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EXTREME POINTS IN MUSIELAK-ORLICZ SEQUENCE SPACES

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Abstract. This paper establishes some characterizations of extreme points and strongly extreme points of the closed unit ball in a Musielak-Orlicz sequence space equipped with the Luxemburg norm. As a consequence of these results, we obtain some geometric properties such as rotundity and strong rotundity in Nakano sequence spaces and Orlicz sequence spaces.

1. Introduction

For a Banach space X, we denote by $S(X)$ and $B(X)$ the unit sphere and the closed unit ball of X, respectively. Recall that a point $x \in S(X)$ is an *extreme point* if $2x = y + z$ for $y, z \in B(X)$ implies $y = z$, and is a *strongly extreme point* if $2x = y_n + z_n$ for all $n \in \mathbb{N}$ and $\|y_n\| \to 1$, $\|z_n\| \to 1$ imply $\|y_n - z_n\| \to 0$. A Banach space X is said to be *rotund* if every point in its unit sphere is an extreme point. If every point in its unit sphere is a strongly extreme point, then X is said to be *strongly rotund*.

Clearly, every strongly extreme point is an extreme point. Thus every strongly rotund space is a rotund space. An example in [8] shows that there is a rotund Banach space which is not strongly rotund.

In this paper, we study extreme points and related properties in Musielak-Orlicz sequence spaces. Before stating our main result we first recall the following definitions:

Let $\mathbb{N}$ and $\mathbb{R}$ stand for the set of natural numbers and the set of real numbers, respectively. A function $\Phi : \mathbb{R} \to [0, \infty)$ is said to be an *Orlicz function* if Φ is even, convex, and vanishes at zero. A sequence $\Phi = (\Phi_k)$ of Orlicz functions Φ_k is called a *Musielak-Orlicz function*. If $\Phi = (\Phi_k)$ is a Musielak-Orlicz function, then the sequence $\Psi = (\Psi_k)$ defined by

$$\Psi_k(v) := \sup\{|v|u - \Phi_k(u) : u \geq 0\}, \quad k = 1, 2, \dots \tag{1.1}$$

is called the *complementary function* of Φ in the sense of Young (see [7]).

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Let $\mathbb{R}^{\mathbb{N}}$ denote the space of all real sequences $x = (x(k))$. For a given Musielak-Orlicz function Φ we define a *convex modular* $I_\Phi : \mathbb{R}^{\mathbb{N}} \to [0, \infty]$ by the formula

$$I_\Phi(x) = \sum_{k=1}^{\infty} \Phi_k(x(k)) \quad \text{for } x \in \mathbb{R}^{\mathbb{N}}. \tag{1.2}$$

The *Musielak-Orlicz sequence space* l_Φ generated by $\Phi = (\Phi_k)$ is defined by

$$l_\Phi := \{x \in \mathbb{R}^{\mathbb{N}} : I_\Phi(\lambda x) < \infty \ \text{ for some } \lambda > 0\}. \tag{1.3}$$

In particular, if $\Phi_k = M$ for every $k \in \mathbb{N}$, then l_M is called the *Orlicz sequence space* generated by M. We consider two norms on l_Φ: The *Luxemburg norm:*

$$\|x\| = \inf\{\lambda > 0 : I_\Phi(x/\lambda) \le 1\} \tag{1.4}$$

and the *Orlicz norm:*

$$\|x\|^o = \inf\left\{\frac{1}{\lambda}(1 + I_\Phi(\lambda x)) : \lambda > 0\right\}, \tag{1.5}$$

where $I_\Phi(\cdot)$ is defined by (1.2).

Let $l_\Phi := (l_\Phi, \|\cdot\|)$ and $l_\Phi^o := (l_\Phi, \|\cdot\|^o)$ denote the space l_Φ equipped with the Luxemburg norm and the Orlicz norm, respectively. It is known (see [7]) that both are Banach spaces. The subspace h_Φ of l_Φ defined by

$$h_\Phi := \{x \in l_\Phi : I_\Phi(\lambda x) < \infty \text{ for all } \lambda > 0\}. \tag{1.6}$$

is called *the space of finite elements*. Let

$$\theta(x) = \inf\{\lambda > 0 : I_\Phi(x/\lambda) < \infty\}. \tag{1.7}$$

It is clear that $x \in h_\Phi$ if and only if $\theta(x) = 0$. If Ψ is the complementary function (see (1.1)) of the Musielak-Orlicz function Φ, then by [7] the space $h_\Psi^o := (h_\Psi, \|\cdot\|^o)$ equipped with the Orlicz norm (1.5) is separable, and its dual is isometrically isomorphic to l_Φ.

