_m \in \mathbb{C}_p$ be such that $|x_i| = r_i$. Consider the following transformations of $\mathbb{C}_p^m$:

$$\varphi(z_{(m)}) = (x_1 z_1, \dots, x_m z_m).$$

Set $x = (x_1, \dots, x_m)$. We have $\varphi(D^m) = D_{r_{(m)}}$, and

$$f_s \circ \varphi(z_{(m)}) = \sum_{|\gamma|=0}^{\infty} \big(a_\gamma^s x^\gamma\big) z^\gamma$$

are non-zero entire functions on $\mathbb{C}_p^m$. By (2.1) there exists $w = w_{(m)}$ such that

$$\begin{aligned} \big|f_s \circ \varphi(w)\big| &= \max_{0 \le |\gamma| < \infty} \big|a_\gamma^s x^\gamma\big| = \max_{0 \le |\gamma| < \infty} \big|a_\gamma^s\big|\big|x_1\big|^{\gamma_1} \cdots \big|x_m\big|^{\gamma_m} \\ &= \max_{0 \le |\gamma| < \infty} \big|a_\gamma^s\big| r^\gamma = \big|f_s\big|_{r_{(m)}}. \end{aligned}$$

Set $u = \varphi(w)$. Then $u \in D_{r_{(m)}}$ and $\big|f_s(u)\big| = \big|f_s\big|_{r_{(m)}}$, $s = 1, \dots, q$. □

Lemma 2.2. *Let $f_s(z_{(m)})$, $s = 1, 2, \dots, q$, be q non-zero holomorphic functions on $D_{r_{(m)}}$. Then there exists $u = u_{(m)} \in D_{r_{(m)}}$ such that*

$$|f_s(u)| = |f_s|_{r_{(m)}}, \quad s = 1, 2, \dots, q.$$

Proof. Let

$$f_s(z_{(m)}) = \sum_{|\gamma|=0}^{\infty} a_\gamma^s z^\gamma.$$

For each $s = 1, 2, \dots, q$, we set

$$k_s = \max_{0 \le |\gamma| < \infty} \Big\{|\gamma| : \big|a_\gamma^s\big| r^\gamma = \big|f_s\big|_{r_{(m)}}\Big\}.$$

Then

$$P_s = \sum_{0 \le |\gamma| \le k_s} a^s_\gamma z^\gamma, \quad s = 1 \dots , q,$$

are non-zero entire functions on $\mathbb{C}_p^m$. By Lemma 2.1, there exists $u_{(m)} = (u_1, \dots , u_m) \in D_{r_{(m)}}$ with $|u_i| = r_i$ such that

$$|P_s(u_{(m)})| = |P_s|_{r_{(m)}}, \quad s = 1, \dots , q.$$

Moreover,

$$|P_s|_{r_{(m)}} = |f_s|_{r_{(m)}}, \quad |P_s(u_{(m)})| = |f_s(u_{(m)})|, \quad s = 1, \dots , q.$$

Thus $|f_s(u_{(m)})| = |f_s|_{r_{(m)}}$, $s = 1, \dots , q$. □

As an immediate consequence of Lemma 2.2 we have

Corollary 2.1. *Let $f(z_{(m)})$ be a non-zero holomorphic function on $D_{r_{(m)}}$. Then*

$$\max_{u \in D_{r_{(m)}}} |f(u)| = |f|_{r_{(m)}}.$$

Let $f = \sum\limits_{|\gamma|=0}^{\infty} a_\gamma z^\gamma$ be a non-zero holomorphic function on $D_{r_{(m)}}$.

