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A REFINEMENT OF OSTROWSKI'S INEQUALITY FOR ABSOLUTELY CONTINUOUS FUNCTIONS AND APPLICATIONS

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Abstract. A refinement of the Ostrowski inequality obtained by Dragomir and Wang in [7] and applications for special means, quadrature formulae, cumulative distribution functions and Jeffreys divergence measure are given.

1. Introduction

In 1997, Dragomir and Wang proved the following Ostrowski type inequality [7].

Theorem 1. *Let $f : [a, b] \to \mathbb{R}$ be an absolutely continuous function on $[a, b]$. Then*

$$\left| f(x) - \frac{1}{b-a} \int_a^b f(t)\, dt \right| \le \left[\frac{1}{2} + \frac{\left| x - \frac{a+b}{2} \right|}{b-a} \right] \|f'\|_1 , \tag{1.1}$$

for all $x \in [a, b]$, where $\|\cdot\|_1$ is the Lebesgue norm on $L_1 [a, b]$, i.e.,

$$\|g\|_1 := \int_a^b |g(t)|\, dt.$$

The constant $\frac{1}{2}$ is the best possible.

Note that the fact that $\frac{1}{2}$ is the best constant was proved in [17] and (1.1) can also be obtained from a more general result given by Fink in [2] choosing $n = 1$ and doing some appropriate computation.

In [7], the authors applied (1.1) for special means and in Numerical Integration, obtaining bounds for the remainder in a Riemann type quadrature formula.

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In this paper, we point out a refinement of (1.1) and apply it for special means, in Numerical Integration, for cumulative density functions in Probability Theory and for Jeffreys divergence measure in Information Theory.

2. Integral inequalities

The following result, which is an improvement on the Dragomir-Wang inequality (1.1), holds.

Theorem 2. *Let $f:[a,b]\to\mathbb{K}$ $(\mathbb{K}=\mathbb{R},\mathbb{C})$ be an absolutely continuous function on $[a,b]$. Then*

$$\left|f(x)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \tag{2.1}$$
$$\leq \frac{x-a}{b-a}\|f'\|_{[a,x],1}+\frac{b-x}{b-a}\|f'\|_{[x,b],1}$$
$$\leq \begin{cases} \frac{1}{2}\left[\|f'\|_{[a,b],1}+\left|\|f'\|_{[a,x],1}-\|f'\|_{[x,b],1}\right|\right] \\ \left(\left|\|f'\|^p_{[a,x],1}\right|+\|f'\|^p_{[x,b],1}\right)^{\frac{1}{p}}\left[\left(\frac{x-a}{b-a}\right)^q+\left(\frac{b-x}{b-a}\right)^q\right]^{\frac{1}{q}} \\ \text{where } p>1 \text{ and } \frac{1}{p}+\frac{1}{q}=1, \\ \left[\frac{1}{2}+\frac{\left|x-\frac{a+b}{2}\right|}{b-a}\right]\|f'\|_{[a,b],1} \end{cases}$$

for all $x\in[a,b]$, where $\|\cdot\|_{[m,n],1}$ denotes the usual norm on $L_1[m,n]$, i.e.

$$\|g\|_{[m,n],1}:=\int_m^n |g(t)|\,dt<\infty.$$

Proof. Using the integration by parts formula for absolutely continuous functions on $[a,b]$, we have

$$\int_a^x (t-a)f'(t)\,dt=(x-a)f(x)-\int_a^x f(t)\,dt \tag{2.2}$$

and

$$\int_x^b (t-b)f'(t)\,dt=(b-x)f(x)-\int_x^b f(t)\,dt \tag{2.3}$$

for all $x \in [a,b]$.

Adding the two inequalities, we obtain the Montgomery identity for absolutely continuous functions (see, for example, [18, p. 565])

$$(b-a) f(x) - \int_a^b f(t)\,dt = \int_a^x (t-a) f'(t)\,dt + \int_x^b (t-b) f'(t)\,dt \tag{2.4}$$

for all $x \in [a,b]$.