We say that a Musielak-Orlicz function $\Phi = (\Phi_k)$ satisfies:

(1.8) *the δ_2-condition*, denoted $\Phi \in \delta_2$, if there exist constants $K \ge 2$, $u_0 > 0$ and a sequence (c_k) of positive numbers, with $\sum_{k=1}^{\infty} c_k < \infty$, such that for $\Phi_k(u) \le u_0$ we have

$$\Phi_k(2u) \le K\Phi_k(u) + c_k \quad \text{for every} \quad k \in \mathbb{N} \quad \text{and} \quad u \in \mathbb{R}.$$

(1.9) *the $(*)$-condition* (see [6]) if for any $\varepsilon \in (0, 1)$ there exists a $\delta > 0$ such that $\Phi_k((1 + \delta)u) \le 1$ whenever $\Phi_k(u) \le 1 - \varepsilon$ for all $k \in \mathbb{N}$ and $u \in \mathbb{R}$.

The following theorem is known (see [5]).

Theorem 1.1. $h_\Phi = l_\Phi$ *if and only if* $\Phi \in \delta_2$.

By [5] and [6] if a Musielak-Orlicz function $\Phi = (\Phi_k)$ satisfies (1.8), (1.9) and $\Phi_k(u) = 0$ if and only if $u = 0$ for every k, then

(1.10) For each $\varepsilon > 0$ and each $c > 0$ there exists a $\delta > 0$ such that

$$|I_\Phi(x+y) - I_\Phi(x)| < \varepsilon \text{ whenever } I_\Phi(x) \le c \text{ and } I_\Phi(y) < \delta.$$

(1.11) For any sequence $(x_n) \subset l_\Phi$, $\|x_n\| \to 1$ implies $I_\Phi(x_n) \to 1$, and
(1.12) $\|x\| = 1$ if and only if $I_\Phi(x) = 1$.

Our paper is organized as follows: In Section 2, we characterize extreme points in Musielak-Orlicz sequence spaces. Strongly extreme points in some subspaces of a Musielak-Orlicz sequence space are investigated in Section 3. Finally, in Section 4 we study geometric properties related to rotundity, strong rotundity and H-points.

2. Extreme points in Musielak-Orlicz sequence spaces

Let M be an Orlicz function. An interval $[a, b]$, $a < b$, is called an *affine interval* of M if

$$M(\lambda a + (1-\lambda)b) = \lambda M(a) + (1-\lambda)M(b) \quad \text{for all } \lambda \in [0,1]. \tag{2.1}$$

In addition, if M is neither affine on $[a-\varepsilon, b]$ nor on $[a, b+\varepsilon]$ for any $\varepsilon > 0$ we call $[a, b]$ a *structural affine interval* of M. Let $\{[a_i, b_i] : i \in I\}$ be the family of all the structural affine intervals of M. The set

$$S_M := \mathbb{R} \setminus \bigcup_{i \in I} (a_i, b_i) \tag{2.2}$$

is called the set of *strictly convex points* of M. Let

$$a_M = \sup\{u \ge 0 : M(u) = 0\}. \tag{2.3}$$

Theorem 2.1. *A point $x = (x(k)) \in S(l_\Phi)$ is an extreme point if and only if*

(i) $I_\Phi(x) = 1$ *and*
(ii) $\#\{k : |x(k)| \in [0, a_{\Phi_k})\} = 0$ *and* $\#\{k : x(k) \notin S_{\Phi_k}\} \le 1$, *where a_{Φ_k} and S_{Φ_k} are defined by* (2.3) *and* (2.2) *respectively, and $\#A$ denotes the cardinality of a set A.*

Proof. Necessity. Let $x = (x(k))$ be an extreme point of $S(l_\Phi)$. We will show that (i) and (ii) must hold. Suppose (i) does not hold, i.e. $I_\Phi(x) = r < 1$. Since Φ_1 is continuous we can choose $\varepsilon > 0$ so small that

$$\Phi_1(x(1) \pm \varepsilon) < \Phi_1(x(1)) + \frac{1-r}{2} \cdot$$

Define sequences $y = (y(k)), z = (z(k)) \in l_\Phi$ by $y(1) = x(1) + \varepsilon, z(1) = x(1) - \varepsilon$ and $y(k) = z(k) = x(k)$ for all $k \ge 2$. Obviously, $y \ne z$ and $2x = y + z$. Moreover,

$$I_\Phi(y) < I_\Phi(x) + \frac{1-r}{2} = \frac{1+r}{2} < 1.$$

Thus $\|y\| \le 1$. Similarly, we also have $\|z\| \le 1$. This contradiction shows that (i) must hold.