We set for simplicity

$$\alpha = n^+_f(t_{(m)}), \quad k = n^-_f(t_{(m)}), \quad \beta = n_f(0, 0).$$

We consider the following holomorphic functions on $D_{r_{(m)}}$

$$P(z_{(m)}) = \sum_{|\gamma|=k} a_\gamma z^\gamma, \quad Q(z_{(m)}) = \sum_{|\gamma|=\alpha} a_\gamma z^\gamma, \quad Q_\beta = \sum_{|\gamma|=\beta} a_\gamma z^\gamma.$$

The functions are not identically zero. For a fixed i ($i = 1, \dots , m$), we set

$$\mathcal{E}_j = \frac{r_j}{r_i}, \quad j = 1, 2, \dots , m, \quad \mathcal{E}_{(m)} = (\mathcal{E}_1, ..., \mathcal{E}_m),$$

$$B^i_{f,r_{(m)}} = \Big\{ w = w_{(m)} \in D_{\mathcal{E}_{(m)}} : |P(w_{(m)})| = |P|_{\mathcal{E}_{(m)}}, \\ |Q(w_{(m)})| = |Q|_{\mathcal{E}_{(m)}}, |Q_\beta(w_{(m)})| = |Q_\beta|_{\mathcal{E}_{(m)}} \Big\}.$$

By Lemma 2.2, $B^i_{f,r_{(m)}}$ is a non-empty set. Set

$$f_{i,w}(z) = f(w_1 z, \dots , w_m z), \quad w \in B^i_{f,r_{(m)}}, \ z \in D_{r_i}.$$

The following theorem shows that we can use the Weierstrass Preparation Theorem [4] to count zeros by slicing with a generic line through the point u.

Theorem 2.2. *Let $f(z_{(m)})$ be a holomorphic function on $D_{r_{(m)}}$. Assume that $f(z_{(m)})$ is not identically zero. Then for each $i = 1, \dots , m$, and for all $w \in B^i_{f,r_{(m)}}$, we have*

1) $H_f(t_{(m)}) = H_{f_{i,w}}(t_i)$,

2) $n_f^-(t_{(m)})$ *is equal to the number of zeros of* $f_{i,w}$ *in* D_{r_i},

3) $n_f^-(t_{(m)}) - n_f^+(t_{(m)})$ *is equal to the number of zeros of* $f_{i,w}$ *at* $v(z) = t_i$.

Proof. Write

$$f(z) = \sum_{|\gamma|=0}^{\infty} a_\gamma z^\gamma, \quad f_{i,w}(z) = \sum_{j=0}^{\infty} b_j z^j,$$

where

$$b_j = \sum_{|\gamma|=j} a_\gamma w^\gamma.$$

Set

$$b_k = \sum_{|\gamma|=k} a_\gamma w^\gamma, \quad b_\alpha = \sum_{|\gamma|=\alpha} a_\gamma w^\gamma.$$

We have

$$|f_{i,w}|_{r_i} \le |f|_{r_{(m)}}.$$

By $w \in B^i_{f,r_{(m)}}$,

$$|b_k| = |a_\gamma||w_1|^{\gamma_1} \cdots |w_m|_p^{\gamma_m}, \quad \gamma_1 + \cdots + \gamma_m = k,$$

and

$$|b_\alpha| = |a_\gamma||w_1|^{\gamma_1} \cdots |w_m|_p^{\gamma_m}, \quad \gamma_1 + \cdots + \gamma_m = \alpha.$$

Therefore,

$$|b_k| r_i^k = |a_\gamma| r^\gamma = |b_\alpha| r_i^\alpha = |f|_{r_{(m)}}.$$

Thus

$$|f_{i,w}|_{r_i} = |f|_{r_{(m)}}.$$

So

$$H_f(t_{(m)}) = H_{f_{i,w}}(t_i)$$

and $n^+_{f_{i,w}}(t_i) \le \alpha$, $k \le n^-_{f_{i,w}}(t_i)$.

Now we consider j such that $|b_j| r_i^j = |f_{i,w}|_{r_i}$. Because $|f_{i,w}|_{r_i} = |f|_{r_{(m)}}$, $|b_j| r_i^j = |f|_{r_{(m)}}$. Since $b_j = \sum_{|\gamma|=j} a_\gamma w^\gamma$, we obtain

$$|b_j| r_i^j \le \max_{0 \le |\gamma| < \infty} |a_\gamma||w_1|^{\gamma_1} \cdots |w_m|^{\gamma_m} \le |f|_{r_{(m)}}.$$

