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# STRATIFICATION OF FAMILIES OF FUNCTIONS DEFINABLE IN O-MINIMAL STRUCTURES

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Abstract. We prove the existence of Thom stratifications for families of functions definable in any o-minimal structure. The theory of o-minimal structures is a generalization of semi-algebraic and sub-analytic geometry. Our result implies Fukuda's Theorem on the finiteness of topological types for polynomials on $\mathbb{R}^n$ with bounded degree.

## Introduction

In this note we will consider the stratification with Thom's conditions of families of functions definable in o-minimal structures. The theory of o-minimal structures is a generalization of semialgebraic and subanalytic geometry. For details we refer the readers to the surveys [D] and [DM].

A *structure* on the real field $(\mathbb{R}, +, \cdot)$ is a sequence $\mathcal{D} = (\mathcal{D}_n)_{n\in\mathbb{N}}$ of subsets of $\mathbb{R}^n$ such that the following conditions are satisfied for all $n \in \mathbb{N}$:

- $\mathcal{D}_n$ is a Boolean algebra .
- If $A \in \mathcal{D}_n$, then $A \times \mathbb{R}$ and $\mathbb{R} \times A \in \mathcal{D}_{n+1}$.
- If $A \in \mathcal{D}_{n+1}$, then $\pi(A) \in \mathcal{D}_n$, where $\pi : \mathbb{R}^{n+1} \to \mathbb{R}^n$ is the projection on the first $n$ coordinates.
- $\mathcal{D}_n$ contains $\{x \in \mathbb{R}^n : P(x) = 0\}$, for every polynomial $P \in \mathbb{R}[X_1, \cdots, X_n]$.

Structure $\mathcal{D}$ is called *o-minimal* if

- Each set in $\mathcal{D}_1$ is a finite union of intervals and points.

A set belonging to $\mathcal{D}$ is called *definable* (in that structure). *Definable maps* in structure $\mathcal{D}$ are maps whose graphs are definable sets in $\mathcal{D}$.

It is worth noting that o-minimal structures share many interesting properties with those of semi-algebraic sets. For example, definable sets admit Whitney stratification (see [L2]), so they can be triangulated. Definable functions are piecewise smooth (see [D]) and can be triangulated (see [C]).

In this note we fix an o-minimal structure on $(\mathbb{R}, +, \cdot)$. "Definable" means definable in this structure. Moreover, we shall need the following notions.

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Let $p$ be a positive integer A *definable $C^p$ Whitney stratification* of $X \subset \mathbb{R}^n$ is a partition $\mathcal{X}$ of $X$ into finitely many subsets, called strata, such that:

- Each stratum is a $C^p$ submanifold of $\mathbb{R}^n$ and also a definable set.
- For every $\Gamma \in \mathcal{S}$, $\overline{\Gamma} \setminus \Gamma$ is a union of some of the strata.
- For every $\Gamma$, $\Gamma' \in \mathcal{X}$, if $\Gamma \subset \overline{\Gamma'}$, then $(\Gamma, \Gamma')$ has the Whitney property.

We say that a stratification $\mathcal{X}$ is *compatible* with a class $\mathcal{A}$ of subsets of $\mathbb{R}^n$, if for each $\Gamma \in \mathcal{X}$ and $S \in \mathcal{A}$, $\Gamma \subset S$ or $\Gamma \cap S = \emptyset$.

Let $f : X \to Y$ be a definable map. A *$C^p$ stratification* of $f$ is a pair $(\mathcal{X}, \mathcal{Y})$, where $\mathcal{X}$ and $\mathcal{Y}$ are definable $C^p$ Whitney stratifications of $X$ and $Y$ respectively, and for each $\Gamma \in \mathcal{X}$, there exists $\Phi \in \mathcal{Y}$, such that $f(\Gamma) \subset \Phi$ and $f|_\Gamma : \Gamma \to \Phi$ is a $C^p$ submersion.

Let $(\mathcal{X}, \mathcal{Y})$ be a $C^p$ stratification of $f : X \to Y$. The map $f$ is called a *Thom map stratified by* $(\mathcal{X}, \mathcal{Y})$ if for all $\Gamma, \Gamma' \in \mathcal{X}$ with $\Gamma \subset \overline{\Gamma'}$, the pair $(\Gamma, \Gamma')$ satisfies the following condition at each $x \in \Gamma$ :

($\mathrm{a}_f$) for every sequence $(x_k)$ in $\Gamma'$ converging to $x$, such that $\ker d(f|_{\Gamma'})(x_k)$ converges to a subspace $\tau$ of $T_x\mathbb{R}^n$, then $\ker d(f|_\Gamma)(x) \subset \tau$.