Suppose the first condition in (ii) does not hold, i.e. $j \in \{k : |x(k)| \in [0, a_{\Phi_k})\}$. Choose $\varepsilon \neq 0$ such that $x(j) \pm \varepsilon \in (-a_{\Phi_k}, a_{\Phi_k})$. Define $y = (y(k)) \in l_\Phi$ by $y(j) = x(j) + \varepsilon$, $y(k) = x(k)$ for all $k \neq j$ and $z = 2x - y$. It is easy to verify that $I_\Phi(y) = I_\Phi(z) = I_\Phi(x) = 1$. Since $y \neq z$, x can not be an extreme point.

Suppose the second condition in (ii) does not hold, i.e. $\#\{k : x(k) \notin S_{\Phi_k}\} \geq 2$. Without loss of generality we assume that $x(1) \notin S_{\Phi_1}$ and $x(2) \notin S_{\Phi_2}$. Then $x(1) \in (a_1, b_1)$ and $x(2) \in (a_2, b_2)$ for some structural affine intervals $[a_1, b_1]$ and $[a_2, b_2]$ of Φ_1 and Φ_2, respectively. Let $\Phi_1(u) = k_1 u + \beta_1$ $(u \in (a_1, b_1))$ and $\Phi_2(u) = k_2 u + \beta_2$ $(u \in (a_2, b_2))$ where $k_1 \neq 0$ and $k_2 \neq 0$. Choose $\varepsilon_1 \neq 0, \varepsilon_2 \neq 0$ such that

$$k_1 \varepsilon_1 = k_2 \varepsilon_2 \quad \text{and} \quad x(k) \pm \varepsilon_k \in (a_k, b_k) \quad \text{for } k = 1, 2.$$

Define $y = (y(k)) \in l_\Phi$ by $y(1) = x(1) + \varepsilon_1, y(2) = x(2) - \varepsilon_2, y(k) = x(k)$ for all $k \geq 3$, and $z = 2x - y$. Then we have $\Phi_1(y(1)) + \Phi_2(y(2)) = k_1 x(1) + \beta_1 + k_2 x(2) + \beta_2 = \Phi_1(x(1)) + \Phi_2(x(2))$. This implies $I_\Phi(y) \leq 1$, so $\|y\| \leq 1$. Similarly we have $\|z\| \leq 1$. This is a contradiction.

Sufficiency. If $2x = y + z$ for some $y, z \in B(l_\Phi)$ then, by (i) and the convexity of the modular $I_\Phi(\cdot)$,

$$1 = I_\Phi(x) \leq \frac{1}{2} I_\Phi(y) + \frac{1}{2} I_\Phi(z) \leq 1.$$

This implies $\Phi_k(x(k)) = \frac{1}{2}\Phi_k(y(k)) + \frac{1}{2}\Phi_k(z(k))$ for all $k \in \mathbb{N}$. By the first condition of (ii), there exists at most one $k \in \mathbb{N}$ such that $x(k) \notin S_{\Phi_k}$. If $x(k) \in S_{\Phi_k}$ then $x(k) = y(k) = z(k)$. Now suppose that there exists $j \in \mathbb{N}$ such that $x(j) \notin S_{\Phi_j}$. Then we have $x(k) = y(k) = z(k)$ for all $k \neq j$ and $x(j), y(j), z(j)$ belong to the same structural affine intervals of Φ_j. Since $\sum_{k=1}^{\infty} \Phi_k(y(k)) = 1 = \sum_{k=1}^{\infty} \Phi_k(z(k))$, we have $\Phi_j(y(j)) = \Phi_j(z(j)) = \Phi_j(x(j))$. If $y(j) \neq z(j)$, then $x(j) \in [-a_{\Phi_j}, a_{\Phi_j}]$. Since $a_{\Phi_j} \in S_{\Phi_j}$, $x(j) \in (-a_{\Phi_j}, a_{\Phi_j})$. This contradicts the second condition of (ii). Hence $y(j) = z(j)$. Therefore x is an extreme point. $\square$

Recall that a *Nakano sequence space* $l^{\{p_k\}}$ is a Musielak-Orlicz sequence space with $\Phi_k(u) = |u|^{p_k}$ for some sequence $\{p_k\}$ in $[1, \infty)$.

