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STRASSEN'S LOCAL LAW FOR DIFFUSION PROCESSES UNDER STRONG TOPOLOGIES

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Abstract. Under the assumption of pathwise uniqueness, we prove Strassen type functional local law of the iterated logarithm for solutions of stochastic differential equations in modulus spaces defined in term of the Young function $M_2(x) = \exp(x^2) - 1$ and the modulus of continuity $\varphi_0(t) = (t \log(1/t))^{1/2}$.

1. Introduction

Let $W = \{(W_1(t), \dots, W_d(t)) : t \geq 0\}$ be a standard Brownian motion on a complete probability space $(\Omega,\ \mathcal{F},\ \mathbb{P})$ and $\mathcal{C}_m([0,1])$ be the space of all $\mathbb{R}^m$–valued continuous functions defined on $[0,1]$. A classical result of Strassen [12] states that $\{W_1(n\cdot)/\sqrt{2n\log\log n} : n \geq 3\}$ is almost surely relatively compact in $\mathcal{C}_1([0,1])$ with limit set points $\mathcal{K} = \{f \in \mathcal{H} : \mu(f) \leq 1\}$, where $\mathcal{H}$ stands for the Cameron–Martin space of absolutely continuous functions with Lebesgue derivative $\dot{f}$ and

$$\mu(f) = \frac{1}{2}\int_0^1 |\dot{f}(s)|^2 ds.$$

An immediate consequence of the above result is the usual law of the iterated logarithm

$$\mathbb{P}\Big[\limsup_{n\to\infty} \frac{W_1(n)}{\sqrt{2n\log\log n}} = 1\Big] = 1. \tag{1}$$

A time inversion argument which consists of noting that $\Big\{tW\Big(\dfrac{1}{t}\Big) : t > 0\Big\}$ is a Brownian motion yields the following local version of (1):

$$\mathbb{P}\left[\limsup_{n\to\infty} \frac{W_1\Big(\dfrac{1}{n}\Big)}{\sqrt{\dfrac{2}{n}\log\log n}} = 1\right] = 1. \tag{2}$$

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In the same spirit, by defining a time inversion transformation on $\mathcal{C}_1([0,\infty[)$, Gantert [6] proved a local version of Strassen's theorem without the help of the connection between large deviations and laws of the iterated logarithm which can be found for instance in Stroock and Varadhan [13] and Baldi [1].

Recently N'zi [10] derived by the way of large deviations an analogous result for Lévy's area process. Indeed, even though this process shares many properties with the Brownian motion, it seems that it doesn't satisfy the time inversion one. This was pointed out by Helmes [7] who studied the non–functional local law of the iterated logarithm for a class of stochastic integrals containing Lévy's area process.

The aim of this paper is to generalize the results of Gantert [6] and N'zi [10] in two directions: we deal with a large class of diffusion processes and consider stronger topologies than the uniform one. The proof follows the same line in Baldi [1] and uses a recent result of Eddahbi [5].

We now give the context of our study.

Let us set $\Phi = \{\varphi \in \mathcal{C}_1([0,1]) : \varphi(t) > 0 \text{ on }]0,1] \text{ and } \varphi(0) = 0\}$. For every φ and ψ in Φ, we denote by $\mathcal{B}^{\varphi}_{\psi,M_2,q}$ the Banach space of Borelian functions $f : [0,1] \longrightarrow \mathbb{R}^m$ such that

$$\|f\|^{\varphi}_{\psi,M_2,q} \;:=\; \|f\|_\infty + \|f\|^{\varphi} + \|f\|_{M_2} + \left(\int_0^1 \left(\frac{\omega_{M_2}(f,t)}{\psi(t)}\right)^q \frac{dt}{t}\right)^{\frac{1}{q}} \;<\; +\infty$$

where

$$\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|,$$

$$\|f\|^{\varphi} = \sup_{0\le s<t\le 1} \frac{|f(t)-f(s)|}{\varphi(|t-s|)},$$

where $\|f\|_{M_2}$ stands for the Orlicz–norm associated with the Young function $M_2(x) = \exp(x^2) - 1$, defined by

$$\|f\|_{M_2} = \sup_{p\ge 1} \frac{\|f\|_{L^p([0,1])}}{\sqrt{p}}$$

and $\omega_{M_2}(f,t)$ is the modulus of continuity in Orlicz norm given by

$$\omega_{M_2}(f,t) = \sup_{0\le h\le t} \|\triangle_h f\|_{M_2}$$

with

$$\triangle_h f(x) = 1\!\!1_{[0,1-h]}(x)[f(x+h) - f(x)], \quad \text{for } h \in [0,1].$$

Let us note that $\mathcal{B}^{\varphi}_{\psi,M_2,q}$ is a subspace of

$$\mathcal{C}^{\varphi}_m([0,1]) = \{f \in \mathcal{C}_m([0,1]) \;:\; \|f\|_\infty + \|f\|^{\varphi} < +\infty\}.$$

If $\varphi(t) = t^\alpha$ then $\mathcal{C}^{\varphi}_m([0,1])$ is the Hölder space of order α. When $q = +\infty$, we simply write $\mathcal{B}^{\varphi}_{\psi,M_2}$ for $\mathcal{B}^{\varphi}_{\psi,M_2,\infty}$. Let

$$\mathcal{B}^{\varphi,0}_{\psi,M_2} = \Big\{ f \in \mathcal{B}^{\varphi}_{\psi,M_2} : |f(t)-f(s)| = o(\varphi(|t-s|)) \text{ as } |t-s| \text{ goes to } 0$$
$$\|f\|_p = o(\sqrt{p}),\ \omega_p(f,t) = o(\sqrt{p}\psi(t)) \text{ as } \max\Big(p, \frac{1}{t}\Big) \text{ goes to } +\infty \Big\}.$$

Then $\mathcal{B}^{\varphi,0}_{\psi,M_2}$ is a separable Banach space. For more details on Besov–Orlicz spaces we refer the reader to Ciesielski et al. [4].