Then there exists $\gamma = (\gamma_1, \dots, \gamma_m)$ with $|\gamma| = j$ such that

$$|a_\gamma||w_1|^{\gamma_1} \cdots |w_m|^{\gamma_m} = |a_\gamma| r^\gamma = |f|_{r_{(m)}}.$$

Hence $\alpha \leq j \leq k$. Therefore

$$n_f^+(t_{(m)}) = \alpha \leq n_{f_{i,w}}^+(t_i) \quad \text{and} \quad n_{f_{i,w}}^-(t_i) \leq k = n_f^-(t_{(m)}).$$

From this it follows that $n_f^+(t_{(m)}) = n_{f_{i,w}}^+(t_i)$ and $n_{f_{i,w}}^-(t_i) = n_f^-(t_{(m)})$.

By Lemma 2.2 and Theorem 2.1, we have $H_f(t_{(m)}) = H_{f_{i,w}}(t_i)$, and $n_f^-(t_{(m)})$ is equal to the number of zeros of $f_{i,w}$ in D_{r_i}, $n_f^-(t_{(m)}) - n_f^+(t_{(m)})$ is equal to the number of zeros of $f_{i,w}$ at $v(z) = t_i$. The proof is complete. □

For each $i = 1, \dots, m$, from Theorem 2.1 we see that $n_f(0,0) = n_{f_{i,w}}(0,0)$ for all $w \in B^i_{f,r_{(m)}}$.

Let f be a non-zero holomorphic function on $D_{r_{(m)}}$. Define $n_f(0, r_{(m)})$ to be the number of zeros with absolute value $\leq r_i$ of the one-variable function $f_{i,w}(z)$. Theorem 2.2 tells us that

$$n_f(0, r_{(m)}) = n_f^-(t_{(m)}).$$

For an element a of $\mathbb{C}_p$ and a holomorphic function f on $D_{r_{(m)}}$, which is not identically equal to a, we define

$$n_f(a, r_{(m)}) = n_{f-a}(0, r_{(m)}), \quad n_f(a,0) = n_{f-a}(0,0), \quad i = 1, \dots, m.$$

Fix real numbers $\rho_1, \dots, \rho_m$ with $0 < \rho_i \leq r_i$, $i = 1, \dots, m$, such that

$$\frac{r_1}{\rho_1} = \frac{r_2}{\rho_2} = \cdots = \frac{r_m}{\rho_m} \cdot$$

Set

$$\frac{r_1}{\rho_1} = r, \quad n_f(a,x) = n_f(a, (\rho_1 x, \dots, \rho_m x)), \text{ with } 0 < x \leq r,$$

$$c_i = -\log \rho_i, \quad i = 1, \dots, m.$$

Define *the counting function* $N_f(a, t_{(m)})$ by

$$N_f(a, t_{(m)}) = \frac{1}{\ln p} \int_1^r \frac{n_f(a,x)}{x} dx.$$

If $a = 0$, then we set $N_f(t_{(m)}) = N_f(0, t_{(m)})$.

Lemma 2.3. *Let f be a non-zero entire function on $\mathbb{C}_p^m$. Then*

$$H_f^+(c_{(m)} + t) = N_f(c_{(m)} + t) + O(1),$$

where $O(1)$ is bounded when $t \to -\infty$.

This lemma can be proved easily by using Theorem 2.2.

Theorem 2.3. *Let f be a non-zero entire function on $\mathbb{C}_p^m$ and γ a multi-index with $\mid \gamma \mid > 0$. Then*

$$H_{\partial^\gamma f}(t_1, \dots, t_m) - H_f(t_1, \dots, t_m) \geq - \mid \gamma \mid T,$$

where $T = \max_{1 \leq i \leq m} t_i$.

The proof of Theorem 2.3 follows immediately from [3, Lemma 4.1].

3. Height of p-adic holomorphic maps

We say that an entire function g *divides* an entire function f if $f = gh$ for some entire function h, and we say that g is a *greatest common divisor* of n entire functions $f_1, \dots, f_n$ if whenever an entire function h divides each of non-zero f_i then h also divides g. We say that n entire functions $f_1, \dots, f_n$ are *without common factors* if 1 is a greatest common divisor.