Corollary 2.1. ([4, Theorem 1]) *A point $x \in S(l^{\{p_k\}})$ is an extreme point if and only if $I_\Phi(x) = 1$ and $\#\{k : x(k) \neq 0$ and $p_k = 1\} \leq 1$.*

Corollary 2.2. ([1, Theorem 2.6]) *A point $x \in S(l_M)$ is an extreme point if and only if $I_M(x) = 1$, $\#\{k : x(k) \notin S_M\} \leq 1$ and $\#\{k : |x(k)| \in [0, a_M)\} = 0$.*

Observe that Corollary 2.1 was proved in [4] under the assumption that $\{p_k\}$ is bounded and Corollary 2.2 was proved in [4] under the assumption that the Orlicz function is an N-function. Our Corollaries 2.1 and 2.2 say that these assumptions can be removed.

3. Strongly extreme points in Musielak-Orlicz sequence spaces

In this section, we investigate strongly extreme points in the Musielak-Orlicz sequence space h_Φ.

Theorem 3.1. *If $x \in S(l_\Phi)$ is a strongly extreme point and $\theta(x) < 1$ (see (1.7)), then $\Phi \in \delta_2$.*

Suppose, in addition, that Φ satisfies the $(*)$-condition (see (1.9)) and each Φ_k vanishes only at zero. Then a point $x \in S(h_\Phi)$ is a strongly extreme point of $B(h_\Phi)$ if and only if it is an extreme point and $\Phi \in \delta_2$. In particular, if $h_\Phi = l_\Phi$, then a point $x \in S(l_\Phi)$ is a strongly extreme point if and only if it is an extreme point.

Proof. Suppose that $\Phi \notin \delta_2$, then by [5] there exists $x_0 = (x_0(k))$ such that

$$I_\Phi(x_0) \le 1 \quad \text{and} \quad I_\Phi(\lambda x_0) = \infty \text{ for any } \lambda > 1.$$

Since $\theta(x) < 1$, we have $I_\Phi(\lambda_0 x) < \infty$ for some $\lambda_0 > 1$. We define (y_n) and (z_n) by

$$y_n = (x(1), \dots, x(n), x(n+1) + \varepsilon_0 x_0(n+1), x(n+2) + \varepsilon_0 x_0(n+2), \dots),$$
$$z_n = (x(1), \dots, x(n), x(n+1) - \varepsilon_0 x_0(n+1), x(n+2) - \varepsilon_0 x_0(n+2), \dots),$$

where $\varepsilon_0 = 1 - 1/\lambda_0$. Clearly, $2x = y_n + z_n$ for all $n = 1, 2, \dots$. Moreover,

$$I_\Phi\left(\frac{y_n - z_n}{\varepsilon_0}\right) = \sum_{k=n+1}^{\infty} \Phi_k(2x_0(k)) = \infty.$$

It follows that $\|y_n - z_n\| > \varepsilon_0$ for all $n \in \mathbb{N}$. We will prove that $\|y_n\| \to 1$ and $\|z_n\| \to 1$. For $\varepsilon \in (0,1)$ let $\lambda = 1 + \varepsilon$. Observe that for each $n \in \mathbb{N}$ we have

$$\begin{aligned} I_\Phi\left(\frac{y_n}{\lambda}\right) &= \sum_{k=1}^{n} \Phi_k\left(\frac{x(k)}{\lambda}\right) + \sum_{k=n+1}^{\infty} \Phi_k\left(\frac{1}{\lambda\lambda_0}\lambda_0 x(k) + \frac{\varepsilon_0}{\lambda} x_0(k)\right) \\ &\le \sum_{k=1}^{n} \Phi_k\left(\frac{x(k)}{\lambda}\right) + \frac{1}{\lambda\lambda_0}\sum_{k=n+1}^{\infty} \Phi_k(\lambda_0 x(k)) + \frac{\varepsilon_0}{\lambda}\sum_{k=n+1}^{\infty} \Phi_k(x_0(k)). \end{aligned}$$