Taking the modulus, we deduce that

$$\begin{aligned}
&\left|(b-a) f(x) - \int_a^b f(t)\,dt\right| \\
\leq\ & \left|\int_a^x (t-a) f'(t)\,dt\right| + \left|\int_x^b (t-b) f'(t)\,dt\right| \\
\leq\ & \int_a^x (t-a) \left|f'(t)\right| dt + \int_x^b (b-t) \left|f'(t)\right| dt \\
\leq\ & (x-a) \int_a^x \left|f'(t)\right| dt + (b-x) \int_x^b \left|f'(t)\right| dt \\
=\ & (x-a) \left\|f'\right\|_{[a,x],1} + (b-x) \left\|f'\right\|_{[x,b],1}
\end{aligned} \tag{2.5}$$

and the first inequality in (2.1) is proved.

Now, let us observe that

$$\begin{aligned}
& (x-a) \left\|f'\right\|_{[a,x],1} + (b-x) \left\|f'\right\|_{[x,b],1} \\
\leq\ & \max\left\{\left\|f'\right\|_{[a,x],1} + \left\|f'\right\|_{[x,b],1}\right\} (b-a) \\
=\ & \frac{1}{2}\left[\left\|f'\right\|_{[a,x],1} + \left\|f'\right\|_{[x,b],1} + \left|\left\|f'\right\|_{[a,x],1} - \left\|f'\right\|_{[x,b],1}\right|\right] (b-a) \\
=\ & \frac{1}{2}\left[\left\|f'\right\|_{[a,b],1} + \left|\left\|f'\right\|_{[a,x],1} - \left\|f'\right\|_{[x,b],1}\right|\right] (b-a)
\end{aligned}$$

and the first part of the second inequality is proved.

For the second inequality, we employ the elementary inequality for real numbers which can be derived from Hölder's discrete inequality

$$0 \leq ms + nt \leq (m^p + n^p)^{\frac{1}{p}} \times (s^q + t^q)^{\frac{1}{q}}, \tag{2.6}$$

provided that $m, s, n, t \geq 0$, $p > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$.

Using (2.6), we obtain

$$\begin{aligned} & (x-a)\left\|f'\right\|_{[a,x],1}+(b-x)\left\|f'\right\|_{[x,b],1} \\ \leq \;& \left(\left\|f'\right\|_{[a,x],1}^{p}+\left\|f'\right\|_{[x,b],1}^{p}\right)^{\frac{1}{p}}\left[(x-a)^{q}+(b-x)^{q}\right]^{\frac{1}{q}} \end{aligned}$$

and the second part of the second inequality in (2.1) is also obtained.

Finally, we observe that

$$\begin{aligned} & (x-a)\left\|f'\right\|_{[a,x],1}+(b-x)\left\|f'\right\|_{[x,b],1} \\ \leq \;& \max\{x-a,b-x\}\left[\left\|f'\right\|_{[a,x],1}+\left\|f'\right\|_{[x,b],1}\right] \\ = \;& \left[\frac{b-a}{2}+\left|x-\frac{a+b}{2}\right|\right]\left\|f'\right\|_{[a,b],1} \end{aligned}$$

and the last part of the second inequality in (2.1) is proved. □

The following corollary is also natural.

Corollary 1. *Under the above assumptions, we have*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{2}\left\|f'\right\|_{[a,b],1}. \tag{2.7}$$

Another interesting result is the following one.

Corollary 2. *Under the above assumptions, if there is an* $x_0 \in [a,b]$ *with*

$$\int_a^{x_0}\left|f'(t)\right|dt=\int_{x_0}^{b}\left|f'(t)\right|dt \tag{2.8}$$

then

$$\left|f(x_0)-\frac{1}{b-a}\int_a^b f(t)\,dt\right| \leq \frac{1}{2}\left\|f'\right\|_{[a,b],1}. \tag{2.9}$$

3. Applications for special means

Let us recall the following means for two positive numbers.

1. *The Arithmetic mean*

$$A=A(a,b):=\frac{a+b}{2}, \quad a,b\geq 0;$$

2. *The Geometric mean*

$$G=G(a,b):=\sqrt{ab}, \quad a,b\geq 0;$$

3. *The Harmonic mean*

$$H = H(a,b) := \frac{2}{\frac{1}{a}+\frac{1}{b}}, \quad a,b>0;$$

4. *The Logarithmic mean*

$$L = L(a,b) := \begin{cases} a & \text{if } \ a=b \\ \dfrac{b-a}{\ln b - \ln a} & \text{if } \ a \neq b; \end{cases}, \quad a,b>0,$$

5. *The Identric mean*

$$I = I(a,b) := \begin{cases} a & \text{if } \ a=b \\ \dfrac{1}{e}\left(\dfrac{b^b}{a^a}\right)^{\frac{1}{b-a}} & \text{if } \ a \neq b \end{cases}, \quad a,b>0;$$

6. *The p-Logarithmic mean*

$$L_p = L_p(a,b) := \begin{cases} a & \text{if } \ a=b \\ \left[\dfrac{b^{p+1}-a^{p+1}}{(p+1)(b-a)}\right]^{\frac{1}{p}} & \text{if } \ a \neq b \end{cases}, \quad a,b>0,$$

where $p \in \mathbb{R} \setminus \{-1,0\}$.