## 1. Main result

Our main result can be formulated as follows.

**Theorem 1.1.** *Let $X \subset \mathbb{R}^n$, $T \subset \mathbb{R}^m$ be definable sets. Let*

$$f : X \times T \to \mathbb{R}, (x, t) \mapsto f(x, t) = f_t(x)$$

*be a continuous definable function. Then for every finite collection $\mathcal{A}$ of definable subsets of $X \times T$ and $p \geq 2$, there exists a finite partition $T = \bigcup_{i=1}^{q} T_i$ into $C^p$ definable manifolds, such that for each $i \in \{1, \dots, q\}$, there exist definable $C^p$ Whitney stratifications $\mathcal{X}$ of $X \times T_i$ and $\mathcal{Y}$ of $\mathbb{R} \times T_i$, such that $\mathcal{X}$ is compatible with $\mathcal{A}$ and the map*

$$X \times T_i \to \mathbb{R} \times T_i, \ (x, t) \mapsto (f(x, t), t)$$

*is a Thom map stratified by $(\mathcal{X}, \mathcal{Y})$, and $(\mathcal{Y}, \{T_i\})$ is a stratification of the projection $\mathbb{R} \times T_i \to T_i, (y, t) \mapsto t$.*

**Corollary 1.1.** *Under the assumptions of the theorem, if $t$ and $t'$ are in the same connected component of $T_i$, then $f_t$ and $f_{t'}$ are topologically equivalent, that is there exist homeomorphisms $h : X \to X$ and $\lambda : \mathbb{R} \to \mathbb{R}$, such that $f_t \circ h = \lambda \circ f_{t'}$.*

The corollary is an extension of [F], where Fukuda proved that the number of topological types of polynomial functions on $\mathbb{R}^n$ of degree $\leq d$ is finite.

## 2. Proof of the main result

We shall need the existence of the stratifications of definable maps. The following theorem is proved in [DM, Theorem 4.8] with a gap.

**Theorem 2.1.** *Let $f : X \to Y$ be a continuous definable map. Let $\mathcal{A}$ and $\mathcal{B}$ be finite collections of definable subsets of $X$ and $Y$ respectively. Then there exists a $C^p$ stratification $(\mathcal{X}, \mathcal{Y})$ of $f$ such that $\mathcal{X}$ is compatible with $\mathcal{A}$ and $\mathcal{Y}$ is compatible with $\mathcal{B}$.*

*Proof.* We follow closely the proof of [S, Theorem I.2.6] for subanalytic maps. Let $m = \dim Y$. We will construct a chain of definable sets

$$Y^m \subset Y^{m-1} \subset \cdots \subset Y^0 = Y \ ,$$

and the pairs $(\mathcal{X}^k, \mathcal{Y}^k)$, $k = m, m-1, \cdots, 0$, satisfying the following conditions

$(F_k)$ $Y \setminus Y^k$ is a closed subset of $Y$ and $\dim(Y \setminus Y^k) < k$; $\mathcal{X}^k$ is a definable $C^p$ Whitney stratification of $X^k = f^{-1}(Y^k)$ compatible with $\mathcal{A}$; $\mathcal{Y}^k$ is a definable $C^p$ Whitney stratification of $Y^k$ compatible with $\mathcal{B}$, and $\dim \Phi \geq k$, $\forall \Phi \in \mathcal{Y}^k$; $\mathcal{X}^{k+1} \subset \mathcal{X}^k$ and $\mathcal{Y}^{k+1} \subset \mathcal{Y}^k$; and $(\mathcal{X}^k, \mathcal{Y}^k)$ is a $C^p$ stratification of $f|_{X^k} : X^k \to Y^k$.

This inductive construction leads to a stratification $(\mathcal{X}, \mathcal{Y}) = (\mathcal{X}^0, \mathcal{Y}^0)$, which satisfies the demands of the theorem.