Note that $I_\Phi(x/\lambda) < 1$. Choose $N > 0$ so that for each $n \ge N$

$$\frac{1}{\lambda\lambda_0}\sum_{k=n+1}^{\infty} \Phi_k(\lambda_0 x(k)) < \frac{1 - I_\Phi(x/\lambda)}{2},$$
$$\frac{\varepsilon_0}{\lambda}\sum_{k=n+1}^{\infty} \Phi_k(x_0(k)) < \frac{1 - I_\Phi(x/\lambda)}{2}.$$

So $I_\Phi(y_n/\lambda) \le 1$ for all $n \ge N$. Then $\|y_n\| \le \lambda = 1 + \varepsilon$ for all $n \ge N$. Therefore $\limsup_{n\to\infty} \|y_n\| \le 1$. Similarly, $\limsup_{n\to\infty} \|z_n\| \le 1$. Hence $\liminf_{n\to\infty} \|y_n\| \ge 2 - \limsup_{n\to\infty} \|z_n\| \ge 1$ which yields $\|y_n\| \to 1$. Similarly, $\|z_n\| \to 1$. Hence, x can not be a strongly extreme point. This contradiction proves the first part of the theorem.

To prove the second part of the theorem observe that, since $\theta(x) = 0$ for every $x \in S(h_\Phi)$, the necessity of the theorem is trivial. To demonstrate the sufficiency of the theorem, assume that x is an extreme point and $\Phi \in \delta_2$. Let (x_n) and (y_n) be sequences in h_Φ such that $\|x_n\| \to 1$, $\|y_n\| \to 1$ and $2x = x_n + y_n$ for all $n \in \mathbb{N}$. By the Banach-Alaoglu Theorem, the unit ball of l_Φ is weakly star compact. Therefore, by passing to subsequences if necessary, we may assume that $x_n \overset{w*}{\to} x_0$, and $y_n \overset{w*}{\to} y_0$, for some $\|x_0\| \le 1$ and $\|y_0\| \le 1$. But since $x_n + y_n = 2x$ we have $x_0 + y_0 = 2x$, which implies $x_0 = y_0 = x$. Therefore

$$x_n(k) \to x(k) \text{ and } y_n(k) \to x(k) \quad \text{for each} \quad k = 1, 2, \dots. \tag{3.1}$$

Given $\varepsilon \in (0,1)$, by (1.10) we can find $\delta \in (0, \varepsilon)$ such that

$$|I_\Phi(x+y) - I_\Phi(x)| < \varepsilon \quad \text{whenever} \quad I_\Phi(x) \le 1 \quad \text{and} \quad I_\Phi(y) < \delta. \tag{3.2}$$

We choose m_0 so that $\sum_{k=m_0+1}^{\infty} \Phi_k(x(k)) < \delta/3$.

By (1.11) and (1.12), we have $I_\Phi(x_n) \to 1 = I_\Phi(x)$. Then $I_\Phi(x_n) < I_\Phi(x) + \delta/3$ for sufficiently large n. From (3.1) we have

$$\left| \sum_{k=1}^{m_0} (\Phi_k(x_n(k)) - \Phi_k(x(k))) \right| < \delta/3 \quad \text{for sufficiently large} \quad n. \tag{3.3}$$

Consequently, for n large enough, we have

$$\begin{aligned} \sum_{k=m_0+1}^{\infty} \Phi_k(x_n(k)) &= I_\Phi(x_n) - \sum_{k=1}^{m_0} \Phi_k(x_n(k)) \\ &< I_\Phi(x) + \delta/3 - \left(\sum_{k=1}^{m_0} \Phi_k(x(k)) - \delta/3 \right) \\ &= \sum_{k=m_0+1}^{\infty} \Phi_k(x(k)) + 2\delta/3 < \delta. \end{aligned}$$

Let

$$\begin{aligned} x' &= (0, \dots, 0, x(m_0+1), x(m_0+2), \dots), \\ x_n' &= (0, \dots, 0, x_n(m_0+1), x_n(m_0+2), \dots). \end{aligned}$$

Then we have $I_\Phi(x') < \delta$ and $I_\Phi(x_n') < \delta$ for all large n. Again, from (3.1) it follows that $\sum_{k=1}^{m_0} \Phi_k(x_n(k) - x(k)) < \varepsilon$ for sufficiently large n.

