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ARCWISE CONNECTEDNESS OF THE SOLUTION SETS OF A SEMISTRICTLY QUASICONCAVE VECTOR MAXIMIZATION PROBLEM

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Abstract. This paper presents some new facts on arcwise connectedness and contractibility of the solution sets in semistrictly quasiconcave vector maximization problems, where at least one of the objective functions is strictly quasiconcave.

1. Introduction

Topological properties of the solution sets of vector optimization (VOP) problems have been investigated intensively (see [1]–[18], [20]–[29], and references therein). The following four fundamental properties are of frequent consideration: compactness, contractibility, arcwise connectedness, and connectedness. Compactness of the weakly efficient solution set of a convex VOP problem has been characterized in [9]. Contractibility of the solution sets in convex VOP was studied in [23], [15], [18] and [2]. Arcwise connectedness of the solution sets in quasiconcave VOP has been addressed in [5]–[7] and [20]. Connectedness of the solution sets in several basic classes of problems such as convex VOP problems, quasiconcave VOP problems, linear fractional VOP problems, strongly convex VOP problems, etc., has been studied by several different methods.

The aim of this paper is to present some new facts on arcwise connectedness and contractibility of the solution sets in semistrictly quasiconcave vector maximization problems, where at least one of the objective functions is strictly quasiconcave.

Some preliminaries will be given in Section 2. The arcwise connectedness of the solution sets is studied in Section 3. In the Section 4 we discuss the contractibility of the solution sets of a bicriteria semistrictly quasiconcave maximization problem.

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2. Preliminaries

Let $\mathbb{R}^m$ be the m-dimensional Euclidean space which is partially ordered by the cone $\mathbb{R}^m_+ = \{u = (u_1, u_2, \dots, u_m) : \ u_i \geq 0 \ \text{ for all } i = 1, 2, \dots, m\}$. For any $u^i = (u^i_1, u^i_2, \dots, u^i_m) \in \mathbb{R}^m$ $(i = 1, 2)$, we write $u^1 \leq u^2$ (resp., $u^1 < u^2$) if $u^2 - u^1 \in \mathbb{R}^m_+$ (resp., $u^2 - u^1 \in \mathbb{R}^m_+ \setminus \{0\}$). If $u^2 - u^1$ belongs to the interior of $\mathbb{R}^m_+$, then we write $u^1 \ll u^2$.

Consider the following VOP problem

$$(P) \qquad \begin{cases} \text{Maximize} & F(x) = (f_1(x), f_2(x), \dots, f_m(x)) \\ \text{subject to} & x \in X, \end{cases}$$

where the *feasible region* $X \subset \mathbb{R}^n$ is nonempty, compact, convex, and the *objective functions* $f_i : X \to \mathbb{R}$ $(i = 1, 2, \dots, m)$ are continuous on X.

Definition 2.1. An *efficient solution* (resp., a *weakly efficient solution*) of (P) is a vector $x \in X$ such that there exists no $y \in X$ satisfying $F(x) < F(y)$ (resp., $F(x) \ll F(y)$). The set of all the efficient solutions (resp., weakly efficient solutions) of (P) is denoted by $E(P)$ (resp., by $E^w(P)$).

Definition 2.2. The set $F(E(P)) = \{F(x) : x \in E(P)\} \subset \mathbb{R}^m$ is called the *efficient frontier set* of (P).

Definition 2.3. [19, p. 238] (cf. [7], [26]) A real function f defined on a convex subset $X \subset \mathbb{R}^n$ is said to be

(i) *quasiconcave* on X, if

$f(tx^1 + (1-t)x^2) \geq \min\{f(x^1), f(x^2)\}$ for all $x^1, \ x^2 \in X$, and $t \in (0, 1)$;

(ii) *semistrictly quasiconcave* on X, if f is quasiconcave and

$f(tx^1 + (1-t)x^2) > \min\{f(x^1), f(x^2)\}$ for all $x^1, \ x^2 \in X$ satisfying $f(x^1) \neq f(x^2)$, and for all $t \in (0, 1)$;

(iii) *strictly quasiconcave* on X, if

$f(tx^1 + (1-t)x^2) > \min\{f(x^1), f(x^2)\}$ for all $x^1, \ x^2 \in X$ satisfying $x^1 \neq x^2$, and for all $t \in (0, 1)$.