Suppose $(\mathcal{X}^{k+1}, \mathcal{Y}^{k+1})$ is constructed. By [L2, Theorem 1.3 and Proposition 1.10], there exists a finite collection $\mathcal{Z}^k$ of disjoint definable submanifolds of dimension $k$, contained in $Y \setminus Y^{k+1}$ such that: $\mathcal{Z}^k$ is compatible with $\mathcal{B}$; $\dim(Y \setminus Y^{k+1} \setminus |\mathcal{Z}^k|) < k$ (where $|\mathcal{Z}^k| = \cup_{Z \in \mathcal{Z}^k} Z$); and $\mathcal{Y}^{k+1} \cup \mathcal{Z}^k$ is a definable $C^p$ Whitney stratification of a subset of $Y$.

We will prove that for each $Z \in \mathcal{Z}^k$, there is a definable closed subset $Z^0$ of $Z$ with $\dim Z^0 < k$, and we will modify $\mathcal{A}|_{f^{-1}(Z \setminus Z^0)}$ to a stratification $\mathcal{W}_Z$ so that the pair $(\mathcal{X}^k = \mathcal{X}^{k+1} \cup \cup_{Z \in \mathcal{Z}^k} \mathcal{W}_Z,\ \mathcal{Y}^k = \mathcal{Y}^{k+1} \cup \{Z \setminus Z^0 : Z \in \mathcal{Z}^k\})$ satisfies $(F_k)$.

For $Z \in \mathcal{Z}^k, f^{-1}(Z) = \emptyset$, let $Z^0 = \emptyset$ and $\mathcal{W}_Z = \emptyset$.

For $Z \in \mathcal{Z}^k, f^{-1}(Z) \neq \emptyset$, by [DM, Theorem 4.2], we may assume that $\mathcal{A}$ is compatible with $f^{-1}(Z)$. Moreover, by [DM, Lemma C.2], for each $A \in \mathcal{A}|_{f^{-1}(Z)}$, there is a definable subset $B_A$ of $A$ such that $A \setminus B_A$ is a submanifold and $f|_{A \setminus B_A}$ is submersive into $Z$ (if $A \setminus B_A \neq \emptyset$), and $\dim f(B_A) < k$. Then $Z \cap \cup_{A \in \mathcal{A}|_{f^{-1}(Z)}} \overline{f(B_A)}$ is of dimension $< k$. By deleting a closed subset of dimemsion $< k$ from $Z$, we may assume that $f|_A : A \to Z$ is submersive for every $A \in \mathcal{A}|_{f^{-1}(Z)}$. Under the above assumptions, let $n = \dim f^{-1}(Z)$, we now construct chains of definable sets

$$\emptyset = Z^m \subset Z^{m-1} \subset \cdots \subset Z^0 \subset Z \text{ and } W^n \subset W^{n-1} \subset \cdots \subset W^0 \subset f^{-1}(Z),$$

and for $l = n, n-1, \ldots, 0$, partitions $\mathcal{W}_Z^l$ of $W^l$ into definable submanifolds satisfying the following conditions

$(G_l)$ $\dim Z^l < k$; $\dim f^{-1}(Z \setminus Z^l) \setminus W^l < l$; $\mathcal{W}_Z^l$ is compatible with $\mathcal{A}$ and $\dim W \geq l, \forall W \in \mathcal{W}_Z^l$; $\mathcal{W}_Z^{l+1} \subset \mathcal{W}_Z^l$; $\mathcal{X}^{k+1} \cup \mathcal{W}_Z^l$ is a definable $C^p$ Whitney stratification; and for each $W \in \mathcal{W}_Z^l$, $f|_W : W \to Z$ is submersive.

Suppose $Z^{l+1}$ and $\mathcal{W}_Z^{l+1}$ are constructed. For each $A \in \mathcal{A}|_{f^{-1}(Z)}$, let $A' = A \setminus f^{-1}(Z^{l+1}) \setminus W^{l+1}$. By [L, Theorem 1.3] and [DM, Lemma C.2], there exist definable subsets $B'_A$ and $B''_A$ of $A'$ such that $A' \setminus (B'_A \cup B''_A)$ is a submanifold of dimension $l$ (if not empty), $\dim B'_A < l$, $\dim f(B''_A) < k$, $f|_{A' \setminus (B'_A \cup B''_A)}$ is submersive, and $\mathcal{X}^{k+1} \cup \mathcal{W}_Z^{l+1} \cup \{A' \setminus (B'_A \cup B''_A), A \in \mathcal{A}\}$ is a definable $C^p$ Whitney stratification. Let $Z^l = Z^{l+1} \cup \left( Z \cap \cup_{A \in \mathcal{A}|_{f^{-1}(Z)}} \overline{f(B''_A)} \right)$, and $\mathcal{W}_Z^l = \mathcal{W}_Z^{l+1} \cup \{A' \setminus (B'_A \cup B''_A), A \in \mathcal{A}|_{f^{-1}(Z)}\}$. Then $Z^l$ and $\mathcal{W}_Z^l$ satisfy $(G_l)$.