Note that greatest common divisors exist in the ring of entire functions on $\mathbb{C}_p^m$ (see [3]). By a *holomorphic map*

$$f : \mathbb{C}_p^m \longrightarrow \mathbb{P}^n(\mathbb{C}_p) = \mathbb{P}^n,$$

we mean an equivalence class of $(n+1)-$tuples of entire functions $(f_1, \dots, f_{n+1})$ such that $f_1, \dots, f_{n+1}$ do not have any common factors in the ring of entire functions on $\mathbb{C}_p^m$ and such that not all of the f_i are identically zero. Two $(n+1)-$tuples entire functions$(f_1, \dots, f_{n+1})$ and $(g_1, \dots, g_{n+1})$ are equivalent if there exists a constant c such that $f_i = cg_i$ for all i. We identify f with its representation by a collection of entire functions on $\mathbb{C}_p^m$

$$f = (f_1, \dots, f_{n+1}).$$

Definition 3.1. *The height* of a holomorphic map f is defined by

$$H_f(t_{(m)}) = \min_{1 \le i \le n+1} H_{f_i}(t_{(m)}).$$

We also use the notation

$$H_f^+(t_{(m)}) = -H_f(t_{(m)}).$$

Let $H_1, \dots, H_q$ $(q \ge n+1)$ be q hyperplanes in $\mathbb{P}^n(\mathbb{C}_p)$ in *general position*. This means that any $n+1$ of these hyperplanes are linearly independent. Let $f : \mathbb{C}_p^m \longrightarrow \mathbb{P}^n$ be a holomorphic map. Suppose that $F = 0$, $F_i = 0$ are the equations defining the hyperplanes H, H_i. We set

$$\begin{aligned} H_f(H, (t_{(m)})) &= H_{F \circ f}(t_{(m)}), \\ H_f(H_i, (t_{(m)})) &= H_{F_i \circ f}(t_{(m)}), \\ N_f(H, (t_{(m)})) &= N_{F \circ f}(t_{(m)}), \\ N_f(H_i, (t_{(m)})) &= N_{F_i \circ f}(t_{(m)}), \end{aligned}$$

$$\begin{aligned} m_f(H, (t_{(m)})) &= \max_{1 \le i \le n+1} H^+_{\frac{f_i}{F \circ f}}(t_{(m)}) \text{ if } F \circ f \not\equiv 0, \\ T_f(H, (t_{(m)})) &= N_f(H, (t_{(m)})) + m_f(H, (t_{(m)})). \end{aligned}$$

Theorem 3.1. (First Main Theorem). *Let $f : \mathbb{C}_p^m \longrightarrow \mathbb{P}^n$ be a holomorphic map. Let H be a hyperplane in $\mathbb{P}^n$ such that the image of f is not contained in H. Then we have*

$$T_f(H, (c_{(m)} + t)) = H_f^+(c_{(m)} + t) + O(1),$$

where $O(1)$ depends on H, but not on t.

Proof. Let $f = (f_1, \dots, f_{n+1})$. By definition,

$$T_f(H, (c_{(m)} + t)) = N_{F\circ f}(c_{(m)} + t) + \max_{1\le i\le n+1}(H^+_{f_i}(c_{(m)} + t) - H^+_{F\circ f}(c_{(m)} + t))$$
$$= H^+_f(c_{(m)} + t) + (N_{F\circ f}(c_{(m)} + t) - H^+_{F\circ f}(c_{(m)} + t)).$$

By Lemma 2.3,

$$N_{F\circ f}(c_{(m)} + t) - H^+_{F\circ f}(c_{(m)} + t) = O(1),$$

Therefore,

$$T_f(H, (c_{(m)} + t)) = H^+_f(c_{(m)} + t) + O(1).$$

and the proof is complete. □

A holomorphic map $f : \mathbb{C}_p^m \longrightarrow \mathbb{P}^n$ is called *linearly non-degenerate* if the image of f is not contained in any hyperplanes of $\mathbb{P}^n$. If $f = (f_1, \dots, f_{n+1})$ is an $(n+1)-$tuple of entire functions and if γ is a multi-index, then by $\partial^\gamma f$ we mean the $(n+1)$-tuple