Note that

strict quasiconcavity $\Rightarrow$ *semistrict quasiconcavity* $\Rightarrow$ *quasiconcavity,*

but the reverse implications are not true in general.

Example 2.1. Let $X = [-2, 2] \subset \mathbb{R}$ and

$$f(x) = \begin{cases} 0 & \text{for every } x \in [-2, 0], \\ x & \text{for every } x \in (0, 1], \\ 1 & \text{for every } x \in (1, 2]. \end{cases}$$

We check at once that f is continuous and quasiconcave on X, but it is not semistrictly quasiconcave on X.

Example 2.2. Let $X = [0, 2] \subset \mathbb{R}$ and

$$f(x) = \begin{cases} x & \text{for every } x \in [0,1], \\ 1 & \text{for every } x \in (1,2]. \end{cases}$$

Note that f is continuous and semistrictly quasiconcave on X, but it is not strictly quasiconcave on X.

Example 2.3. Let $X = [-1, 1] \subset \mathbb{R}$ and $f(x) = -x^2 + 1$ for every $x \in X$. It is clear that f is continuous and strictly quasiconcave on X. Note that the function $g(x) = -|x| + 1$ is also continuous and strictly quasiconcave on X.

We observe that some authors call the property described in part (ii) (resp., in part (iii)) of Definition 2.3 strict quasiconcavity (resp., strong quasiconcavity).

Lemma 2.1. (See [7, Theorem 5]) *If f_1 and f_2 are semistrictly quasiconcave functions on X, then the efficient frontier set of (P), where $m = 2$, is arcwise connected.*

Recall that a set $A \subset \mathbb{R}^n$ is said to be *arcwise connected* if for any $u \in A$ and $v \in A$ there exists a continuous mapping $\gamma : [0, 1] \longrightarrow A$ satisfying $\gamma(0) = u$, and $\gamma(1) = v$. If γ is such a mapping, then we say that γ is a continuous curve in A joining u and v.

Definition 2.4. A set $A \subset R^n$ is said to be *contractible* if there exists a continuous mapping $H : A\times[0, 1] \longrightarrow A$ and a point $x^0 \in A$ such that $H(x, 0) = x$ and $H(x, 1) = x^0$ for every $x \in A$.

Definition 2.5. A subset $B \subset A$ is said to be a *retract* of A if there exists a continuous map h, called a *retraction*, from A into B such that $h(x) = x$ whenever $x \in B$.

It is well known that any convex set is contractible, and any retract of a contractible set is contractible. It is also well known that any contractible set is arcwise connected.

3. Arcwise connectedness of the solution sets

Unless otherwise stated, in the sequel we shall assume that the functions f_i ($i = 1, 2, \ldots, m$) in the definition of (P) are quasiconcave on X.

Define $I = \{1, 2, \ldots, m\}$. Given any $i \in I,\ j \in I,\ 2 \le j \le i$, and $\alpha \in R$, we consider the following VOP problem:

$$(P_j^i\alpha) \quad \begin{cases} \text{Maximize } (f_1(x), \ldots, f_{j-1}(x), f_{j+1}(x), \ldots, f_i(x)) \\ \text{subject to } \ x \in X, \quad f_j(x) \ge \alpha. \end{cases}$$

It is understood that if $j = i$ then the symbol $f_{j+1}(x)$ is absent in the description of this problem.

Let $E(P_j^i\alpha)$ (resp., $E^w(P_j^i\alpha)$) stand for the efficient solution set (resp., the weakly efficient solution set) of $(P_j^i\alpha)$.