Now let us recall the large deviations principle in modulus spaces obtained by Eddahbi [5].

Let σ_ε (resp. b_ε) be a $\mathbb{R}^m \times \mathbb{R}^d$ (resp. $\mathbb{R}^m$)–valued field defined on $\mathbb{R}_+ \times \mathbb{R}^m$. We consider the Itô's stochastic differential equation for every $t \geq 0$:

$$X^\varepsilon_t \;=\; x^\varepsilon(t) + \varepsilon \int_0^t \sigma_\varepsilon(s, X^\varepsilon_s) dW_s + \int_0^t b_\varepsilon(s, X^\varepsilon_s) ds. \tag{3}$$

In the sequel, we make the following assumptions:

(H1) (i) $(t,x) \mapsto \sigma_\varepsilon(t,x)$ and $(t,x) \mapsto b_\varepsilon(t,x)$ are measurable functions, continuous in x

uniformly with respect to t.

(ii) σ_ε converges uniformly to a $\mathbb{R}^m \times \mathbb{R}^d$–valued matrix field σ as ε goes to 0,

(iii) b_ε converges uniformly to a $\mathbb{R}^m$–valued vector field b as ε goes to 0,

(iv) $x^\varepsilon(\cdot)$ converges in $\mathcal{B}^{\varphi,0}_{\psi,M_2}$ to a function $x(\cdot)$ as ε goes to 0.

(H2) The equation (3) admits an unique solution adapted to the Brownian filtration.

(H3) $\displaystyle\int_{0^+} \frac{dr}{\omega_\sigma(r) + \omega_b(r)} = \infty$, where ω_σ and ω_b denote respectively the modulus of continuity of σ and b.

Some conditions ensuring the existence and the uniqueness of solutions of (3) can be found in Yamada and Watanabe [14], Watanabe and Yamada [15], Ikeda and Watanabe [8], Barlow and Perkins [2] and Rutkowski [11].

For every $h \in \mathcal{H}$, $S(h)$ stands for the solution of the deterministic differential equation

$$S(h)_t = y(t) + \int_0^t \sigma(s, S(h)_s)\dot{h}_s \, ds + \int_0^t b(s, S(h)_s)\, ds. \tag{4}$$

It is clear that (H3) implies the existence and uniqueness of solution of (4).

Now, we define the Cramer transform

$$\lambda(f) = \left\{ \inf_{+\infty}\{\mu(h) : h \in \mathcal{H},\ S(h) = f\} \underset{\text{otherwise}}{\text{if}} (S(h))^{-1}(\{f\}) \neq \emptyset, \right\}$$

and the Cramer functional

$$\Lambda(A) = \inf_{f \in A} \lambda(f), \quad A \subset \mathcal{C}_m([0,1]).$$

In what follows, we assume that φ and ψ satisfy the conditions below: φ and ψ are increasing functions null in zero, $\varphi_0(t) = o(\varphi(t))$ as t goes to zero, $\dfrac{\varphi_0(t)}{\psi(t)}$ is bounded near zero and the maps $t \longmapsto \dfrac{\varphi(t)}{\sqrt{t}}$ and $t \longmapsto \dfrac{\psi(t)}{\sqrt{t}}$ are decreasing functions converging to infinity as t goes to zero.

Theorem 1.1. *Under assumptions (H1)–(H3), for every $a > 0$, $\rho > 0$ and $R > 0$ there exist $\varepsilon_0 > 0$, $\alpha_0 > 0$ and $\eta > 0$ such that for every $h \in \mathcal{H}$ with $\mu(h) \le a$, every function $y(\cdot) \in \mathcal{B}^{\varphi,0}_{\psi,M_2}$ such that $\|x - y\|^{\varphi}_{\psi,M_2} \le \eta$,*

$$\mathbb{P}\big[\|X^\varepsilon - S(h)\|^{\varphi}_{\psi,M_2} \ge \rho,\ \|\varepsilon W - h\| \le \alpha\big] \le \exp\Big(-\frac{R}{\varepsilon^2}\Big)$$

for all $\varepsilon \in]0, \varepsilon_0]$ and $\alpha \in]0, \alpha_0]$.

Under the assumptions (H1)–(H3), using the technics in Ciesielski and Kamont [3] and Mellouk [9], we can prove that $X^\varepsilon_\cdot$ belongs to $\mathcal{B}^{\varphi,0}_{\psi,M_2}$.