Denoting $L_0 := I$ and $L_{-1} := L$, then it is well known that L_p is monotonic increasing over $p \in \mathbb{R}$ and the following particular inequalities hold

$$H \le G \le L \le I \le A. \tag{3.1}$$

1. Consider the function $f : [a,b] \to \mathbb{R}$ $(0<a<b<\infty)$, $f(x) = x^p$, $p \in \mathbb{R} \setminus \{-1,0\}$. Then

$$\begin{aligned} \frac{1}{b-a}\int_a^b f(t)\,dt &= L_p^p(a,b) = L_p^p, \\ \|f'\|_{[a,b],1} &= (b-a)\,|p|\,L_{p-1}^{p-1}(a,b). \end{aligned}$$

Applying Theorem 2 for the function $f(x) = x^p$, we get

$$
\begin{aligned}
(3.2)\quad 0 &\leq \left|x^p - L_p^p(a,b)\right| \\
&\leq \frac{|p|}{b-a}\left[(x-a)^2 L_{p-1}^{p-1}(a,x) + (b-x)^2 L_{p-1}^{p-1}(x,b)\right] \\
&\leq \begin{cases}
\frac{|p|}{2}\Big[(b-a) L_{p-1}^{p-1}(a,b) \\
\qquad + \left|(x-a) L_{p-1}^{p-1}(a,x) - (b-x) L_{p-1}^{p-1}(x,b)\right|\Big]; \\
\\
|p|\left[(x-a)^s L_{p-1}^{s(p-1)}(a,x) + (b-x)^s L_{p-1}^{s(p-1)}(x,b)\right]^{\frac{1}{s}} \\
\times\left[\left(\frac{x-a}{b-a}\right)^q + \left(\frac{b-x}{b-a}\right)^q\right]^{\frac{1}{q}}, \quad \text{where } s>1,\ \frac{1}{s}+\frac{1}{q}=1; \\
\\
\left[\frac{1}{2} + \frac{|x-A(a,b)|}{b-a}\right](b-a) L_{p-1}^{p-1}(a,b)
\end{cases}
\end{aligned}
$$

for all $x \in [a,b]$, which improves the inequality (3.1) from [7].

2. Consider the function $f : [a,b] \to \mathbb{R}$ $(0 < a < b < \infty)$, $f(x) = \frac{1}{x}$. Then

$$
\begin{aligned}
\frac{1}{b-a}\int_a^b f(t)\,dt &= L^{-1}(a,b), \\
\left\|f'\right\|_{[a,b],1} &= \frac{b-a}{G^2(a,b)}.
\end{aligned}
$$

Applying Theorem 2 for the function $f(x) = \frac{1}{x}$, we get

$$
\begin{aligned}
(3.3)\quad & |x-L| \\
&\leq \frac{1}{(b-a)}\left[\frac{(x-a)^2}{G^2(a,x)} + \frac{(b-x)^2}{G^2(x,b)}\right]\cdot xL \\
&\leq \begin{cases}
\frac{1}{2}\left[\frac{b-a}{G^2(a,b)} + \left|\frac{x-a}{G^2(a,x)} - \frac{b-x}{G^2(x,b)}\right|\right]\cdot xL; \\
\\
\left[\frac{(x-a)^s}{G^{2s}(a,x)} + \frac{(b-x)^s}{G^{2s}(x,b)}\right]^{\frac{1}{s}}\left[\left(\frac{x-a}{b-a}\right)^q + \left(\frac{b-x}{b-a}\right)^q\right]^{\frac{1}{q}}\cdot xL, \\
\text{where } s>1,\ \frac{1}{s}+\frac{1}{q}=1; \\
\\
\left[\frac{1}{2} + \frac{|x-A(a,b)|}{b-a}\right]\frac{b-a}{G^2(a,b)}\cdot xL
\end{cases}
\end{aligned}
$$

for all $x \in [a,b]$, improving the similar result in [7].