$$(\partial^\gamma f_1, \dots, \partial^\gamma f_{n+1}).$$

Lemma 3.1. [3] *Let $f = (f_1, \dots, f_{n+1})$ be a linearly non-degenerate holomorphic map from $\mathbb{C}_p^m$ to $\mathbb{P}^n$. Then there exist multi-indexes $\gamma_1, \dots, \gamma_n$ such that $\mid \gamma_i \mid\le i$ and $f, \partial^{\gamma_1} f, \dots, \partial^{\gamma_n} f$ are linearly independent over the field of meromorphic functions on $\mathbb{C}_p^m$.*

Let $f = (f_1, \dots, f_{n+1})$ be a linearly non-degenerate holomorphic map. By Lemma 3.1, we can always find such γ_i with $\mid \gamma_i \mid\le i$ that the Wronskian

$$W = \det\begin{pmatrix} f_1 & \dots & f_{n+1} \\ \vdots & \ddots & \vdots \\ \partial^{\gamma_n} f_1 & \dots & \partial^{\gamma_n} f_{n+1}\end{pmatrix}$$

is not identically zero.

Set $B = \sum_{1\le i\le n} \mid \gamma_i \mid$. Note that $n \le B \le n(n+1)/2$.

For a linearly non-degenerate holomorphic map f from $\mathbb{C}_p^m$ to $\mathbb{P}^n$, we define the ramification term $N_{f,Ram}(t_{(m)})$ by

$$N_{f,Ram}(t_{(m)}) = N_W(t_{(m)}).$$

For different choices of the γ_i one gets different ramification terms.

Theorem 3.2. *Let $H_1, \dots, H_q$ be q hyperplanes in general position, and f be a linearly non-degenerate holomorphic map from $\mathbb{C}_p^m$ to $\mathbb{P}^n$. Then we have*

$$(q - n - 1)H^+_f(t_{(m)}) + H^+_W(t_{(m)}) \le \sum_{j=1}^{q} H^+_f(H_j, (t_{(m)})) + BT + O(1),$$

where $O(1)$ is bounded when $T = \max_{1\le i\le m} t_i \to -\infty$.

Proof. We first consider the case $q > n+1$.

Let $G_i = F_i \circ f$, $i = 1, \dots, q$, and $\beta_1, \dots, \beta_{q-n-1}$ be distinct numbers in the set $\{1, 2, \dots, q\}$.

Let $G = (\dots, G_{\beta_1} \dots G_{\beta_{q-n-1}}, \dots)$, where $(\beta_1, \dots, \beta_{q-n-1})$ is taken by all possible choices.

We need the following lemmas.

Lemma 3.2. *G determines a holomorphic map from $\mathbb{C}_p^m$ to $\mathbb{P}^{k-1}$, where $k = C_q^{q-n-1}$.*

Proof. Assume that the functions $G_{\beta_1} \dots G_{\beta_{q-n-1}}$ have a non-constant greatest common divisor. Then the functions $G_{\beta_1} \dots G_{\beta_{q-n-1}}$ have common zeros. Because $q > n+1$, there exist G_{α_i}, $i = 1, \dots, n+1$ and $(z_{(m)}) \in \mathbb{C}_p^m$ such that $G_{\alpha_i}(z_{(m)}) = 0$. Then

$$f(z_{(m)}) \in H_{\alpha_i}, \quad i = 1, \dots, n+1.$$

Since $H_{\alpha_1}, \dots, H_{\alpha_{n+1}}$ are in general position, we have a contradiction. $\square$

Lemma 3.3. *We have*

$$H_G(t_{(m)}) \leq (q-n-1) H_f(t_{(m)}) + O(1),$$

where $O(1)$ does not depend on $(t_{(m)})$.