Theorem 1.2. *Under assumptions* (H1)–(H3)*, for every Borel set A of $\mathcal{B}^{\varphi,0}_{\psi,M_2}$*

$$-\Lambda(\overset{\circ}{A}) \le \liminf_{\varepsilon \searrow 0} \varepsilon^2 \log \mathbb{P}\left[X^\varepsilon_\cdot \in A\right] \ \le\ \limsup_{\varepsilon \searrow 0} \varepsilon^2 \log \mathbb{P}\left[X^\varepsilon_\cdot \in A\right] \ \le\ -\Lambda(\bar{A})$$

where $\overset{\circ}{A}$ (resp. $\bar{A}$) stands for the interior (resp. adherence) of A in the topology of $\mathcal{B}^{\varphi,0}_{\psi,M_2}$.

2. Main result

This section is devoted to the proof of a local version of Strassen's law in $\mathcal{B}^{\varphi,0}_{\psi,M_2}$ for diffusion processes. The ideas of the proof are similar to that of Baldi [1].

First of all we introduce a system of contractions by adapting the definition of Baldi [1] to our purpose. Let $\mathcal{U}$ be an open subset of $\mathbb{R}^m$ and $y \in \mathcal{U}$.

Definition 2.1. A family $\big(\Gamma_\alpha\big)_{\alpha \in \mathbb{R}_+}$ of continuous bijective transformations $\Gamma_\alpha : \mathcal{U} \longrightarrow \mathcal{U}$ is said to be a system of contractions centered at y if

(i) $\Gamma_\alpha(y) = y$ for all $\alpha \in \mathbb{R}_+$,

(ii) for $\alpha > \beta$,

$$|\Gamma_\alpha(z_1) - \Gamma_\alpha(z_1') - \Gamma_\alpha(z_2) + \Gamma_\alpha(z_2')| \le |\Gamma_\beta(z_1) - \Gamma_\beta(z_1') - \Gamma_\beta(z_2) + \Gamma_\beta(z_2')|$$

for every z_1, z_2, z_1' and z_2' in $\mathcal{U}$,

(iii) Γ_1 is the identical mapping on $\mathcal{U}$ and $\Gamma_\alpha^{-1} \ = \ \Gamma_{\alpha^{-1}}$.

Moreover, for every $\varepsilon > 0$ and every compact subset K of $\mathcal{B}^{\varphi,0}_{\psi,M_2}$, there exists $\delta > 0$ such that

$$\text{if } |\alpha\beta - 1| \ < \ \delta \ \text{ then } \ \|\Gamma_\alpha o \Gamma_\beta(f(\cdot)) - f(\cdot)\|^{\varphi}_{\psi,M_2} \ < \ \varepsilon$$

for every f in K.

Let $\bar{\sigma}$ be a $\mathbb{R}^m \times \mathbb{R}^d$–valued matrix field and $\bar{b}$ a $\mathbb{R}^m$–valued vector field defined on $\mathcal{U}$. We put $\bar{a} \ = \ \bar{\sigma}^t\bar{\sigma}$.

Let $\bar{L}$ denote the differential operator on $\mathcal{U}$ defined by

$$\bar{L} := \frac{1}{2}\sum_{i,j=1}^{m} \bar{a}_{i,j}(z)\frac{\partial}{\partial x_i \partial x_j} + \sum_{j=1}^{m} \bar{b}_j(z)\frac{\partial}{\partial x_j}.$$

Let

$$\phi(u) = \sqrt{\frac{\mathcal{L}_2(u)}{u}},$$

where

$$\mathcal{L}_2(u) = \begin{cases} \log\log u & \text{if } \ u \geq 3, \\ 1 & \text{if } \ 0 < u < 3. \end{cases}$$

For every $\alpha > 0$, we set

$$\bar{\sigma}_\alpha(z) = \phi(\alpha)(\mathrm{grad}\Gamma_{\phi(\alpha)})(\Gamma^{-1}_{\phi(\alpha)}(z)).\bar{\sigma}(\Gamma^{-1}_{\phi(\alpha)}(z)), \tag{5}$$

$$\bar{b}_\alpha(z) = \frac{1}{\alpha}(\bar{L}\Gamma_{\phi(\alpha)})o\Gamma^{-1}_{\phi(\alpha)}(z), \tag{6}$$

where (Γ_α) is a system of contractions centered at y.

Definition 2.2. We say that the triple $(\bar{\sigma}_\alpha, \ \bar{b}_\alpha, \ \Gamma_\alpha)_{\alpha\in\mathbb{R}_+}$ satisfies the assumption (K) if Γ_α is twice continuously differentiable for all $\alpha > 0$ and there exist a $\mathbb{R}^m \times \mathbb{R}^d$–valued matrix field $\bar{\sigma}$ and a $\mathbb{R}^m$–valued vector field $\bar{b}$ on $\mathcal{U}$ such that

$$\lim_{\alpha\nearrow+\infty} \bar{\sigma}_\alpha(z) = \bar{\sigma}(z) \ , \quad \lim_{\alpha\nearrow+\infty} \bar{b}_\alpha(z) \ = \ \bar{b}(z)$$

uniformly on compact subsets of $\mathcal{U}$.

In the sequel, $(\bar{\sigma}_\alpha, \ \bar{b}_\alpha, \ \Gamma_\alpha)_{\alpha\in\mathbb{R}_+}$ satisfies the assumption (K) in Definition 2.2.

