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ON A FIXED POINT THEOREM OF D. W. BOYD AND J. S. WONG

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Abstract. We present a generalization of a well-known fixed point theorem due to D. W. Boyd and J. S. Wong (see [3]). We also provide some improvements to this theorem.

1. Introduction

One of the main generalizations of the well-known Banach principle is the following theorem established by D. W. Boyd and J. S. Wong in [3].

Theorem 1.1. *Let (M, d) be a complete metric space, and let $P := \{d(x, y) : x, y \in M\}$. Let $T : M \to M$ be a self-mapping satisfying*

$$d(Tx, Ty) \leq \gamma(d(x, y)) \quad \text{for all } x, y \in M, \tag{1.1}$$

where $\gamma : \bar{P} \mapsto [0, \infty[$ is upper semicontinuous from the right on $\bar{P}$ and satisfies $\gamma(t) < t$ for all $t \in \bar{P} \setminus \{0\}$ ($\bar{P}$ denotes the closure of P). Then T has a unique fixed point z and $d(T^n x, z)$ tends to zero for every $x \in M$ (T^n means the $n-th$ iterate of T).

We remark that the contractive condition (1.1) forces T to be continuous. The purpose of this note is to change this condition and introduce a more general contractive condition from which no information on the continuity of T could be derived. Denote by Φ the set of continuous functions $\phi : [0, \infty[\longrightarrow [0, \infty[$ satisfying the following conditions:
$(C1)$ $\phi(t) = 0$ if and only if $t = 0$, and
$(C2)$ For all sequence $\{t_n\}$ of elements in $[0, \infty[$, if $\{\phi(t_n)\}$ is decreasing then $\sup_n t_n < \infty$.

Observe that if $\phi : [0, \infty[\longrightarrow [0, \infty[$ is a continuous function satisfying one of the following properties then it must belong to the class Φ:
$(C3)$ ϕ is nondecreasing in $[0, \infty[$;
$(C4)$ $\phi(t) \geq Mt^u$ for every $t > 0$, where M and u are strictly positive constants.

We can now state the first main result of this paper.

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Theorem 1.2. *Let (M,d) be a complete metric space, and let $P := \{d(x,y) : x,y \in M\}$. Let $\phi \in \Phi$ and let $\gamma : \overline{\phi(P)} \to [0,\infty[$ be upper semicontinuous from the right on $\overline{\phi(P)}$ and satisfies $\gamma(t) < t$ for all $t \in \overline{\phi(P)} \setminus \{0\}$, where $\overline{\phi(P)}$ denotes the closure of $\phi(P)$. Let T be a self-mapping of M, satisfying the following contractive condition*

$$\phi\left(d(Tx,Ty)\right) \le \gamma\left(\phi\left(d(Tx,Ty)\right)\right) \quad \textit{for all } x,y \in M. \tag{1.2}$$

Then T has a unique fixed point z and $d(T^n x, z)$ tends to zero for every $x \in M$.

Theorem 1.2 will be proved in Section 2. Since no information on the continuity of T is given by the assumptions of Theorem 1.2, we must use arguments different from those utilized in [3]. Theorem 1.2 can be considered as a generalization of Therem 1.1 of Boyd and Wong. In Section 3, we propose some complements to Theorem 1.1, when γ satisfies some supplementary but natural conditions (see Theorem 3.1).

2. Proof of Theorem 1.2

(a) Let x_0 be some point in M. For every integer $n \ge 0$, we set $x_n := T^n x_0$ and put $t_n := d(x_n, x_{n+1})$. Then for every integer n, we have

$$\begin{aligned}\phi(t_{n+1}) &= \phi(d(Tx_n, Tx_{n+1})) \\ &\le \gamma(\phi(d(x_n, x_{n+1}))) \\ &\le \phi(d(x_n, x_{n+1})) = \phi(t_n).\end{aligned} \tag{2.1}$$