Proof. By the definition,

$$\begin{aligned} H_G(t_{(m)}) &= \min_{(\beta_1, \dots, \beta_{q-n-1})} H_{G_{\beta_1} \dots G_{\beta_{q-n-1}}}(t_{(m)}) \\ &= \min_{(\beta_1, \dots, \beta_{q-n-1})} \sum_{i=1}^{q-n-1} H_{G_{\beta_i}}(t_{(m)}). \end{aligned}$$

Assume that for a fixed $(t_{(m)})$, the following inequalities hold

$$\begin{aligned} H_{G_{\beta_1}}(t_{(m)}) \leq H_{G_{\beta_2}}(t_{(m)}) &\leq \dots \\ &\leq H_{G_{\beta_q}}(t_{(m)}). \end{aligned}$$

Then

$$H_G(t_{(m)}) = H_{G_{\beta_1}}(t_{(m)}) + H_{G_{\beta_2}}(t_{(m)}) + \dots + H_{G_{\beta_{q-n-1}}}(t_{(m)}).$$

On the other hand, due to the hypothesis of general position, we can represent f_i by a linear combination of $G_{\beta_{q-n}}, \dots, G_{\beta_q}$:

$$f_i = \sum_{0 \leq j \leq n} a_{ij} G_{\beta_{q-j}}.$$

It follows that

$$H_{f_i}(t_{(m)}) \geq \min_{0 \leq j \leq n} H_{G_{\beta_{q-j}}}(t_{(m)}) + O(1).$$

Therfore, we obtain

$$H_{f_i}(t_{(m)}) \geq H_{G_{\beta_j}}(t_{(m)}) + O(1),$$

for $j = 1, \dots, q-n-1$. Hence,

$$H_f(t_{(m)}) = \min_{1\le i\le n+1} H_{f_i}(t_{(m)}) \geq H_{G_{\beta_j}}(t_{(m)}) + O(1),$$

for $j = 1, \dots, q-n-1$. Lemma 3.3 is then proved by summarizing $(q-n-1)$ inequalities. □

Proof of Theorem 3.2. For $(n+1)$ $g_1, \dots, g_{n+1}$ we denote by $W(g_1, \dots, g_{n+1})$ their Wronskian with respect to the γ_i as in the statement of Lemma 3.1.

Let $(\alpha_1, \dots, \alpha_{n+1})$ be distinct numbers in $\{1, \dots, q\}$ and $(\beta_1, \dots, \beta_{q-n-1})$ be the rest. Note that the functions f_i can be represented as linear combinations of $G_{\alpha_1}, \dots, G_{\alpha_{n+1}}$. Then we have

$$W(G_{\alpha_1}, \dots, G_{\alpha_{n+1}}) = c_{(\alpha_1,\dots,\alpha_{n+1})} W(f_1, \dots, f_{n+1}),$$

where $c_{(\alpha_1,\dots,\alpha_{n+1})} = c$ is constant depending only on $(\alpha_1, \dots, \alpha_{n+1})$.

We set

$$A = A(\alpha_1, \dots, \alpha_{n+1}) = \frac{W(G_{\alpha_1}, \dots, G_{\alpha_{n+1}})}{G_{\alpha_1} \dots G_{\alpha_{n+1}}} = \det \begin{pmatrix} 1 & \dots & 1 \\ \dfrac{\partial^{\gamma_1} G_{\alpha_1}}{G_{\alpha_1}} & \dots & \dfrac{\partial^{\gamma_1} G_{\alpha_{n+1}}}{G_{\alpha_{n+1}}} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^{\gamma_n} G_{\alpha_1}}{G_{\alpha_1}} & \dots & \dfrac{\partial^{\gamma_n} G_{\alpha_{n+1}}}{G_{\alpha_{n+1}}} \end{pmatrix}.$$

Then

$$\frac{G_1 \dots G_q}{W(f_1, \dots, f_{n+1})} = \frac{CG_{\beta_1} \dots G_{\beta_{q-n-1}}}{A}. \tag{3.1}$$

Let S be the set of all permutations of $\{0, \dots, n\}$. We set $\dfrac{\partial^0 G_{\alpha_i}}{G_{\alpha_i}} = 1$, $i = 1, 2, \dots, n+1$, and

$$G_\sigma = \frac{\partial^{\gamma_{\sigma(0)}} G_{\alpha_1}}{G_{\alpha_1}} \dots \frac{\partial^{\gamma_{\sigma(n)}} G_{\alpha_{n+1}}}{G_{\alpha_{n+1}}}, \quad \sigma \in S.$$