Let $\mathcal{Y}$ be the Itô's process with values in $\mathcal{U}$ and defined by

$$\mathcal{Y}_t \ = \ y + \int_0^t \bar{\sigma}(\mathcal{Y}_s)dW_s + \int_0^t \bar{b}(\mathcal{Y}_s)ds.$$

For every $u > 0$, we put

$$W^u_t = \sqrt{u}W(\frac{t}{u}),$$

$$\mathcal{Y}^u_t = \mathcal{Y}_{\frac{t}{u}} \ \text{ and } \ \mathcal{Z}^u_t = \Gamma_{\phi(u)}(\mathcal{Y}^u_t).$$

Itô's formula yields

$$\mathcal{Z}_t^u \;=\; y + \frac{1}{\sqrt{\mathcal{L}_2(u)}} \int_0^t \bar{\sigma}_u(\mathcal{Z}_s^u) dW_s^u + \int_0^t \bar{b}_u(\mathcal{Z}_s^u)\, ds$$

where $\bar{\sigma}_u$ and $\bar{b}_u$ are given by (5) and (6).

Applying the large deviations principle in Theorem 1.2 for $\bar{\sigma}_u = \sigma_{\frac{1}{u}}$ and $\bar{b}_u = b_{\frac{1}{u}}$, we have

$$-\Lambda(\overset{\circ}{A}) \leq \liminf_{u \nearrow +\infty} \frac{1}{\mathcal{L}_2(u)} \log \mathbb{P}\left[\mathcal{Z}_\cdot^u \in A\right] \leq \limsup_{u \nearrow +\infty} \frac{1}{\mathcal{L}_2(u)} \log \mathbb{P}\left[\mathcal{Z}_\cdot^u \in A\right] \leq -\Lambda(\bar{A})$$

for every Borel subset of $\mathcal{B}_{\psi,M_2}^{\varphi,0}$.

Now, we are able to state our main result.

Theorem 2.1. *Under assumptions* (H1)–(H3) *and (K), the process* $\{\mathcal{Z}_\cdot^u : u > 0\}$ *is* $\mathbb{P}$*–almost surely relatively compact in the topology of* $\mathcal{B}_{\psi,M_2}^{\varphi,0}$ *as u goes to* $+\infty$*, with limit set points* $\mathcal{K}_\lambda(1) = \{f \in \mathcal{C}_m([0,1]) : \lambda(f) \leq 1\}$·

The proof of Theorem 2.1 consists of a suitable combination of Propositions 2.1 and 2.2 below.

Proposition 2.1. *For every* $\varepsilon > 0$*, there exists* $\mathbb{P}$*–almost surely* $u^0 > 0$ *such that if* $u > u^0$ *then*

$$d(\mathcal{Z}_\cdot^u, \mathcal{K}_\lambda(1)) \leq \varepsilon$$

where $d(g, \mathcal{K}_\lambda(1)) := \inf_{h \in \mathcal{K}_\lambda(1)} \|g - h\|_{\psi,M_2}^{\varphi}$.

To prove Proposition 2.1 we need some technical lemmas.

Lemma 2.1. *For every* $c > 1$ *and* $\varepsilon > 0$*, there exists a.s.* $j_0 = j_0(\omega)$ *such that if* $j > j_0$ *then*

$$d(\mathcal{Z}_\cdot^{c^j}, \mathcal{K}_\lambda(1)) \leq \varepsilon.$$

Proof. Let $\mathcal{K}_\varepsilon = \{g \in \mathcal{B}_{\psi,M_2}^{\varphi,0},\ d(g, \mathcal{K}_\lambda(1)) \geq \varepsilon\}$ and $\delta > 0$ be such that $\Lambda(\mathcal{K}_\varepsilon) > 1 + 2\delta$. By virtue of Theorem 1.2, we have

$$\limsup_{u \nearrow +\infty} \frac{1}{\mathcal{L}_2(u)} \log \mathbb{P}[\mathcal{Z}_\cdot^u \in \mathcal{K}_\varepsilon] \leq -(1 + 2\delta).$$

It follows that for j sufficiently large, we have

$$\mathbb{P}[\mathcal{Z}_\cdot^{c^j} \in \mathcal{K}_\varepsilon] \leq \exp(-(1+\delta)\mathcal{L}_2(c^j)) \leq \frac{C}{j^{1+\delta}}.$$

Now, noting that $\sum_j \mathbb{P}[\mathcal{Z}_\cdot^{c^j} \in \mathcal{K}_\varepsilon] < +\infty$ we see that the conclusion is an immediate consequence of the Borel–Cantelli lemma. $\square$

Lemma 2.2. *Let*

$$Y_j = \sup_{c^{j-1} \le u \le c^j} \|\mathcal{Z}^u_\cdot - \Gamma_{\phi(u)} o \Gamma^{-1}_{\phi(c^j)}(\mathcal{Z}^{c^j}_\cdot)\|^{\varphi}_{\psi,M_2}.$$

For every $\varepsilon > 0$, there exists $c_\varepsilon > 1$ such that for every $1 < c < c_\varepsilon$ there exists $j_0 = j_0(\omega)$ satisfying $Y_j(\omega) < \varepsilon$ for every $j \ge j_0$.