Lemma 3.1. *Suppose that there exists $i_0 \in I$ such that f_{i_0} is a strictly quasiconcave function on X. Let $i \in I$ and $j \in I$ be such that $i_0 \le i$, $j \ne i_0$, $2 \le j \le i$. Then, for any $\alpha \in \mathbb{R}$,*

$$E(P_j^i \alpha) \subset E(P).$$

Proof. Let $i_0,\ i,\ j,\ \alpha$ be as in the statement of the lemma. Let $\bar{x} \in E(P_j^i \alpha)$. We have to show that $\bar{x} \in E(P)$. To obtain a contradiction, suppose that there exist $i_1 \in I$ and $y \in X$ such that $f_i(y) \ge f_i(\bar{x})$ for every $i \in I$, and $f_{i_1}(y) > f_{i_1}(\bar{x})$. Define $z = \dfrac{1}{2}y + \dfrac{1}{2}\bar{x}$. By the convexity of X, $z \in X$. As f_i is quasiconcave and f_{i_0} is strictly quasiconcave, we have

$$f_i(z) \ge \min\{f_i(y), f_i(\bar{x})\} = f_i(\bar{x}) \quad (\text{for every } i \in I), \tag{3.1}$$

$$f_{i_0}(z) > \min\{f_{i_0}(y), f_{i_0}(\bar{x})\} = f_{i_0}(\bar{x}). \tag{3.2}$$

From (3.1) we deduce that $f_j(z) \ge f_j(\bar{x}) \ge \alpha$. This implies that z is a feasible point of $(P_j^i \alpha)$. Then, from (3.1), (3.2) and the assumption that $j \ne i_0$ it follows that $\bar{x} \notin E(P_j^i \alpha)$, a contradiction. We have thus proved that $E(P_j^i \alpha) \subset E(P)$. □

Lemma 3.2. *Assume that there exists $i_0 \in I$ such that f_{i_0} is a strictly quasiconcave function. Then, $E(P)$ is homeomorphic to $F(E(P))$.*

Proof. Since the map $F:\ X \longrightarrow \mathbb{R}^m$ is continuous, the restriction

$$F_*:\ E(P) \longrightarrow F(E(P)) \tag{3.3}$$

of F to $E(P)$ with values in $F(E(P))$ is also continuous. We claim that the map in (3.3) is one-to-one. It suffices to prove that for any $\bar{x}$, $\hat{x} \in E(P)$, $\bar{x} \ne \hat{x}$, we have $F(\bar{x}) \ne F(\hat{x})$. On the contrary, suppose there exist $\bar{x}$, $\hat{x} \in E(P)$, $\bar{x} \ne \hat{x}$, such that $F(\bar{x}) = F(\hat{x})$. Clearly, $z := \dfrac{1}{2}\bar{x} + \dfrac{1}{2}\hat{x}$ belongs to X. By the quasiconcavity of f_i $(i \in I)$ and the strict quasiconcavity of f_{i_0}, we have

$$f_i(z) \ge \min\{f_i(\hat{x}), f_i(\bar{x})\} = f_i(\bar{x}) \quad (\text{for every } i \in I),$$
$$f_{i_0}(z) > \min\{f_{i_0}(\hat{x}), f_{i_0}(\bar{x})\} = f_{i_0}(\bar{x}).$$

This implies that $\bar{x} \notin E(P)$, a contradiction. Our claim has been proved.

Consider the inverse map of the one in (3.3):

$$G_*:\ F(E(P)) \longrightarrow E(P). \tag{3.4}$$

We proceed to prove that the map in (3.4) is continuous. Let there be given any point $\bar{u} \in F(E(P))$ and any sequence $\{u^k\}$ in $F(E(P))$ such that $u^k \longrightarrow \bar{u}$ as $k \to \infty$. We set $\bar{x} = G_*(\bar{u})$ and $x^k = G_*(u^k)$ for every $k \in N$. Then $\bar{x} \in E(P) \subset X$ and $x^k \in E(P) \subset X$ for every $k \in N$. It suffices to show that the sequence $\{x^k\}$ converges in $E(P)$ to $\bar{x}$.