By (3.2) and (3.3), for all large n we have

$$\begin{aligned} I_\Phi(x_n - x) &= \sum_{k=1}^{m_0} \Phi_k(x_n(k) - x(k)) + I_\Phi(x_n' - x') \\ &< \varepsilon + I_\Phi(x_n') + \varepsilon < 3\varepsilon. \end{aligned}$$

This implies $I_\Phi(x_n - x) \to 0$, i.e. $x_n \to x$. Therefore $\|x_n - y_n\| \to 0$, so x is a strongly extreme point. The proof is complete. $\square$

Remark 3.1. (1) By [3], if $x \in l_M$ is a strongly extreme point then $\theta(x) = 0$.

(2) The assumption $\theta(x) < 1$ in Theorem 3.1 is essential as we can see in the following example.

Example 3.1. We consider a Nakano sequence space $l^{\{k^2\}}$. Observe that $\Phi_k(u) = |u|^{k^2}$. Let $x = (x(k))$, where $x(k) = (1/2)^{1/k}$. Clearly, $\Phi = (\Phi_k) \notin \delta_2$. We also have $I_\Phi(x) = 1$ and $I_\Phi(\lambda x) = \sum_{k=1}^{\infty} \frac{\lambda^{k^2}}{2^k} = \sum_{k=1}^{\infty} \big(\frac{\lambda^k}{2}\big)^k = \infty$ for any $\lambda > 1$, so $\theta(x) = 1$. By Corollary 2.1, x is an extreme point. Next, we prove that x is a strongly extreme point. Suppose $(x_n), (y_n) \subset l_\Phi$, $x_n + y_n = 2x$ for all $n \in \mathbb{N}$, $\|x_n\| \to 1$ and $\|y_n\| \to 1$. As in the proof of Theorem 3.1 we may assume that

$$x_n(k) \to x(k) \quad \text{and} \quad y_n(k) \to x(k) \quad \text{for each} \quad k = 1, 2, \ldots.$$

It suffices to prove that $\|x_n - x\| \to 0$. Given $\varepsilon > 0$, we choose integers K and N_1 so that

$$1/K < \varepsilon \text{ and } \|x_n\| < 1 + \varepsilon \quad \text{for all} \quad n > N_1. \tag{3.4}$$

This implies $\sum_{k=1}^{\infty} \big|\frac{x_n(k)}{1+\varepsilon}\big|^{k^2} < 1$ for all $n > N_1$. In particular,

$$|x_n(k)| < 1 + \varepsilon \quad \text{for all } n > N_1 \quad \text{and} \quad k = 1, 2, \ldots. \tag{3.5}$$

Again, choose $N_2 > N_1$ so that

$$|x_n(k) - x(k)| < \varepsilon \text{ and } |y_n(k) - x(k)| < \varepsilon \tag{3.6}$$

for all $n > N_2$, and $k = 1, \ldots, K$. Let $\Gamma_n = \{k \in \mathbb{N} : x_n(k) > 1 \text{ or } y_n(k) > 1\}$. We consider two cases.

Case 1. $k \in \Gamma_n$. If $x_n(k) > 1$, then $x_n(k) - 1 < \varepsilon$ for all $n > N_1$. Note that

$$1 - \Big(\frac{1}{2}\Big)^{1/k} \le \frac{1}{k} \quad \text{for all} \quad k \in \mathbb{N}.$$

This means for all $n > N_1$ and $k \in \Gamma_n$ we have

$$|x_n(k) - x(k)| \le |x_n(k) - 1| + |1 - x(k)| < \varepsilon + 1/k. \tag{3.7}$$

Similarly, if $y_n(k) > 1$ then

$$|y_n(k) - x(k)| < \varepsilon + 1/k \quad \text{for all} \quad n > N_1.$$

Case 2. $k \notin \Gamma_n$. In this case we have

$$|x_n(k) - x(k)| \leq 1/k \quad \text{and} \quad |y_n(k) - x(k)| \leq 1/k \quad \text{for all} \quad n \in \mathbb{N}. \tag{3.8}$$