The inequalities in (2.1) show that the sequence $\{\phi(t_n)\}$ is decreasing. Let θ be the limit of $\{\phi(t_n)\}$. We observe that θ belongs to the closure of $\phi(P)$. Let us show that $\theta = 0$. Suppose on the contrary that $\theta > 0$. Then from the inequalities (2.1), for every integer n, we get $\theta \le \gamma(\phi(t_n))$. By letting $n \longrightarrow \infty$ and using the continuity of ϕ and the upper semicontinuity from the right of γ at the point θ, we obtain

$$0 < \theta \le \limsup_n \gamma(\phi(t_n)) \le \gamma(\theta). \tag{2.2}$$

It is clear that (2.2) contradicts the assumptions on γ. Thus $\theta = 0$. Since ϕ satisfies condition $(C2)$, the sequence $\{t_n\}$ is bounded. Let us show that $\{t_n\}$ converges to 0. Indeed, consider a convergent subsequence $\{t_m\}$ of $\{t_n\}$, say $\lim_m t_m = t$. By the continuity of ϕ, we get $\lim_m \phi(t_m) = \phi(t) = 0$, and then, in view of $(C1)$, we obtain $t = 0$. Since any convergent subsequence of the bounded sequence $\{t_n\}$ converges to 0, we conclude that the whole sequence $\{t_n\}$ converges to 0.

(b) Now, we shall prove that $\{x_n\}$ is a Cauchy sequence. To obtain a contradiction, suppose that we can find a number $\epsilon > 0$ and two sequences $\{p(n)\}$, $\{q(n)\}$ such that, for every integer n, we have

$$n \le p(n) < q(n), \quad d(x_{p(n)}, x_{q(n)}) > \varepsilon, \quad \text{and} \quad d(x_{p(n)}, x_{q(n)-1}) \le \varepsilon. \tag{2.3}$$

For each n, we set $s_n := d(x_{p(n)}, x_{q(n)})$, and $r_n := d(x_{p(n)+1}, x_{q(n)+1})$. By using the triangular inequalities, we obtain

$$\begin{aligned} &\varepsilon < s_n \leq \varepsilon + t_{q(n)-1}, \\ &|r_n - s_n| \leq t_{p(n)} + t_{q(n)}. \end{aligned} \tag{2.4}$$

Since the sequence $\{t_n\}$ converges to 0, we deduce from (2.4) that the sequences $\{s_n\}$ and $\{r_n\}$ converge to ε. Now, for every n, we have

$$\begin{aligned} \phi(r_n) &= \phi(d(x_{p(n)+1}, x_{q(n)+1})) \\ &= \phi(d(Tx_{p(n)}, Tx_{q(n)})) \\ &\leq \gamma(\phi(s_n)). \end{aligned} \tag{2.5}$$

We let $n \to \infty$ in (2.5) and use the properties of γ and ϕ to get

$$0 < \phi(\epsilon) \leq \limsup_n \gamma(\phi(s_n)) \leq \gamma(\phi(\epsilon)). \tag{2.6}$$

Since $\phi(\epsilon) \in \overline{\phi(P)} \setminus \{0\}$, (2.6) contains a contradiction. Therefore $\{x_n\}$ is a Cauchy sequence in the complete metric space (M, d). Let z be the limit of the sequence $\{x_n\}$. We proceed to prove that z is a fixed point for T.

(c) For every n, we have

$$\phi(d(x_{n+1}, Tz)) = \phi(d(Tx_n, Tz)) \leq \gamma(\phi(d(x_n, z))). \tag{2.7}$$

By letting $n \longrightarrow \infty$ in (2.7) we get

$$\begin{aligned} \phi(d(z, Tz)) &= \lim_n \phi(d(x_{n+1}, Tz)) \\ &\leq \limsup_n \gamma(\phi(d(x_n, z))) \\ &\leq \gamma(\phi(0)) = \gamma(0) = 0. \end{aligned} \tag{2.8}$$

We deduce that $\phi(d(z, Tz)) = 0$. Since ϕ satisfies $(C1)$, we obtain $z = Tz$.

(d) Suppose that there exists another fixed point $y \neq z$ of T. Using (1.2) we have

$$0 < \phi(d(y, z)) = \phi(d(Ty, Tz)) \leq \gamma(\phi(d(y, z))) < \phi(d(y, z)). \tag{2.9}$$

Since $\phi(d(y, z)) \in \overline{\phi(P)} \setminus \{0\}$, (2.9) contains a contradiction. Consequently, there exists a unique point $z \in M$ to which every Picard sequence converges. The proof is complete. □

3. Complements to the theorem of Boyd and Wong

If the function γ in Theorem 1.1 satisfies some natural additional conditions, then one could obtain some information about the diameters of level sets of the function $F : x \mapsto d(Tx, x)$, and a result of approximation characterizing the fixed point of T. This observation has been used in [8]. Before stating our result (see Theorem 3.1 below), we need to introduce the following notations and definitions.