To obtain a contradiction, suppose that $\{x^k\}$ does not converge in $E(P)$ to $\bar{x}$. Then there exist $\varepsilon > 0$ and a subsequence $\{x^{k'}\}$ of $\{x^k\}$ such that $\|x^{k'} - \bar{x}\| \ge \varepsilon$ for all k'. As X is compact, there is no loss of generality in assuming that $\{x^{k'}\}$

converges to a point $\hat{x} \in X$. Obviously, $\|\hat{x} - \bar{x}\| \geq \varepsilon$. Since $\bar{x} = G_*(\bar{u})$ and $\bar{x} \in E(P)$, we have

$$\bar{u} = F_*(\bar{x}) = F(\bar{x}). \tag{3.5}$$

Similarly, since $x^k = G_*(u^k)$ and $x^k \in E(P)$ for every $k \in N$, we have

$$u^{k'} = F_*(x^{k'}) = F(x^{k'}) \quad \text{for every } k'. \tag{3.6}$$

On one hand, from (3.5) and (3.6) we obtain

$$F(x^{k'}) = u^{k'} \longrightarrow \bar{u} = F(\bar{x}) \quad (\text{as } k' \to \infty).$$

On the other hand, from (3.6) and the continuity of F we deduce that

$$\bar{u} = F(\hat{x}).$$

Consequently,

$$F(\bar{x}) = \bar{u} = F(\hat{x}). \tag{3.7}$$

Since $\bar{x} \in E(P)$, (3.7) implies that there exists no $y \in X$ with the property that $F(y) > F(\hat{x})$. This means that $\hat{x} \in E(P)$. Hence, on account of (3.7), we have $F_*(\bar{x}) = F_*(\hat{x})$. Because F_* is an one-to-one map, we obtain $\hat{x} = \bar{x}$. This contradicts the fact that $\|\hat{x} - \bar{x}\| \geq \varepsilon > 0$.

We have thus shown that F_* is a homeomorphism. □

The following lemma follows directly from Lemmas 2.1 and 3.2.

Lemma 3.3. *Let $m = 2$. If the functions f_i $(i = 1, 2)$ are semistrictly quasiconcave on X, and one of them is strictly quasiconcave, then the efficient solution set $E(P)$ is arcwise connected.*

Now we are in the position to establish the main result of this section.

Theorem 3.1. *Suppose that the functions f_i $(i = 1, 2, \ldots, m)$ are semistrictly quasiconcave on X. Suppose that $m \geq 2$. If there exists $i_0 \in I$ such that f_{i_0} is strictly quasiconcave, then the efficient solution set $E(P)$ is arcwise connected.*

Proof. We prove this theorem by induction on the number of the objective functions.

For $m = 2$, the assertion of the theorem follows from Lemma 3.3. By renumbering the objective functions, if necessary, we can assume that $i_0 = 1$.

Suppose that the assertion is true for all the integers $m \leq k$, where $k \geq 2$ is a given integer. We have to prove that the assertion is true for $m = k + 1$, that is the efficient solution set $E(P^{k+1})$ of the VOP problem

$$(P^{k+1}) \qquad \begin{cases} \text{Maximize} \;\; (f_1(x), f_2(x), \ldots, f_{k+1}(x)) \\ \text{subject to} \;\; x \in X \end{cases}$$

is arcwise connected.

We define

$$\underline{f}_2 = \min_{x\in X} f_2(x), \quad \bar{f}_2 = \max_{x\in X} f_2(x),$$

and consider the VOP problem

$$(P_2^{k+1}\alpha) \qquad \begin{cases}\text{Maximize} & (f_1(x), f_3(x), \dots, f_{k+1}(x))\\ \text{subject to} & x \in X,\ f_2(x) \geq \alpha,\end{cases}$$

where $\alpha \in [\underline{f}_2, \bar{f}_2]$.