Proof. Note that

$$Y_j = \sup_{c^{j-1} \le u \le c^j} \|\Gamma_{\phi(u)}(\mathcal{Y}^u_\cdot) - \Gamma_{\phi(u)}(\mathcal{Y}^{c^j}_\cdot)\|^{\varphi}_{\psi,M_2}$$

By virtue of Lemma 2.1 there exists a constant $C > 0$ such that for j sufficiently large, $\|\mathcal{Z}^{c^j}_\cdot\|^{\varphi}_{\psi,M_2} \le C$ a.s. In view of the Borel–Cantelli lemma, it suffices to prove that

$$\sum_j \mathbb{P}[Y_j \ge \varepsilon,\ \|\mathcal{Z}^{c^j}_\cdot\|^{\varphi}_{\psi,M_2} \le C] < +\infty.$$

In view of the definition of a system of contractions, it is easy to derive the following inclusions

$$\begin{aligned}
\{Y_j \ge \varepsilon\} &\subset \left\{ \sup_{c^{j-1} \le u \le c^j} \|\Gamma_{\phi(c^{j-1})}(\mathcal{Y}_{\frac{\cdot}{u}}) - \Gamma_{\phi(c^{j-1})}(\mathcal{Y}_{\frac{\cdot}{c^j}})\|^{\varphi}_{\psi,M_2} \ge \varepsilon \right\} \\
&\subset \left\{ \sup_{1 \le v \le c} \|\Gamma_{\phi(c^{j-1})}(\mathcal{Y}_{\frac{\cdot}{vc^j}}) - \Gamma_{\phi(c^{j-1})}(\mathcal{Y}_{\frac{\cdot}{c^j}})\|^{\varphi}_{\psi,M_2} \ge \varepsilon \right\} \\
&= \left\{ \sup_{1 \le v \le c} \|\Gamma_{\phi(c^{j-1})} o \Gamma^{-1}_{\phi(c^j)}(\mathcal{Z}^{c^j}_{\frac{\cdot}{v}}) - \Gamma_{\phi(c^{j-1})} o \Gamma^{-1}_{\phi(c^j)}(\mathcal{Z}^{c^j}_\cdot)\|^{\varphi}_{\psi,M_2} \ge \varepsilon \right\}. \qquad (7)
\end{aligned}$$

Therefore

$$\{Y_j \ge \varepsilon\} \subset A^j_{\varepsilon,1} \cup A^j_{\varepsilon,2} \cup A^j_{\varepsilon,3}, \qquad (8)$$

where

$$\begin{aligned}
A^j_{\varepsilon,1} &= \left\{ \sup_{1 \le v \le c} \|\Gamma_{\phi(c^{j-1})} o \Gamma^{-1}_{\phi(c^j)}(\mathcal{Z}^{c^j}_{\frac{\cdot}{v}}) - \mathcal{Z}^{c^j}_{\frac{\cdot}{v}}\|^{\varphi}_{\psi,M_2} \ge \frac{\varepsilon}{3} \right\}, \\
A^j_{\varepsilon,2} &= \left\{ \sup_{1 \le v \le c} \|\mathcal{Z}^{c^j}_{\frac{\cdot}{v}} - \mathcal{Z}^{c^j}_\cdot\|^{\varphi}_{\psi,M_2} \ge \frac{\varepsilon}{3} \right\}, \qquad (9) \\
A^j_{\varepsilon,3} &= \left\{ \sup_{1 \le v \le c} \|\Gamma_{\phi(c^{j-1})} o \Gamma^{-1}_{\phi(c^j)}(\mathcal{Z}^{c^j}_\cdot) - \mathcal{Z}^{c^j}_\cdot\|^{\varphi}_{\psi,M_2} \ge \frac{\varepsilon}{3} \right\}.
\end{aligned}$$

Since for every $\delta > 0$ and j sufficiently large it holds

$$0 \le \frac{\phi(c^j)}{\phi(c^{j-1})} = \frac{1}{\sqrt{c}}\sqrt{\frac{\mathcal{L}_2(c^j)}{\mathcal{L}_2(c^{j-1})}} \le \frac{1}{\sqrt{c}}(1+\delta),$$

we deduce from (7)–(9) and (iii) of Definition 2.1 that for c close enough to 1,

$$\mathbb{P}[\{Y_j \ge \varepsilon, \|\mathcal{Z}^{c^j}_\cdot\|^{\varphi}_{\psi,M_2} \le C\}] \le \mathbb{P}[\mathcal{Z}^{c^j}_\cdot \in A_{\varepsilon,c}],$$

where

$$A_{\varepsilon,c} = \left\{ g \in \mathcal{C}_m([0,1]) : \sup_{1 \le v \le c} \|g(\cdot) - g(\frac{\cdot}{v})\|^{\varphi}_{\psi,M_2} \ge \frac{\varepsilon}{3},\ \|g(\cdot)\|^{\varphi}_{\psi,M_2} \le C \right\}.$$

By virtue of the closedness of $A_{\varepsilon,c}$ in $\mathcal{B}^{\varphi,0}_{\psi,M_2}$ and Theorem 1.2, for every $\delta > 0$ and for j sufficiently large we have

$$\mathbb{P}[\mathcal{Z}^{c^j}_{\cdot} \in A_{\varepsilon,c}] \le \exp\left[-(\Lambda(A_{\varepsilon,c}) - \delta)\mathcal{L}_2(c^j)\right].$$

It remains to prove that we can choose $\delta > 0$ such that, for c close to 1, $\Lambda(A_{\varepsilon,c}) > 1 + 2\delta$.