Then we have

$$H_A(t_{(m)}) \geq \min_{\sigma\in S} H_{G_\sigma}(t_{(m)}).$$

By Theorem 2.3,

$$H_{\frac{G_{\alpha_i}^{\gamma_{\sigma(i)}}}{G_{\alpha_i}}}(t_{(m)}) \geq - \mid \gamma_{\sigma_{(i)}} \mid T + O(1),$$

where $T = \max_{1\le i\le m} t_i$. Then

$$H_A(t_{(m)}) \ge - \mid \gamma \mid T + O(1) = -BT + O(1). \tag{3.2}$$

By (3.1) and (3.2) we get

$$\sum_{i=1}^{q} H_{G_i}(t_{(m)}) - H_W(t_{(m)}) = H_{G_{\beta_1}\dots G_{\beta_{q-n-1}}}(t_{(m)}) - H_A(t_{(m)}) + O(1).$$

This implies that

$$\begin{aligned} H_G(t_{(m)}) &= \min_{(\beta_1,\dots,\beta_{q-n-1})} H_{G_{\beta_1}\dots G_{\beta_{q-n-1}}}(t_{(m)}) \\ &\ge \sum_{i=1}^{q} H_{G_i}(t_{(m)}) - H_W(t_{(m)}) - BT + O(1). \end{aligned}$$

Therefore

$$(q-n-1)H_f(t_{(m)}) \ge \sum_{i=1}^{q} H_{G_i}(t_{(m)}) - H_W(t_{(m)}) - BT + O(1).$$

Hence

$$(q-n-1)H_f^+(t_{(m)}) + H_W^+(t_{(m)}) \le \sum_{j=1}^{q} H_f^+(H_j, (t_{(m)})) + BT + O(1).$$

If $q = n+1$, then we have

$$\frac{G_1 \dots G_{n+1}}{W} = \frac{c}{A}.$$

From this and (3.2) we obtain

$$H_W^+(t_{(m)}) \le \sum_{i=1}^{n+1} H_f^+(H_i, t_{(m)}) + BT + 0(1).$$

Theorem 3.2 is proved. □

Theorem 3.3. (Second Main Theorem). *Let $H_1, \dots, H_q$ be q hyperplanes in general position and f be a linearly non-degenerate holomorphic map from $\mathbb{C}_p^m$ to $\mathbb{P}^n$. Then*

$$\begin{aligned} &(q-n-1)H_f^+(c_{(m)}+t) + N_{f,Ram}(c_{(m)}+t) \\ &\le \sum_{j=1}^{q} N_f(H_j, (c_{(m)}+t)) + BT + O(1), \end{aligned}$$

where $T = \max_{1\le i\le m}(c_i + t)$, and $O(1)$ is bounded when $T \to -\infty$.

Proof. By Lemma 2.3,

$$H_W^+(c_{(m)}+t) = N_W(c_{(m)}+t) + O(1),$$

and

$$H_f^+(H_j,(c_{(m)}+t)) = N_f(H_j,(c_{(m)}+t)) + O(1).$$

Then, by Theorem 3.2, we have

$$(q-n-1)H_f^+(c_{(m)}+t) + N_{f,Ram}(c_{(m)}+t) \leq \sum_{1\leq j\leq q} N_f(H_j,(c_{(m)}+t)) + BT + O(1).$$

which completes the proof. □

In particular, for $c_1 = c_2 = \ldots = c_m$, we obtain Cherry-Ye's theorem.

Corollary 3.1. (see [3]) *Let $H_1,\ldots,H_q$ be q hyperplanes in general position in $\mathbb{P}^n$, and f a linearly non-degenerate holomorphic map from $\mathbb{C}_p^m$ to $\mathbb{P}^n$. Then we have*

$$(q-n-1)H_f^+(t,\ldots,t) + N_{f,Ram}(t,\ldots,t) \leq \sum_{1\leq j\leq q} N_f(H_j,(t,\ldots,t)) + Bt + O(1),$$

where $O(1)$ is bounded when $t \to -\infty$.

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