Let $\bar{x} \in E(P^{k+1})$ and $\bar{y} \in E(P^{k+1})$. We set $\bar{\alpha} = f_2(\bar{x})$ and $\bar{\beta} = f_2(\bar{y})$. Then we have $\bar{x} \in E(P_2^{k+1}\bar{\alpha})$. On the contrary, suppose that $\bar{x} \notin E(P_2^{k+1}\bar{\alpha})$. It is clear that $\bar{x}$ is a feasible point of $(P_2^{k+1}\bar{\alpha})$. Since $\bar{x} \notin E(P_2^{k+1}\bar{\alpha})$, there exist $i_1 \in \{1, 2, \dots, k+1\} \setminus \{2\}$ and $y \in X$ such that $f_2(y) \geq \bar{\alpha} = f_2(\bar{x})$,

$$f_i(y) \geq f_i(\bar{x}) \text{ for every } i \in \{1, 2, \dots, k+1\} \setminus \{2\}, \text{ and } f_{i_1}(y) > f_{i_1}(\bar{x}).$$

From this we see that $\bar{x} \notin E(P^{k+1})$, a contradiction. We have thus proved that

$$\bar{x} \in E(P_2^{k+1}\bar{\alpha}).$$

Similarly,

$$\bar{y} \in E(P_2^{k+1}\bar{\beta}).$$

Consider the bicriteria optimization problem

$$\text{(3.8)} \qquad \begin{cases}\text{Maximize} & (f_1(x), f_2(x))\\ \text{subject to} & x \in X,\end{cases}$$

and the scalar optimization problems:

$$\text{(3.9)} \qquad \begin{cases}\text{Maximize} & f_1(x)\\ \text{subject to} & x \in X,\ f_2(x) \geq \bar{\alpha},\end{cases}$$

$$\text{(3.10)} \qquad \begin{cases}\text{Maximize} & f_1(x)\\ \text{subject to} & x \in X,\ f_2(x) \geq \bar{\beta}.\end{cases}$$

Since $\bar{x}$ is a feasible point for (3.9), from the compactness of X and the continuity of f_2 we deduce that the feasible region of (3.9) is nonempty and compact. Note that (3.9) is a weighted problem of $(P_2^{k+1}\bar{\alpha})$ with the weight $(1, 0, \dots, 0)$. Since f_1 is strictly quasiconcave, (3.9) has a unique solution $\widetilde{x}$. We check at once that $\widetilde{x}$ is an efficient solution of $(P_2^{k+1}\bar{\alpha})$. Similarly, since $\widetilde{x}$ is an efficient solution for the section

$$\{x \in X : f_2(x) \geq \bar{\alpha}\},$$

it is an efficient solution of (3.8). Likewise, there exists a unique solution $\widetilde{y}$ of (3.10), which is an efficient solution of both the problems $(P_2^{k+1}\bar{\beta})$ and (3.8). Applying Lemma 3.3 to problem (3.8) we deduce that there exists a continuous curve in the solution set of (3.8) joining $\widetilde{x}$ and $\widetilde{y}$. Since f_1 is strictly quasiconcave and f_2 is semistrictly quasiconcave, the efficient solution set of (3.8) is a subset of $E(P^{k+1})$. So the just mentioned curve is contained in $E(P^{k+1})$. Since $\bar{x}$ and $\widetilde{x}$ belong to $E(P_2^{k+1}\bar{\alpha})$, by the induction hypothesis, there exists a continuous curve in $E(P_2^{k+1}\bar{\alpha})$ joining $\bar{x}$ and $\widetilde{x}$. Similarly, there exists a continuous curve in $E(P_2^{k+1}\bar{\beta})$ joining $\bar{y}$ and $\widetilde{y}$. According to Lemma 3.1, we have $E(P_2^{k+1}\bar{\alpha}) \subset E(P^{k+1})$ and $E(P_2^{k+1}\bar{\beta}) \subset E(P^{k+1})$. Hence the just mentioned two curves are contained in $E(P^{k+1})$. From what has been said, we conclude that there exists a continuous curve in $E(P^{k+1})$ joining $\bar{x}$ and $\bar{y}$. The proof of the theorem is complete. □

Since F is a continuous map, the following corollary follows directly from Theorem 3.1.