As before (M,d) is a complete metric space and $P=\{d(x,y):x,y\in M\}$. For every subset B of M, the closure of B is denoted by $\overline{B}$. Let $T:M\to M$ be a self-mapping. For every $x\in M$, we set $F(x)=d(Tx,x)$. If the orbit of x is bounded, then we set $D(x):=\operatorname{diam}(O(x))$. For each $c>0$, let $L_c:=\{x\in M:F(x)\le c\}$.

We recall (see [2] and [8]) that a function $G:M\longrightarrow\mathbb{R}$ is said to be a regular-global-inf (r.g.i.) at $x\in M$ if $G(x)>\inf\limits_M(G)$ implies the existence of $\varepsilon>0$ such that $\varepsilon<G(x)-\inf\limits_M(G)$ and a neighborhood N_x of x such that $G(y)>G(x)-\varepsilon$ for every $y\in N_x$. If this condition holds for each $x\in M$, then G is said to be an r.g.i. on M.

Definition 3.1. We denote by Υ the set of functions $\gamma:\overline{P}\longrightarrow[0,\infty[$ such that $\gamma(t)\le t$ for all $t\in\overline{P}$, and there exists an associated positive function ψ defined on $[0,\infty[$ satisfying the following two properties:

$(S1)$ $\lim\limits_{t\in P,\,t\to 0}\psi(t)=0$, and

$(S2)$ $\forall t\in\overline{P},\ \forall s\ge 0,\ s-\gamma(s)\le t\Longrightarrow s\le\psi(t)$.

If γ is defined on $[0,\infty[$ and if the function $x\mapsto\mu(x):=x-\gamma(x)$ is continuous and strictly increasing from $[0,\infty[$ onto itself, then by taking ψ as the inverse mapping of μ, we see that $(S1)$ and $(S2)$ are satisfied.

Now we are ready to state our second main result.

Theorem 3.1. *Let (M,d) be a complete metric space and let $T:M\longrightarrow M$ be a self-mapping satisfying the condition*

$$d(Tx,Ty)\le\gamma(d(x,y))\quad\text{for all } x,y\in M, \tag{3.1}$$

where $\gamma\in\Upsilon$ is upper semicontinuous from the right on $\overline{P}$ and satisfies $\gamma(t)<t$ for all $t\in\overline{P}\setminus\{0\}$. Then T has bounded orbits and the following five equivalent assertions hold:

(i) *T has a unique fixed point $z\in M$, and $\lim\limits_{k\to+\infty}T^k(x)=z$ for each $x\in M$;*

(ii) *$\forall c>0$, the set L_c is nonempty and $\lim\limits_{c\to 0^+}\operatorname{diam}(L_c)=0$;*

(iii) *There exists a unique point $z\in M$, such that, for each sequence $\{x_n\}\subset M$, $\lim\limits_n d(x_n,Tx_n)=0$ if and only if $\{x_n\}$ converges to z;*

(iv) *There exists a unique point $z\in M$ such that, for each sequence $\{x_n\}\subset M$, $\lim\limits_n D(x_n)=0$ if and only if $\{x_n\}$ converges to z;*

(v) *The mapping $D:x\longmapsto\operatorname{diam}(O(x))$ is an r.g.i. on M.*

Proof. (a) Let us prove that T has bounded orbits. For every $x\in M$ and every positive integer n we set $O_n(x):=\{x,Tx,...,T^n(x)\}$. It is easy to verify that, for each $n\ge 1$,

$$\operatorname{diam}(O_n(Tx))\le\gamma(\operatorname{diam}(O_{n+1}(x))) \tag{3.2}$$

and there exists an integer $k_n\in\{1,2,\dots,n\}$ such that

$$\operatorname{diam}(O_n(x))=d(x,T^{k_n}(x)). \tag{3.3}$$

From (3.2) and (3.3) it follows that

$$\begin{aligned}\operatorname{diam}(O_n(x)) = d(x, T^{k_n}(x)) &\leq d(x, Tx) + d(Tx, T^{k_n}(x)) \\ &\leq d(x, Tx) + \operatorname{diam}(O_{n-1}(Tx)) \\ &\leq d(x, Tx) + \gamma(\operatorname{diam}(O_n(x))).\end{aligned}$$