If $n > N_2$ and $\lambda > 8\varepsilon$, then from (3.4)-(3.8) we obtain

$$\begin{aligned} I_\Phi\Big(\frac{x_n - x}{\lambda}\Big) &= \sum_{k=1}^{\infty} \Big(\frac{|x_n(k) - x(k)|}{\lambda}\Big)^{k^2} \\ &= \Bigg(\sum_{k=1}^{K} + \sum_{k \in \Gamma_n \setminus \{1,\dots,K\}} + \sum_{k \notin (\Gamma_n \cup \{1,\dots,K\})}\Bigg) \Big(\frac{|x_n(k) - x(k)|}{\lambda}\Big)^{k^2} \\ &< \sum_{k=1}^{K} \Big(\frac{\varepsilon}{\lambda}\Big)^{k^2} + \sum_{k=K+1}^{\infty} \Big(\frac{\varepsilon + 1/k}{\lambda}\Big)^{k^2} \\ &< \sum_{k=1}^{K} \Big(\frac{1}{8}\Big)^{k^2} + \sum_{k=K+1}^{\infty} \Big(\frac{1}{4}\Big)^{k^2} < 1. \end{aligned}$$

This means $\|x_n - x\| \leq \lambda$ for all $n > N_2$. Letting $\lambda \downarrow 8\varepsilon$ we get $\|x_n - x\| \leq 8\varepsilon$ for all $n > N_2$, i.e. $\|x_n - x\| \to 0$.

Let

$$h^{\{p_k\}} = \Big\{x = (x(k)) \in l^{\{p_k\}} : \sum_{k=1}^{\infty} |\lambda x(k)|^{p_k} < \infty \text{ for all } \lambda > 0\Big\}.$$

From Theorem 3.1 we get

Corollary 3.1. *A point $x \in S(h^{\{p_k\}})$ is a strongly extreme point if and only if it is an extreme point and the sequence $\{p_k\}$ is bounded.*

Proof. It is easy to verify that the δ_2-condition is equivalent to the boundedness of the sequence $\{p_k\}$ (see [5]). □

The following corollary follows immediately from Remark 3.1(1) and Theorem 3.1.

Corollary 3.2. ([1, Theorem 2.10] and [3, Corollary 1]) *Suppose that M vanishes only at zero. Then $x \in S(l_M)$ is a strongly extreme point if and only if x is an extreme point and $M \in \delta_2$.*

4. The rotundity and strong rotundity in Musielak-Orlicz sequence spaces

Theorem 4.1. *The Musielak-Orlicz sequence space l_Φ is rotund if and only if*

(i) $\Phi \in \delta_2$,
(ii) *each Φ_k vanishes only at zero, and*
(iii) *there exists at most one k such that $[0, \Phi_k^{-1}(\frac{1}{2})]$ contains an affine interval and if $[0, \Phi_{k_0}^{-1}(\frac{1}{2})]$ contains an affine interval $[a, b]$ for some k_0, then*

$[0, \Phi_k^{-1}(1 - \Phi_{k_0}(a))]$ *does not contain any affine interval for any* $k \neq k_0$, *i.e.,* $[0, \Phi_k^{-1}(1 - \Phi_{k_0}(a))] \subset S_{\Phi_k}$ *for every* $k \neq k_0$.

Proof. Necessity. If (i) does not hold, then we can construct an element $x = (x_k)$ such that $\|x\| = 1$ but $I_\Phi(x) < 1$. By Theorem 2.1 x is not an extreme point.

If (ii) does not hold, then we can construct an element $x \in S(l_\Phi)$ which is not an extreme point. If (iii) does not hold, then we can construct an element $x \in S(l_\Phi)$ such that $\#\{k : x(k) \notin S_{\Phi_k}\} \geq 2$.

Sufficiency. It suffices to prove that $\#\{k : x(k) \notin S_{\Phi_k}\} \leq 1$ for any $x \in S(l_\Phi)$. From (i) we have $I_\Phi(x) = 1$. Then $\Phi_k(x(k)) > \frac{1}{2}$ for at most one k. By (iii) we conclude that x is an extreme point. □

Remark 4.1. (1) In [5], condition (iii) in Theorem 4.1 is replaced by

(iii′) there exists a sequence $\{a_k\} \subset [0, \infty)$ such that $\Phi_n(a_n) + \Phi_m(a_m) \geq 1$ for all $n \neq m$ and Φ_k is strictly convex on $[0, a_k]$ for all $k \in \mathbb{N}$.

(2) By Theorem 1.1, l_Φ is rotund if and only if $l_\Phi = h_\Phi$ and h_Φ is rotund.

(3) Observe that for every $x \in S(h_\Phi)$ we have $I_\Phi(x) = 1$. Therefore h_Φ is rotund if and only if (ii) and (iii) are satisfied.