Corollary 3.1. *Under the assumptions of Theorem 3.1, the set $F(E(P))$ is arcwise connected.*

Theorem 3.2. *Under the assumptions of Theorem 3.1, the weakly efficient solution set $E^w(P)$ is arcwise connected.*

Proof. Let $a \in E^w(P)$ and $b \in E^w(P)$. Consider the scalar optimization problem

$$(3.11) \qquad \begin{cases} \text{Maximize} \;\; g(x) := & f_1(x) + f_2(x) + \cdots + f_m(x) \\ \text{subject to} \;\; x \in X, & f_1(x) \geq f_1(a),\; f_2(x) \geq f_2(a),\; \ldots, \\ & f_m(x) \geq f_m(a). \end{cases}$$

Note that a is a feasible point for (3.11). Since the feasible region of (3.11) is compact, from the continuity of $g(\cdot)$ it follows that (3.11) has a solution $\widetilde{x}$.

We claim that $\widetilde{x} \in E(P)$. Otherwise there exist $i_1 \in I$ and $y \in X$ such that

$$f_i(y) \geq f_i(\tilde{x}) \quad \text{for every } \; i \in I \setminus \{i_1\}, \quad f_{i_1}(y) > f_{i_1}(\widetilde{x}).$$

Then $f_i(y) \geq f_i(\widetilde{x}) \geq f_i(a)$ for all $i \in I$. So y is a feasible point for (3.11). Since

$$\begin{aligned} g(y) &= f_1(y) + f_2(y) + \cdots + f_m(y) \\ &> f_1(\widetilde{x}) + f_2(\widetilde{x}) + \cdots + f_m(\widetilde{x}) \\ &= g(\widetilde{x}), \end{aligned}$$

we see that $\widetilde{x}$ cannot be a solution of (3.11), a contradiction. We have thus proved that $\widetilde{x} \in E(P)$.

Fix any $t \in [0,1]$. It is clear that $x_t := t\widetilde{x} + (1-t)a$ belongs to X. We have $x_t \in E^w(P)$. On the contrary, suppose that there exists $y \in X$ such that $f_i(y) > f_i(x_t)$ for all $i \in I$. Combining this with the semistrict quasiconcavity of f_i $(i \in I)$ we deduce that

$$f_i(y) > \min\{f_i(\widetilde{x}),\; f_i(a)\} = f_i(a)$$

for all $i \in I$. Then $a \notin E^w(P)$, a contradiction. Therefore $x_t \in E^w(P)$ for any $t \in [0,1]$. This means that line-segment $[a, \widetilde{x}]$ is contained in $E^w(P)$.

Similarly, there exists $\widetilde{y} \in E(P)$ such that $[b, \widetilde{y}] \subset E^w(P)$.

According to Theorem 3.1, there exists continuous curve in $E(P)$ joining $\widetilde{x}$ and $\widetilde{y}$.

Since $E(P) \subset E^w(P)$, from what has been said we conclude that there exists a continuous curve in $E^w(P)$ joining a and b. The proof is complete. □

Note that if all the objective functions f_i $(i = 1, \dots, m)$ are strictly quasiconcave then the efficient solution set $E(P)$ is contractible (see [16]).

4. Contractibility of the solution sets in the case $m = 2$

In this section we consider problem (P) under the assumption that $m = 2$, f_1 and f_2 are semistrictly quasiconcave continuous functions on X. Let $\underline{f}_2$ and $\bar{f}_2$ be defined as in the preceding section. For every $\alpha \in [\underline{f}_2, \bar{f}_2]$, we consider the scalar optimization problem

$$(4.1) \qquad \begin{cases} \text{Maximize} & f_1(x) \\ \text{subject to} & x \in X, \ f_2(x) \geq \alpha. \end{cases}$$

Denote the solution set of (4.1) by $S(\alpha)$.