Let $g \in A_{\varepsilon,c}$ be such that $\lambda(g) < +\infty$. Since $\|f\|^{\varphi}_{\psi,M_2} \le D(\|f\|^{\varphi} + \|f\|^{\psi})$ for f null in zero, we have

$$\begin{aligned} D\left(\sup_{1\le v\le c} \|g(\cdot) - g(\frac{\cdot}{v})\|^{\varphi} + \sup_{1\le v\le c} \|g(\cdot) - g(\frac{\cdot}{v})\|^{\psi} \right) &\ge \sup_{1\le v\le c} \|g(\cdot) - g(\frac{\cdot}{v})\|^{\varphi}_{\psi,M_2} \\ &\ge \frac{\varepsilon}{3}\,. \end{aligned}$$

So, there exists $s \in [0,1]$, $t \in [0,1]$ and $v \in [1,c]$ satisfying

$$\frac{\varepsilon}{6D}\theta(|t-s|) \le \left| \left[g(t) - g\Big(\frac{t}{v}\Big) \right] - \left[g(s) - g\Big(\frac{s}{v}\Big) \right] \right|,$$

where $\theta = \varphi$ or ψ. Since

$$\begin{aligned} \left| \left[g(t) - g\Big(\frac{t}{v}\Big) \right] - \left[g(s) - g\Big(\frac{s}{v}\Big) \right] \right| &= \left| \int_{s \vee \frac{t}{v}}^{t} \dot{g}(u)du - \int_{\frac{s}{v}}^{s \wedge \frac{t}{v}} \dot{g}(u)du \right| \\ &\le \left| \int_{s \vee \frac{t}{v}}^{t} \dot{g}(u)du \right| + \left| \int_{\frac{s}{v}}^{s \wedge \frac{t}{v}} \dot{g}(u)du \right|, \end{aligned}$$

we have

$$\frac{\varepsilon}{6D}\theta(|t-s|) \le \left| \int_{s \vee \frac{t}{v}}^{t} \dot{g}(u)du \right| + \left| \int_{\frac{s}{v}}^{s \wedge \frac{t}{v}} \dot{g}(u)du \right|. \tag{10}$$

Now, let $h \in \mathcal{H}$ be such that $\lambda(g) = \mu(h)$ and $S(h) = g$. Since $\|g\| \le \|f\|^{\varphi}_{\psi,M_2} \le C$, σ and b are locally bounded, we deduce that

$$\begin{aligned} \left| \int_s^t \dot{g}(u)du \right| &= \left| \int_s^t \sigma(g(s))\dot{h}_s\, ds + \int_s^t b(g(s))\, ds \right| \\ &\le \left| \int_s^t \sigma(g(s))\dot{h}_s\, ds \right| + \left| \int_s^t b(g(s))\, ds \right|. \end{aligned}$$

Thus

$$\left|\int_s^t \dot{g}(u)du\right| \;\le\; M_1\sqrt{|t-s|}\,\|h\|_{\mathcal{H}} + M_2\,|t-s|\,. \tag{11}$$

It follows from (10) and (11) that

$$\begin{aligned}\frac{\varepsilon}{6D}\theta(|t-s|) \le M_1\,\|h\|_{\mathcal{H}}\left(\left|t-s\vee\frac{t}{v}\right|^{\frac12}+\left|s\wedge\frac{t}{v}-\frac{s}{v}\right|^{\frac12}\right)\\ + M_2\left(\left|t-s\vee\frac{t}{v}\right|+\left|s\wedge\frac{t}{v}-\frac{s}{v}\right|\right).\end{aligned}$$

So

$$\|h\|_{\mathcal{H}}=\sqrt{2\lambda(g)}\ge\frac{\frac{\varepsilon}{6D}\theta(|t-s|)-M_2\left(\left|t-s\vee\frac{t}{v}\right|+\left|s\wedge\frac{t}{v}-\frac{s}{v}\right|\right)}{M_1\left(\left|t-s\vee\frac{t}{v}\right|^{\frac12}+\left|s\wedge\frac{t}{v}-\frac{s}{v}\right|^{\frac12}\right)}\,.$$

Therefore

$$\sqrt{2\lambda(g)}\ge\frac{\frac{\varepsilon}{6D}\theta(|t-s|)-M_2\left(\left|t-s\vee\frac{t}{v}\right|+\left|s\wedge\frac{t}{v}-\frac{s}{v}\right|\right)}{M_1\left(\left|t-s\vee\frac{t}{v}\right|^{\frac12}+\left|s\wedge\frac{t}{v}-\frac{s}{v}\right|^{\frac12}\right)}\,. \tag{12}$$

We consider two cases:

1) $s<\dfrac{t}{v}$. We have

$$\begin{aligned}\sqrt{2\lambda(g)}&\ge\frac{\frac{\varepsilon}{6D}\theta(t-s)-M_2\left(\left(t-\frac{t}{v}\right)+\left(s-\frac{s}{v}\right)\right)}{M_1\left(\left(t-\frac{t}{v}\right)^{\frac12}+\left(s-\frac{s}{v}\right)^{\frac12}\right)}\\ &\ge\frac{\frac{\varepsilon}{6D}\theta\left(t\left(1-\frac{s}{t}\right)\right)-M_2\left(\left(1-\frac{1}{v}\right)(t+s)\right)}{M_1\left(\left(1-\frac{1}{v}\right)^{\frac12}\left(t^{\frac12}+s^{\frac12}\right)\right)}\,.\end{aligned}$$

The increasing property of θ and the decreasing one of $t\longmapsto\dfrac{\theta(t)}{\sqrt{t}}$ lead to

$$
\begin{aligned}
\sqrt{2\lambda(g)} &\geq \frac{\frac{\varepsilon}{6D}\theta\left(t\Big(1-\frac{1}{v}\Big)\right)-2M_2\left(t\Big(1-\frac{1}{v}\Big)\right)}{2M_1\left(t\Big(1-\frac{1}{v}\Big)\right)^{\frac{1}{2}}} \\
&= \frac{\varepsilon\theta\left(t\Big(1-\frac{1}{v}\Big)\right)}{12DM_1\left(t\Big(1-\frac{1}{v}\Big)\right)^{\frac{1}{2}}}-\frac{M_2\left(t\Big(1-\frac{1}{v}\Big)\right)}{M_1\left(t\Big(1-\frac{1}{v}\Big)\right)^{\frac{1}{2}}} \\
&= \frac{\varepsilon\theta\left(t\Big(1-\frac{1}{v}\Big)\right)}{12DM_1\left(t\Big(1-\frac{1}{v}\Big)\right)^{\frac{1}{2}}}-\frac{M_2}{M_1}\left(t\Big(1-\frac{1}{v}\Big)\right)^{\frac{1}{2}} \\
&\geq \frac{\varepsilon\theta\Big(1-\frac{1}{c}\Big)}{12DM_1\Big(1-\frac{1}{c}\Big)^{\frac{1}{2}}}-\frac{M_2}{M_1}\Big(1-\frac{1}{c}\Big)^{\frac{1}{2}}.
\end{aligned}
$$

Since $\dfrac{\theta(t)}{\sqrt{t}}$ converges to infinity as t goes to zero, there exist $c_\varepsilon > 1$ and $\delta > 0$ such that if $1 < c < c_\varepsilon$ then

$$
\sqrt{2\lambda(g)} \geq \frac{\varepsilon\theta\Big(1-\frac{1}{c}\Big)}{12DM_1\Big(1-\frac{1}{c}\Big)^{\frac{1}{2}}}-\frac{M_2}{M_1}\Big(1-\frac{1}{c}\Big)^{\frac{1}{2}} \geq 1+2\delta.
$$

2) $s > \dfrac{t}{v}\cdot$ In view of (12) we have

$$
\begin{aligned}
\sqrt{2\lambda(g)} &\geq \frac{\frac{\varepsilon}{6D}\theta(|t-s|)-M_2\Big(|t-s|+\Big|\frac{t}{v}-\frac{s}{v}\Big|\Big)}{M_1\left(|t-s|^{\frac{1}{2}}+\Big|\frac{t}{v}-\frac{s}{v}\Big|^{\frac{1}{2}}\right)} \\
&\geq \frac{\frac{\varepsilon}{6D}\theta(|t-s|)-M_2\left(|t-s|\Big(1+\frac{1}{v}\Big)\right)}{M_1\left(|t-s|^{\frac{1}{2}}\Big(1+\Big(\frac{1}{v}\Big)^{\frac{1}{2}}\Big)\right)} \\
&\geq \frac{\frac{\varepsilon}{6D}\theta\left(t\Big(1-\frac{s}{t}\Big)\right)-M_2\left(t\Big(1-\frac{s}{t}\Big)\Big(1+\frac{1}{v}\Big)\right)}{2M_1\left(t\Big(1-\frac{s}{t}\Big)\right)^{\frac{1}{2}}}.
\end{aligned}
$$

Hence

$$\sqrt{2\lambda(g)} \geq \frac{\varepsilon\theta\left(t\left(1-\frac{s}{t}\right)\right)}{12DM_1\left(t\left(1-\frac{s}{t}\right)\right)^{\frac{1}{2}}} - \frac{M_2}{M_1}\left(t\left(1-\frac{s}{t}\right)\right)^{\frac{1}{2}}.$$

By virtue of the decreasing property of $\frac{\theta(t)}{\sqrt{t}}$, we have

$$\sqrt{2\lambda(g)} \geq \frac{\varepsilon\theta\left(1-\frac{1}{c}\right)}{12DM_1\left(1-\frac{1}{c}\right)^{\frac{1}{2}}} - \frac{M_2}{M_1}\left(1-\frac{1}{c}\right)^{\frac{1}{2}}.$$

Finally, letting $c \to 1$ yields the existence of $c_\varepsilon > 1$ and $\delta > 0$ such that for every $1 < c < c_\varepsilon$ we have $\Lambda(A_{\varepsilon,c}) > 1 + 2\delta$.