Obviously, $Z^0$ and $\mathcal{W}_Z = \mathcal{W}_Z^0|_{f^{-1}(Z \setminus Z^0)}$ have the desired properties. □

Now we will use the notations of Theorem 1.1. For $\Gamma \subset X \times T$ and $t \in T$, we set $\Gamma_t = \{x \in X : (x,t) \in \Gamma\}$. Let $\pi : X \times T \to T$ be the natural projection.

**Lemma 2.1.** *There exists a $C^p$ stratification of $(f, \pi) : X \times T \to \mathbb{R} \times T$, compatible with $\mathcal{A}$.*

*Proof.* This follows from Theorem 2.1. □

**Lemma 2.2.** *Let $\Gamma_t$ and $\Gamma'_t$ be definable, $C^p$ submanifolds of $\mathbb{R}^n$, and $\Gamma_t \subset \overline{\Gamma'_t}$. Let $f_t : \Gamma_t \cup \Gamma'_t \to \mathbb{R}$ be a continuous definable function. Suppose that the restrictions of $f_t$ to $\Gamma_t$ and $\Gamma'_t$ are of class $C^p$, and have constant ranks. Then the set*

$$A = A(f_t, \Gamma_t, \Gamma'_t) = \{x \in \Gamma_t : (\Gamma_t, \Gamma'_t) \text{ satisfies } (\mathrm{a}_{f_t}) \quad \text{at } x\}$$

*is definable and* $\dim(\Gamma_t \setminus A) < \dim \Gamma_t$.

*Proof.* See [L1]. □

**Lemma 2.3.** *Let $\Gamma$ and $\Gamma'$ be definable, $C^p$ submanifolds of $X \times T$. Suppose that the restrictions $\pi|_\Gamma$ and $\pi|_{\Gamma'}$ have constant ranks, and* $\operatorname{rank} f_t|_{\Gamma_t}$ *and* $\operatorname{rank} f_t|_{\Gamma'_t}$ *are constant for all $t \in \pi(\Gamma)$. Then the set*

$$A((f,\pi), \Gamma, \Gamma') = \{(x,t) \in \Gamma : x \in A(f_t, \Gamma_t, \Gamma'_t)\}$$

*is definable and* $\dim(\Gamma \setminus A((f,\pi), \Gamma, \Gamma')) < \dim \Gamma$.

*Proof.* Obviously, $A((f,\pi),\Gamma,\Gamma')$ is definable by definition. By Lemma 2.2,

$$\dim(\Gamma \setminus A((f,\pi),\Gamma,\Gamma')) < \dim \Gamma.$$

□

*Proof of the Theorem 1.1.* We use induction on $\dim T$. Let $N = n + m$. By Lemma 2.1, we may suppose that $T$ is a $C^p$ manifold, and that there are definable $C^p$ Whitney stratifications $\mathcal{X}^N$ of $X \times T$ and $\mathcal{Y}^N$ of $\mathbb{R} \times T$, compatible with $\mathcal{A}$, such that for each $\Gamma \in \mathcal{X}^N$, $\pi|_\Gamma$ has constant rank, the restriction $f|_\Gamma$ is of class $C^p$, and $\operatorname{rank}(f_t|_{\Gamma_t})$ is constant for all $t \in \pi(\Gamma)$.