Lemma 4.1. *If f_1 is strictly quasiconcave then it holds*

$$(4.2) \qquad E(P) = \bigcup \{S(\alpha) : \ \alpha \in [\underline{f}_2, \bar{f}_2]\}.$$

Besides, the map $S : \ [\underline{f}_2, \bar{f}_2] \longrightarrow 2^{E(P)}$, $\alpha \longrightarrow S(\alpha)$, is single-valued and continuous on $[\underline{f}_2, \bar{f}_2]$.

Proof. By [24, Theorem 1], the representation (4.2) holds. The strict quasiconcavity of f_1 implies that, for every $\alpha \in [\underline{f}_2, \bar{f}_2]$, the solution set $S(\alpha)$ is a singleton. From the upper semicontinuity of $S(\cdot)$ (see [24, Lemma 3]) we deduce that $S(\cdot)$ is continuous on $[\underline{f}_2, \bar{f}_2]$. □

Theorem 4.1. *If f_1 is strictly quasiconcave on X then $E(P)$ is a retract of X. In particular, $E(P)$ is contractible.*

Proof. First we recall that the map $S(\cdot)$ in Lemma 4.1 is single-valued. For every $x \in X$, it holds $f_2(x) \in [\underline{f}_2, \bar{f}_2]$. By (4.2), vector $S(f_2(x))$ belongs to $E(P)$.

We will show that the map $h : \ X \longrightarrow E(P)$ defined by setting $h(x) = S(f_2(x))$ for all $x \in X$, is a retraction. By Lemma 4.1, h is continuous on X. It suffices to prove that $h(\bar{x}) = \bar{x}$ for every $\bar{x} \in E(P)$. Let $\bar{x} \in E(P)$, and let $\alpha = f_2(\bar{x})$. We claim that $\bar{x}$ is a solution of (4.1). Indeed, if there exists $y \in X$ such that $f_2(y) \geq \alpha = f_2(\bar{x})$ and $f_1(y) > f_1(\bar{x})$ then $\bar{x} \notin E(P)$, a contradiction. As $\bar{x}$ is the unique solution of (4.1), we have $S(\alpha) = \bar{x}$. Therefore $h(\bar{x}) = S(f_2(\bar{x})) = \bar{x}$. We have thus proved that $E(P)$ is a retract of X. From the convexity of X it follows that $E(P)$ is contractible. □

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Note added in revision

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References

[1] J. Benoist, *Connectedness of the efficient set for strictly quasiconcave sets*, J. Optim. Theory Appl. **96** (1998), 627–654.

[2] J. Benoist, *The structure of the efficient frontier of finite-dimensional completely-shaded sets*, J. Math. Anal. Appl. **250** (2000), 98–117.

[3] G. R. Bitran and T. L. Magnanti, *The structure of admissible points with respect to cone dominance*, J. Optim. Theory Appl. **29** (1979), 573–614.

[4] R. Cambini, *Generalized concavity for bicriteria functions*, In: "Generalized Convexity, Generalized Monotonicity: Recent Results", J. P. Crouzeix, Ed., Kluwer Academic Publishers, Netherlands (1998), 439–451.

[5] E. U. Choo and D. R. Atkins, *Bicriteria linear fractional programming*, J. Optim. Theory Appl. **36** (1982), 203–220.

[6] E. U. Choo and D. R. Atkins, *Connectedness in multiple linear fractional programming*, Management Science **29** (1983), 250–255.

[7] E. U. Choo, S. Schaible and K. P. Chew, *Connectedness of the efficient set in three criteria quasiconcave programming*, Cahiers du Centre d'Etudes de Recherche Opérationnelle **27** (1985), 213–220.

[8] A. Daniilidis, N. Hadjisavvas and S. Schaible, *Connectedness of the efficient set for three-objective quasiconcave maximization problems*, J. Optim. Theory Appl. **93** (1997), 517–524.

[9] S. Deng, *Characterizations of the nonemptiness and compactness of solution sets in convex vector optimization*, J. Optim. Theory Appl. **96** (1998), 123–131.