3. Consider the function $f:[a,b]\to\mathbb{R}$ $(0<a<b<\infty)$, $f(x)=\ln x$. Then

$$\frac{1}{b-a}\int_a^b f(t)\,dt = \ln I(a,b),$$
$$\left\|f'\right\|_{[a,b],1} = \frac{b-a}{L(a,b)}.$$

Applying Theorem 2 for the function $f(x)=\ln x$, we get

$$\left|\ln\left(\frac{x}{I}\right)\right| \leq \frac{1}{(b-a)}\left[\frac{(x-a)^2}{L(a,x)}+\frac{(b-x)^2}{L(x,b)}\right] \tag{3.4}$$
$$\leq \begin{cases} \dfrac{1}{2}\left[\dfrac{b-a}{L(a,b)}+\left|\dfrac{x-a}{L(a,x)}-\dfrac{b-x}{L(x,b)}\right|\right]; \\ \left[\dfrac{(x-a)^s}{L^s(a,x)}+\dfrac{(b-x)^s}{L^s(x,b)}\right]^{\frac{1}{s}}\left[\left(\dfrac{x-a}{b-a}\right)^q+\left(\dfrac{b-x}{b-a}\right)^q\right]^{\frac{1}{q}}, \\ \text{if } s>1,\ \dfrac{1}{s}+\dfrac{1}{q}=1; \\ \left[\dfrac{1}{2}+\dfrac{|x-A(a,b)|}{b-a}\right]\dfrac{b-a}{L(a,b)} \end{cases}$$

for all $x\in[a,b]$, improving the corresponding result from [7].

4. Error estimate in the Riemann quadrature formula

Let $I_n: a=x_0<x_1<...<x_{n-1}<x_n=b$ be a partitioning of the interval $[a,b]$ and define $h_i:=x_{i+1}-x_i$, $\nu(h)=\max\{h_i|i=0,...,n-1\}$. Consider the following quadrature of the Riemann type [7]:

$$A_R(f,I_n,\xi):=\sum_{i=0}^{n-1}h_i f(\xi_i), \tag{4.1}$$

where $\xi=(\xi_0,...,\xi_{n-1})$ and $\xi_i\in[x_i,x_{i+1}]$ $(i=0,...,n-1)$ are intermediate (arbitrarily chosen) points.

The following theorem improves the corresponding result in [7].

Theorem 3. *Let $f:[a,b]\to\mathbb{R}$ be an absolutely continuous function on $[a,b]$. Then*

$$\int_a^b f(t)\,dt = A_R(f,I_n,\xi)+R_R(f,I_n,\xi), \tag{4.2}$$

where $A_R(f, I_n, \xi)$ is the Riemann quadrature given by (4.1) and the remainder $R_R(f, I_n, \xi)$ in (4.2) satisfies the bound

$$|R_R(f, I_n, \xi)| \leq \sum_{i=0}^{n-1} (\xi_i - x_i) \|f'\|_{[x_i,\xi_i],1} + \sum_{i=0}^{n-1} (x_{i+1} - \xi_i) \|f'\|_{[\xi_i,x_{i+1}],1} . \tag{4.3}$$

Proof. We apply the first inequality in (2.2) on the interval $[x_i, x_{i+1}]$ to obtain

$$\left| h_i f(\xi_i) - \int_{x_i}^{x_{i+1}} f(t)\, dt \right| \leq (\xi_i - x_i) \|f'\|_{[x_i,\xi_i],1} + (x_{i+1} - \xi_i) \|f'\|_{[\xi_i,x_{i+1}],1} \tag{4.4}$$

for all $i \in \{0, ..., n-1\}$.

Summing over i from 0 to $n-1$ and using the generalised triangle inequality, we get the desired estimate (4.3). □

Corollary 3. *With the assumptions of Theorem 3, we have the midpoint quadrature formula*

$$\int_a^b f(t)\, dt = A_M(f, I_n) + R_M(f, I_n), \tag{4.5}$$

where $A_M(f, I_n)$ is the midpoint formula, i.e.

$$A_M(f, I_n) := \sum_{i=0}^{n-1} f\left(\frac{x_i + x_{i+1}}{2}\right) h_i$$

and the remainder $R_M(f, I_n)$ satisfies the estimate

$$|R_M(f, I_n)| \leq \frac{1}{2} \sum_{i=0}^{n-1} h_i \|f'\|_{[x_i,x_{i+1}],1} \leq \frac{1}{2} \nu(h) \|f'\|_{[a,b],1} . \tag{4.6}$$

Remark 1. Similar bounds for the value $R_R(f, I_n, \xi)$ can be stated if we use other inequalities in the second part of (2.1), but we omit the details.