By $(S2)$, we have $\operatorname{diam}(O_n(x)) \leq \psi\left(d(x, Tx)\right)$ for every integer $n \geq 1$. Since $O(x) = \bigcup_n O_n(x)$, we deduce that

$$\begin{aligned}F(x) \leq \operatorname{diam}(O(x)) &= \sup_n \operatorname{diam}(O_n(x)) \\ &\leq \psi\left(d(x, Tx)\right) \\ &< \infty. \qquad (3.4)\end{aligned}$$

Hence T has bounded orbits.

(b) Let us prove that (i) implies (ii). For every $c > 0$, the set L_c contains the fixed point z. For every $x \in M$, we have

$$\begin{aligned}d(x, Tx) \leq d(x, z) + d(Tz, Tx) &\leq d(x, z) + \gamma(d(Tz, Tx)) \\ &\leq 2d(x, z). \qquad (3.5)\end{aligned}$$

On the other hand, for every $x \in M$, we have

$$d(x, z) \leq d(x, Tx) + d(Tz, Tx) \leq d(x, Tx) + \gamma(d(z, x)). \qquad (3.6)$$

Using $(S2)$, we deduce from (3.6) that

$$d(x, z) \leq \psi(d(x, Tx)). \qquad (3.7)$$

Now, let $\varepsilon > 0$ and $\delta > 0$ be such that $s \in P$, and $s \leq \delta$ implies $\psi(s) \leq \dfrac{\varepsilon}{2}$. Let $c \in [0, \delta]$. Then, for all $x, y \in L_c$, we have

$$\begin{aligned}d(x, y) \leq d(x, z) + d(z, y) &\leq \psi(d(x, Tx)) + \psi(d(y, Ty)) \\ &\leq \varepsilon.\end{aligned}$$

So, we have proved that $\lim_{c \to 0^+} \operatorname{diam}(L_c) = 0$.

(c) Let us prove that (ii) implies (iii). Let $\{c_n\}$ be a strictly decreasing sequence of positive numbers converging to zero, and set $A := \bigcap_n \overline{L_{c_n}}$, (where $\overline{L_{c_n}}$ means the closure of L_{c_n}). An application of Cantor's intersection theorem implies the existence of a unique element $z \in A$. For every nonzero integer n, since $z \in \overline{L_{c_n}}$, we can find $y_n \in L_{c_n}$ such that $d(y_n, z) \leq \dfrac{1}{n}$. Therefore $\{y_n\}$ converges to z. Since T is continuous, we deduce that the sequence $\{Ty_n\}$ converges to Tz. Since, for each integer n, we have $0 \leq F(y_n) \leq c_n$, we deduce that $\lim_n F(y_n) = 0$, and therefore we get $Tz = z$. Hence z is the unique fixed point of T. Let $\{x_n\}$ be a sequence in M such that $\lim_n F(x_n) = 0$. According to (3.7) and Property $(S1)$, we get $\lim_n d(x_n, z) \leq \lim_n \psi(F(x_n)) = 0$. Conversely, let $\{x_n\}$ be a sequence in

M converging to the fixed point z. Since T is continuous, $\{Tx_n\}$ converges to $Tz = z$. Then we have $\lim_n F(x_n) = 0$.

(d) By the inequalities stated in (3.4), we see that (iii) and (iv) are equivalent. So, let us prove that (iv) implies (v). Note that the point z involved in the assertion (iv) must be a fixed point of T. By (3.1), this fixed point is unique. It follows that $\inf_M D = 0$. To prove that D is an r.g.i., we use Proposition 1.2, of [8]. Let $\{x_n\}$ be a sequence such that $\lim_n D(x_n) = 0$ and $\lim_n x_n = x$. Since $F(x) \leq D(x)$ for all $x \in M$, we have $\lim_n F(x_n) = 0$. By the continuity of T, we obtain $Tx = x$, and therefore $x = z$. Thus D is an r.g.i. on M.