[10] W. T. Fu and K. P. Zhou, *Connectedness of the efficient solution sets for a strictly path quasiconvex programming problem*, Nonlinear Analysis **21** (1993), 903–910.

[11] X. H. Gong, *Connectedness of the efficient solution sets of a convex vector optimization in normed space*, Nonlinear Analysis **23** (1994), 1105–1114.

[12] Y. D. Hu and E. J. Sun, *Connectedness of the efficient set in strictly quasiconcave vector maximization*, J. Optim. Theory Appl. **78** (1993), 613–622.

[13] N. Q. Huy, T. D. Phuong and N. D. Yen, *On the contractibility of the efficient and weakly efficient sets in* $\mathbb{R}^2$, Preprint 5/2000, Hanoi Institute of Mathematics, 2000. To appear in: "Equilibrium Problems and Variational Models" (A. Maugeri, Ed.), Kluwer Academic Publishers, September 2002.

[14] G. M. Lee, D. S. Kim, B. S. Lee and N. D. Yen, *Vector variational inequality as a tool for studying vector optimization problems*, Nonlinear Analysis **34** (1998), 745–765.
[15] D. T. Luc, *Structure of the efficient point sets*, Proc. Amer. Math. Soc. **95** (1985), 433–440.
[16] D. T. Luc, *Connectedness of the efficient set in quasiconcave vector maximization*, J. Math. Anal. Appl. **122** (1987), 346–354.
[17] D. T. Luc, *Theory of Vector Optimization*, Lecture Notes in Economics and Mathematical Systems, Vol. 319, Springer-Verlag, Berlin, 1989.
[18] D. T. Luc, *Contractibility of efficient sets in normed space*, Nonlinear Analysis **15** (1990), 527–535.
[19] D. T. Luc, *Generalized monotone set valued maps and support bifunctions*, Acta Math. Vietnam. **21** (1996), 213–252.
[20] E. K. Makarov, N. N. Rachkovski and W. Song, *Arcwise connectedness of closed efficient point sets*, J. Math. Anal. Appl. **247** (2000), 377–383.
[21] E. Molho and A. Zaffaroni, *Quasiconcavity of sets and connectedness of the efficient frontier in ordered vector spaces*, In: Generalized Convexity, Generalized Monotonicity: Recent Results, J. P. Crouzeix, Ed., Kluwer Academic Publishers, Netherlands, 1998, pp. 407–424.
[22] P. H. Naccache, *Connectedness of the set of nondominated outcomes in multicriteria optimization*, J. Optim. Theory Appl. **25** (1978), 459–467.
[23] B. Peleg, *Topological properties of the efficient point set*, Proc. Amer. Math. Soc. **35** (1972), 531-536.
[24] S. Schaible, *Bicriteria quasiconcave programs*, Cahiers du Centre d'Etudes de Recherche Opérationnelle **25** (1983), 93–101.
[25] W. Song, *A note on connectivity of efficient solution sets*, Arch. Math. **65** (1995), 540–545.
[26] E. J. Sun, *On the connectedness of the efficient set for strictly quasiconvex vector minimization problems*, J. Optim. Theory Appl. **89** (1996), 475-581.
[27] A. R. Warburton, *Quasiconcave vector maximization: Connectedness of the set of Pareto-optimal and weak Pareto-optimal alternatives*, J. Optim. Theory Appl. **40** (1983), 537–557.
[28] R. Wernsdorf, *On the connectedness of the set of efficient points in convex optimization problems with multiple or random objectives*, Math. Operationsforsch. u. Statist., Ser. Optimization **15** (1984), 379–387.
[29] N. D. Yen and T. D. Phuong, *Connectedness and stability of the solution set in linear fractional vector optimization problems*, In: "Vector Variational Inequalities and Vector Equilibria. Mathematical Theories" (F. Giannessi, Ed.), Kluwer Academic Publishers, 2000, pp. 479–489.

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