5. Applications for cumulative density function

Let X be a random variable taking values in the finite interval $[a, b]$, with the cumulative distribution function $F(x) = \Pr(X \leq x)$ and the probability function $f : [a, b] \to \mathbb{R}_+$.

Theorem 4. *Assume that $f \in L_1[a,b]$. Then we have*

$$\left|F(x) - \frac{b - E(X)}{b-a}\right| \tag{5.1}$$
$$\leq \frac{x-a}{b-a}F(x) + \frac{b-x}{b-a}R(x)$$
$$\leq \begin{cases} \frac{1}{2}\left[1 + \left|F(x) - R(x)\right|\right]; \\ \left([F(x)]^p + [R(x)]^p\right)^{\frac{1}{p}} \left[\left(\frac{x-a}{b-a}\right)^q + \left(\frac{b-x}{b-a}\right)^q\right]^{\frac{1}{q}} \\ \text{where } p > 1, \quad \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} + \frac{\left|x - \frac{a+b}{2}\right|}{b-a} \end{cases}$$

for all $x \in [a,b]$, where

$$R(x) = 1 - F(x), \; x \in [a,b].$$

The proof follows by Theorem 2 applied for the cumulative function F and taking into account that

$$\int_a^b F(t)\,dt = F(t)\,t\Big|_a^b - \int_a^b tf(t)\,dt = b - E(X)$$

and

$$F'(t) = f(t), \quad t \in (a,b).$$

We now give an example for a Beta Random Variable.

We recall that a Beta Random variable with parameters (p,q) has the probability density function

$$f(t;p,q) = \frac{t^{p-1}(1-t)^{q-1}}{B(p,q)}, \quad 0 < t < 1,$$

where

$$B(p,q) := \int_0^1 t^{p-1}(1-t)^{q-1}\,dt$$

is the Euler Beta function.

Using Theorem 4 and the fact that for a Beta random variable

$$E(X) = \frac{p}{p+q},$$

we can state the following proposition.

Proposition 1. *Let X be a Beta random variable with the parameters (p,q), $p,q \geq 1$. Then we have*

$$\left|\Pr(X \leq x) - \frac{q}{p+q}\right| \leq x\Pr(X \leq x) + (1-x)\Pr(X \geq x) \leq \begin{cases} \frac{1}{2}\left[1 + \left|\Pr(X \leq x) - \Pr(X \geq x)\right|\right]; \\ \left([\Pr(X \leq x)]^p + [\Pr(X \geq x)]^p\right)^{\frac{1}{p}} \left[x^q + (1-x)^q\right]^{\frac{1}{q}} \\ \text{where } p > 1,\ \frac{1}{p} + \frac{1}{q} = 1; \\ \frac{1}{2} + \left|x - \frac{1}{2}\right| \end{cases} \tag{5.2}$$

for all $x \in [a,b]$.

6. Applications for Jeffreys distance in information theory

Assume that a set χ and the $\sigma-$finite measure μ are given. Consider the set of all probability densities on μ to be $\Omega := \left\{p | p : \chi \to \mathbb{R},\, p(x) \geq 0,\, \int_\chi p(x)\, d\mu(x) = 1\right\}$. The Kullback-Leibler divergence [19] is well known among the information divergences. It is defined as:

$$D_{KL}(p,q) := \int_\chi p(x) \log\left[\frac{p(x)}{q(x)}\right] d\mu(x), \quad p,q \in \Omega, \tag{6.1}$$

where log is to base 2.