(e) Let us prove that (v) implies (i). Let x_0 be some point in M. For every integer $n \geq 0$, we set $x_n := T^n x_0$ and put $t_n := d(x_n, x_{n+1})$. Then for every integer n, we have

$$\begin{aligned} t_{n+1} = d(Tx_n, Tx_{n+1}) &\leq \gamma(d(x_n, x_{n+1})) \\ &\leq d(x_n, x_{n+1}) \\ &= t_n. \end{aligned} \tag{3.8}$$

The inequalities in (3.8) show that the sequence $\{t_n\}$ is decreasing. Let t be its limit. We observe that t belongs to the closure of P. Let us show that $t = 0$. Indeed, from the inequalities (3.8), for every integer n, we get $t \leq \gamma(t_n)$. Letting $n \longrightarrow \infty$ and using the upper semicontinuity from the right at the point t, we obtain

$$t \leq \limsup_n \gamma(t_n) \leq \gamma(t). \tag{3.9}$$

In view of the assumptions made on γ, (3.9) shows that we must have $t = 0$. Now, from (3.4) and Property $(S1)$ we deduce that $\lim_n D(x_n) = 0$. This fact implies that $\{x_n\}$ is a Cauchy sequence. Let z be its limit in M. According to the assumption (v), we have $D(z) = \inf_M D = 0$. Therefore z is the unique fixed point of T to which every Picard sequence converges.

(f) Thus the five properties are equivalent. They are verified by applying the theorem of Boyd and Wong or our result stated in Theorem 1.2. □

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References

[1] M. Akkouchi *On a result of W. A. Kirk and L. M. Saliga,* J. Comput. Appl. Math. **142** (2) (2002), 445-448.

[2] M. Angrisani, M. Clavelli, *Synthetic approaches to problems of fixed points in metric space,* Ann. Mat. Pura Appl. (IV) **CLXX** (1996), 1-12.

[3] D. W. Boyd and J. S. Wong, *On nonlinear contractions,* Proc. Amer. Math. Soc. **20** (1969), 458-469.

[4] D. Delbosco, *Un estensione di un teorema sul punto fisso di S. Reich,* Rend. Sem. Mat. Univers. Politecn. Torino **35** (1976- 77), 233-239.
[5] M. Edelstein, *On fixed and periodic points under contractive mappings,* J. London Math. Soc. **37** (1962), 74-79.
[6] B. Fisher, *A fixed point mapping,* Bull. Calcutta Math. Soc. **68** (1976), 265-266.
[7] M. S. Khan, M. Swaleh and S. Sessa, *Fixed point theorems by altering distance between the points,* Bull. Austral. Math. Soc. **30** (1984), 1-9.
[8] W. A. Kirk and L. M. Saliga, *Some results on existence and approximation in metric fixed point theory,* J. Comput. Appl. Math. **113** (2000), 141-152.
[9] J. Matkowski, *Some inequalities and a generalization of Banach's principle,* Bull. Polish. Acad. Sci. Ser. Sci. Math. Astronom. Phys. **21** (1973), 323-325.
[10] J. Matkowski, *Integrable solutions of functional equations,* Dissertationes Math. **127** (1975).
[11] J. Matkowski and R. Wegrzyk, *On equivalence of some fixed point theorems for selfmappings of metrically convex space,* Boll. Un. Mat. Ital. (5) **15-A** (1978), 359-369.
[12] A. Meir and E. Keeler, *A theorem on contraction mappings,* J. Math. Anal. Appl. **28** (1969), 326-329.
[13] E. Rakotch, *A note on contractive mappings,* Proc. Amer. Math. Soc. **13** (1962), 459-465.
[14] S. Reich, *Kannan's fixed point theorem,* Boll. Un. Mat. Ital. (4) **4** (1971), 1-11.
[15] F. Skof, *Teorema di punti fisso per applicazioni negli spazi metrici,* Atti. Accad. Aci. Torino. **111** (1977), 323-329.
[16] W. Walter, *Remarks on a paper by F. Browder about contractions,* Nonlinear Anal. TMA **5** (1981), 21-25.

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