The same argument as in the proof of Lemma 2.1, and the Borel–Cantelli lemma lead to the conclusion of the proof. □

Proof of Proposition 2.1. Let $c > 1$ and $c^{j-1} \leq u \leq c^j$. We have

$$\begin{aligned} d(\mathcal{Z}^u_\cdot, \mathcal{K}_\lambda(1)) &\leq d(\mathcal{Z}^u_\cdot, \Gamma_{\phi(u)} o \Gamma^{-1}_{\phi(c^j)}(\mathcal{Z}^{c^j}_\cdot)) \\ &\quad + d(\mathcal{Z}^{c^j}_\cdot, \Gamma_{\phi(u)} o \Gamma^{-1}_{\phi(c^j)}(\mathcal{Z}^{c^j}_\cdot)) + d(\mathcal{Z}^{c^j}_\cdot, \mathcal{K}_\lambda(1)) \\ &= I_1 + I_2 + I_3. \end{aligned}$$

By virtue of Lemma 2.1, for j sufficiently large, $I_3 \leq \frac{\varepsilon}{3}$.

Since for every $\delta > 0$ and j sufficiently large,

$$0 \leq \frac{\phi(c^j)}{\phi(u)} \leq \frac{\phi(c^j)}{\phi(c^{j-1})} = \frac{1}{\sqrt{c}}\sqrt{\frac{\mathcal{L}_2(c^j)}{\mathcal{L}_2(c^{j-1})}} \leq \frac{1}{\sqrt{c}}(1+\delta),$$

and $\{\|\mathcal{Z}^{c^j}_\cdot\| : j \geq 0\}$ is bounded, we deduce that, for c close to 1 and j sufficiently large, $I_2 \leq \frac{\varepsilon}{3}$.

Finally, in view of Lemma 2.2, $I_1 \leq \frac{\varepsilon}{3}$ for j sufficiently large and c close enough to 1. □

Proposition 2.2. *For every $g \in \mathcal{K}_\lambda(1)$ and every $\varepsilon > 0$, there exists $c = c_\varepsilon > 1$ such that*

$$\mathbb{P}\left[\|\mathcal{Z}^{c^j}_\cdot - g\|^{\varphi}_{\psi,M_2} \leq \varepsilon \;\; i.o.\right] = 1.$$

Proof. . Let $g \in \mathcal{K}_\lambda(1)$ and $h \in \mathcal{H}$ be such that $S(h) = g$ and $\sqrt{2\lambda(g)} = \mu(h)$. In view of Theorem 1.1, for j sufficiently large and α sufficiently small we have

$$\mathbb{P}\left[\|\mathcal{Z}^{c^j}_\cdot - g\|^{\varphi}_{\psi,M_2} > \varepsilon, \; \left\|\frac{1}{\sqrt{\mathcal{L}_2(c^j)}} W^{c^j}_\cdot - h\right\| \leq \alpha\right] \leq \exp(-2\mathcal{L}_2(c^j)) = \frac{C}{j^2}\cdot$$

By Strassen's law for the Brownian motion in the uniform topology,

$$\mathbb{P}\left[\left\|\frac{1}{\sqrt{\mathcal{L}_2(c^j)}}W_{\cdot}^{c^j} - h\right\| \le \alpha \;\; i.o.\right] = 1. \tag{13}$$

Now, since $\sum_j \frac{1}{j^2}$ is finite, the Borel–Cantelli lemma and (13) lead to the conclusion. □

Remark. An analogue of Theorem 2.1 can be proved for Brownian functionals $F(W)$ with values in $\mathcal{B}^{\varphi,0}_{\psi,M_2}$ such that

(i) For every $a > 0$ the restriction of the functional F on $\{h \in \mathcal{H} : \mu(h) \le a\}$ is continuous with respect to the topology of $\mathcal{B}^{\varphi,0}_{\psi,M_2}$,

(ii) For every $R > 0$, $a > 0$, and $\rho > 0$, there exist $\eta > 0$ and $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0]$, every $h \in \mathcal{H}$ such that $\mu(h) \le a$ we have

$$\mathbb{P}\left[\|F(\varepsilon W) - F(h)\|^{\varphi}_{\psi,M_2} > \rho,\ \|\varepsilon W - h\| \le \eta\right] \le \exp\left(-\frac{R}{\varepsilon^2}\right),$$

(iii) There exists $\tau > 0$ such that for every $\varepsilon > 0$, $(u,t) \in \mathbb{R}^2_+$

$$F\left(\varepsilon W\left(\frac{\cdot}{u}\right)\right)(t) = \varepsilon^{\tau} F(W)\left(\frac{t}{u}\right),$$

(iv) For every $\varepsilon > 0$, there exists $c_\varepsilon > 1$ such that if $1 < c < c_\varepsilon$ then

$$\Lambda(A_{\varepsilon,c}) > 1,$$

where

$$A_{\varepsilon,c} = \left\{g \in \mathcal{C}_m([0,1]) : \sup_{1 \le v \le c} \|g(\cdot) - g(\frac{\cdot}{v})\|^{\varphi}_{\psi,M_2} \ge \varepsilon\sqrt{c^{\tau}}\right\}.$$

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