In Information Theory and Statistics, various divergences are applied in addition to the Kullback-Leibler divergence. These are the: *variation distance* D_v, *Hellinger distance* D_H [20], χ^2-*divergence* D_{χ^2}, $\alpha-$*divergence* D_α, *Bhattacharyya distance* D_B [21], *Harmonic distance* D_{Ha}, *Jeffreys distance* D_J [22], *triangular discrimination* D_Δ [23], etc. They are defined as follows:

$$D_v(p,q) := \int_\chi |p(x) - q(x)|\, d\mu(x), \quad p,q \in \Omega; \tag{6.2}$$

$$D_H(p,q) := \int_\chi \left[\sqrt{p(x)} - \sqrt{q(x)}\right]^2 d\mu(x), \quad p,q \in \Omega; \tag{6.3}$$

$$D_{\chi^2}(p,q) := \int_\chi p(x)\left[\left(\frac{q(x)}{p(x)}\right)^2 - 1\right] d\mu(x), \quad p,q \in \Omega; \tag{6.4}$$

$$D_{\alpha}(p,q) := \frac{4}{1-\alpha^2}\left[1-\int_{\chi}[p(x)]^{\frac{1-\alpha}{2}}[q(x)]^{\frac{1+\alpha}{2}}d\mu(x)\right], \quad p,q\in\Omega; \tag{6.5}$$

$$D_B(p,q) := \int_{\chi}\sqrt{p(x)q(x)}d\mu(x), \quad p,q\in\Omega; \tag{6.6}$$

$$D_{Ha}(p,q) := \int_{\chi}\frac{2p(x)q(x)}{p(x)+q(x)}d\mu(x), \quad p,q\in\Omega; \tag{6.7}$$

$$D_J(p,q) := \int_{\chi}[p(x)-q(x)]\ln\left[\frac{p(x)}{q(x)}\right]d\mu(x), \quad p,q\in\Omega; \tag{6.8}$$

$$D_{\Delta}(p,q) := \int_{\chi}\frac{[p(x)-q(x)]^2}{p(x)+q(x)}d\mu(x), \quad p,q\in\Omega. \tag{6.9}$$

For other divergence measures, see the paper [24] by Kapur or the book on line [25] by Taneja. For a comprehensive collection of preprints available on line, see the RGMIA web site `http://rgmia.vu.edu.au/papersinfth.html`

The following inequalities involving the Jeffreys divergence are known (see for example the book on line by Taneja [25])

$$D_{Ha}(p,q) \geq \exp\left[-\frac{1}{2}D_J(p,q)\right], \quad p,q\in\Omega, \tag{6.10}$$

$$D_{Ha}(p,q) \geq 1-\frac{1}{4}D_J(p,q), \quad p,q\in\Omega \tag{6.11}$$

and

$$D_J(p,q) \geq 4[1-D_B(p,q)], \quad p,q\in\Omega, \tag{6.12}$$

where $D_{Ha}(\cdot,\cdot)$ is the Harmonic distance and $D_B(\cdot,\cdot)$ is the Bhattacharyya distance.

The following result holds (see also [26]).

Theorem 5. *We have*

$$2D_{\Delta}(p,q) \leq D_J(p,q) \leq \frac{1}{2}\left[D_{\chi^2}(p,q)+D_{\chi^2}(q,p)\right], \quad p,q\in\Omega, \tag{6.13}$$

where D_{χ^2} is the chi-square distance and D_{Δ} is the triangular discrimination.

Proof. We use the celebrated Hermite-Hadamard inequality for convex functions

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(t)\,dt \leq \frac{f(a)+f(b)}{2} \tag{6.14}$$

and choose $f(t) = \frac{1}{t}$ to get

$$\frac{2}{a+b} \le \frac{\ln b - \ln a}{b-a} \le \frac{a+b}{2ab},$$

which is equivalent to

$$\frac{2(b-a)^2}{a+b} \le (b-a)(\ln b - \ln a) \le \frac{a+b}{2ab}(b-a)^2. \tag{6.15}$$

If in (6.15) we choose $b = q(x)$, $a = p(x)$, $x \in \chi$, then we obtain

$$\begin{aligned} \frac{2(q(x)-p(x))^2}{p(x)+q(x)} &\le (q(x)-p(x))(\ln q(x) - \ln p(x)) \\ &\le \frac{p(x)+q(x)}{2p(x)q(x)}(q(x)-p(x))^2 \end{aligned}$$

and integrating over x on χ we deduce that

$$\begin{aligned} 2D_\Delta(p,q) &\le D_J(p,q) \\ &\le \frac{1}{2}\left[\int_\chi \frac{(q(x)-p(x))^2}{q(x)} d\mu(x) + \int_\chi \frac{(q(x)-p(x))^2}{p(x)} d\mu(x)\right] \\ &= \frac{1}{2}\left[\int_\chi \frac{q^2(x) - 2p(x)q(x) + p^2(x)}{q(x)} d\mu(x)\right. \\ &\quad \left. + \int_\chi \frac{q^2(x) - 2p(x)q(x) + p^2(x)}{p(x)} d\mu(x)\right] \\ &= \frac{1}{2}\left[\int_\chi \frac{p^2(x)}{q(x)} d\mu(x) - 1 + \int_\chi \frac{q^2(x)}{p(x)} d\mu(x) - 1\right] \\ &= \frac{1}{2}\left[D_{\chi^2}(q,p) + D_{\chi^2}(p,q)\right]. \end{aligned}$$