Corollary 4.1. *The Nakano sequence space* $l^{\{p_k\}}$ *is rotund if and only if* $\{p_k\}$ *is bounded and* $\#\{k : p_k = 1\} \leq 1$.

Corollary 4.2. ([1, Theorem 2.7]) *The Orlicz sequence space* l_M *is rotund if and only if* $M \in \delta_2$, M *vanishes only at zero and* M *is strictly convex on* $[0, M^{-1}(1/2)]$.

Corollary 4.3. *Suppose that* Φ *satisfies the* $(*)$*-condition (see* (1.9)*). Then the Musielak-Orlicz sequence space* l_Φ *is strongly rotund if and only if it is rotund.*

Proof. The necessity of the condition is obvious. We prove the sufficiency. Let $x \in S(l_\Phi)$. By Theorem 4.1 and the definition of rotundity, we have $\Phi \in \delta_2$ and x is an extreme point. By Theorem 3.1, x is a strongly extreme point. □

Corollary 4.4. ([1, Theorem 2.30] and [4, Theorem 21]) *The rotundity and strong rotundity are equivalent in Orlicz sequence spaces and in Nakano sequence spaces.*

A point $x \in S(X)$ is called an *H-point* if for any sequence $(x_n) \subset X$, $\|x_n\| \to 1$ and $x_n \xrightarrow{w} x$ we have $x_n \to x$.

Theorem 4.2. *Suppose that a Musielak-Orlicz function* Φ *satisfies the* $(*)$*-condition (see* (1.9)*) and each* Φ_k *vanishes only at zero, then* $x \in S(l_\Phi)$ *is an* H*-point if and only if* $\Phi \in \delta_2$.

Proof. Sufficiency. Suppose $\Phi \in \delta_2$. Let $(x_n) \subset l_\Phi$ such that $\|x_n\| \to 1$ and $x_n \xrightarrow{w} x$. Then $x_n \to x$ coordinatewise. From the proof of Theorem 3.1 we have $I_\Phi(x_n - x) \to 0$, which implies $\|x_n - x\| \to 0$.

Necessity. Suppose $x = (x(k))$ is an H-point, but $\Phi \notin \delta_2$. Then there exists an $x_0 = (x_0(k)) \in S(l_\Phi)$ such that $I_\Phi(x_0) \leq 1$ and $I_\Phi(\lambda x_0) = \infty$ for all $\lambda > 1$. Consequently, there is a sequence $i_1 < i_2 < \cdots$ such that

$$\|(0, 0, \ldots, x_0(i_n + 1), \ldots, x_0(i_{n+1}), 0, \ldots)\| \geq \frac{1}{2},$$

for all $n = 1, 2, \ldots$. Let

$$u_n = (x(1), \ldots, x(i_n), x(i_n + 1) - |x_0(i_n + 1)|(\mathrm{sgn} x(i_n + 1), \ldots, x(i_{n+1}) - |x_0(i_{n+1})|(\mathrm{sgn} x(i_n + 1), x(i_{n+1} + 1), \ldots).$$

It was proved in [2] that $u_n \xrightarrow{w} x_0$ and $\|u_n - x_0\| \geq \frac{1}{2}$. Moreover,

$$\|x_0\| \leq \liminf_{n\to\infty} \|u_n\| \leq \limsup_{n\to\infty} \|u_n\| \leq \|x_0\|.$$

So $\|u_n\| \to 1$. This contradicts to the definition of an H-point. □

Recall that a Banach space X is said to possess *property* (H) if every point in $S(X)$ is an H-point.

Corollary 4.5. ([2, Theorem 2]) *Suppose that a Musielak-Orlicz function Φ satisfies the $(*)$-condition (see* (1.9)*) and each Φ_k vanishes only at zero. Then the Musielak-Orlicz sequence space l_Φ possesses property (H) if and only if $\Phi \in \delta_2$.*

Corollary 4.6. ([4, Theorem 6]) *The Nakano sequence space $l^{\{p_k\}}$ possesses property (H) if and only if the sequence $\{p_k\}$ is bounded. In fact, $x \in S(l^{\{p_k\}})$ is an H-point if and only if the sequence $\{p_k\}$ is bounded.*

Corollary 4.7. ([1, Theorem 3.17, 3.18]) *Suppose that M vanishes only at zero. Then the Orlicz sequence space l_M possesses property (H) if and only if $M \in \delta_2$. Furthermore, if $M \notin \delta_2$ then $S(l_M)$ contains no H-points.*

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