We will construct the stratifications $(\mathcal{X}^k, \mathcal{Y}^k)$ of $(f,\pi) : X \times T \to \mathbb{R} \times T$, by decreasing $k = N, N-1, \ldots, 0$, such that $\mathcal{X}^k$ is compatible with $\mathcal{A}$ and satisfies the following condition

($*_k$) If $\Gamma, \Gamma' \in \mathcal{X}^k, \Gamma \subset \overline{\Gamma'}$ and $\dim \Gamma \geq k$, then $\pi|_\Gamma$ has constant rank, and for all $t \in \pi(\Gamma), \operatorname{rank} f_t|_{\Gamma_t}$ is constant and $(\Gamma_t, \Gamma'_t)$ satisfies $(\mathrm{a}_{f_t})$ at each point of $\Gamma_t$.

Suppose $(\mathcal{X}^k, \mathcal{Y}^k)$ is constructed. We will construct $(\mathcal{X}^{k-1}, \mathcal{Y}^{k-1})$. For each $\Gamma \in \mathcal{X}^k$, let

$$B_\Gamma = \bigcup \{\Gamma \setminus A((f,\pi), \Gamma, \Gamma') : \Gamma' \in \mathcal{X}^k, \Gamma \subset \overline{\Gamma'}\}$$

By Lemma 2.3, $\dim B_\Gamma < \dim \Gamma$. By Lemma 2.1, there exists a stratification $(\mathcal{T}^{k-1}, \mathcal{Y}^{k-1})$ of $(f, \pi)$ compatible with $\{\Gamma \setminus B_\Gamma : \Gamma \in \mathcal{X}^k, \dim \Gamma = k-1\}$ and $\{\Gamma : \Gamma \in \mathcal{X}^k, \dim \Gamma < k\}$, such that for each $\Gamma^1 \in \mathcal{T}^{k-1}$, $\pi|_{\Gamma^1}$ has constant rank. Now let

$$\mathcal{X}^{k-1} = \{\Gamma \in \mathcal{X}^k : \dim \Gamma \geq k\} \bigcup \{\Gamma^1 \in \mathcal{T}^{k-1} : \dim \Gamma^1 < k\}.$$

It is easy to check that $\mathcal{X}^{k-1}$ has ($*_{k-1}$). Let $\mathcal{P}$ be a definable $C^p$ stratification of $T$ compatible with $\{\pi(\Gamma) : \Gamma \in \mathcal{X}^0\}$. Let $T' = T \setminus \bigcup \{\tau : \tau \in \mathcal{T}, \dim \tau < \dim T\}$. Then the restriction of $(\mathcal{X}^0, \mathcal{Y}^0)$ to $(X \times T', \mathbb{R} \times T')$ is a stratification satisfying the demands of the theorem. Since $\dim(T \setminus T') < \dim T$, the theorem is followed from the induction hypothesis. □

*Proof of the Corollary 1.1.* We make a compactification in a familiar way.
Let

$$\theta : \mathbb{R} \to (-1, 1),\ \ \theta(x) = \frac{x}{1 + |x|},$$

and

$$\theta_n : \mathbb{R}^n \to (-1, 1)^n,\ \ \theta_n(x_1, \cdots, x_n) = (\theta(x_1), \cdots, \theta(x_n)).$$

Let

$$\hat{f} : [-1, 1]^n \times [-1, 1] \times T \to \mathbb{R}, \quad \hat{f}(y, s, t) = s,$$

and

$$\pi : [-1, 1]^n \times [-1, 1] \times T \to T, \quad \pi(y, s, t) = t.$$

Then $(\hat{f}, \pi)$ is proper. Applying Theorem 1.1 to $\hat{f}$ and $\mathcal{A} = \{\theta \circ f \circ \theta_n^{-1} \times T\}$, we derive the corollary from Thom's second isotopy lemma [T], [M]. □

**Remark**

1. If the structure is polynomially bounded (see [DM] for the definition), Theorem 1.1 can be strengthened by replacing the condition $(\mathrm{a}_f)$ by the condition $(\mathrm{w}_f)$. In this case, the same proof goes through if we replace Lemma 2.2 by [L2, Proposition 2.7].

2. Since the proof of Corollary 1.1 is based on Thom's isotopy lemma, the homeomorphisms $h$ and $\lambda$, which are obtained by integrating vector fields, are not necessarily definable. In [C], based on the theory of the real spectrum, it is proved by triangulation that the homeomorphisms can be taken to be definable. Moreover, in semialgebraic or fewnomial case, [BS] and [C] give effective bounds

for the number of topological types in terms of the additive complexity and the number of variables.

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