The inequality (6.13) is proved. □

The following results are also known (see [26]).

Theorem 6. *For all $p, q \in \Omega$, we have*

$$0 \le D_J(p,q) - 2D_\Delta(p,q) \le \frac{1}{6} D_*(p,q), \tag{6.16}$$

where

$$D_*(p,q) := \int_\chi \frac{(p(x)-q(x))^4}{\sqrt{p^3(x)q^3(x)}} d\mu(x).$$

Theorem 7. *For each $p, q \in \Omega$, we have*

$$0 \leq \frac{1}{2}\left[D_{\chi^2}(p,q) + D_{\chi^2}(q,p)\right] - D_J(p,q) \leq \frac{1}{6} D_*(p,q). \tag{6.17}$$

Now, using the inequality (3.3), we can write

$$\left|\frac{1}{x} - \frac{\ln b - \ln a}{b-a}\right| \leq \frac{1}{b-a}\left[\frac{(x-a)^2}{ax} + \frac{(b-x)^2}{bx}\right], \tag{6.18}$$

for all $x \in [a,b] \subset (0,\infty)$.

If in this inequality we put $x = \frac{a+b}{2}$, then we get

$$0 \leq \frac{\ln b - \ln a}{b-a} - \frac{2}{a+b} \leq \frac{b-a}{2ab}. \tag{6.19}$$

The following theorem complements Theorem 6.

Theorem 8. *For all $p, q \in \Omega$, we have*

$$0 \leq D_J(p,q) - 2D_{\Delta}(p,q) \leq \frac{1}{2} D_{@}(p,q), \tag{6.20}$$

where $D_{@}(p,q)$ is given by:

$$D_{@}(p,q) = \int_{\chi} \frac{|q(x) - p(x)|^3}{p(x)\, q(x)} d\mu(x),$$

provided that all the integrals exist.

Proof. If we multiply (6.19) by $(b-a)^2 \geq 0$, then we get

$$0 \leq (b-a)(\ln b - \ln a) - \frac{2(b-a)^2}{a+b} \leq \frac{1}{2} \cdot \frac{|b-a|^3}{ab} \tag{6.21}$$

for all $a, b \in (0,\infty)$.

If in (6.21) we choose $b = q(x)$, $a = p(x)$, $x \in \chi$, we obtain

$$\begin{aligned} 0 &\leq (q(x) - p(x))[\ln q(x) - \ln p(x)] - 2 \cdot \frac{(q(x) - p(x))^2}{p(x) + q(x)} \\ &\leq \frac{1}{2} \cdot \frac{|q(x) - p(x)|^3}{p(x)\, q(x)}. \end{aligned} \tag{6.22}$$

Integrating (6.22) on χ, we deduce (6.20). □

Remark 2. It is still not clear which bound from (6.16) and (6.20) is better.

Now, if in (6.18) we put $x = \sqrt{ab}$, then we obtain

$$0 \leq \frac{1}{\sqrt{ab}} - \frac{\ln b - \ln a}{b-a} \leq \frac{2\left(\sqrt{b} - \sqrt{a}\right)^2}{(b-a)\sqrt{ab}}, \tag{6.23}$$

for $0 < a < b < \infty$.

Using (6.23), we can state the following theorem.

Theorem 9. *For all $p, q \in \Omega$, we have*

$$0 \leq \int_{\chi} \frac{(q(x)-p(x))^2}{\sqrt{p(x)q(x)}} d\mu(x) - D_J(p,q) \leq 2\int_{\chi} \frac{|q(x)-p(x)|\left(\sqrt{q(x)}-\sqrt{p(x)}\right)^2}{\sqrt{p(x)q(x)}} d\mu(x), \tag{6.24}$$

provided that all the integrals exist.

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