ACTA MATHEMATICA VIETNAMICA
Volume 27, Number 2, 2002, pp. 141-150

FIXED POINTS OF CONVEX-VALUED GENERALIZED UPPER HEMICONTINUOUS MAPS, REVISITED

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Abstract. We give new fixed point theorems for a generalized upper hemicontinuous multimap whose domain and range may have different topologies. These include known theorems which appeared in nearly 50 published works.

1. Introduction

The celebrated Kakutani fixed point theorem in 1941 for convex-valued upper semicontinuous multimaps initiated the study of fixed points of multimaps in the last six decades. The Kakutani theorem and its numerous generalizations were applied to game theory, mathematical economics, systems and control theory, coincidence theory, minimax theory, variational inequalities, convex analysis, and many equilibrium theorems. Moreover, the compactness, convexity, upper semicontinuity, selfmapness, and finite dimensionality related to the Kakutani theorem are all extended, and further, for the case of infinite dimension, it is known that the domain and range of the multimap may have different topologies. This is why the Kakutani theorem has so many generalizations; see [P7].

In our previous works [P5, 6], we unified, improved, and generalized a lot of fixed point theorems on Kakutani maps or acyclic maps defined on convex subsets of topological vector spaces. One of the main fixed point theorems in [P6] is concerned with convex-valued generalized upper hemicontinuous maps whose domains and ranges may have different topologies. After the author published the paper [P6], he became aware that there still appear a number of results of this kind of generalization.

In the present paper, we obtain some refined and generalized versions of the main theorems in [P5, 6] with slightly different proofs. We also show that some old or recent results of others are consequences of ours. Our results contain known theorems of Sehgal and Singh [SS], Roux and Singh [RS], Kim and Tan [KT], Ding and Tan [DT], Yuan, Smith, and Lou [YSL], and many others.

Received January 31, 2001; in revised form April 19, 2001.

The main content of this paper was presented at the Colloquium of the Hanoi Institute of Mathematics (Vietnam) on Feb. 4, 1999, and the author would like to express his gratitude to the kind hospitality of the Institute.

2. Preliminaries

A *convex space* X is a nonempty convex set with any topology that induces the Euclidean topology on the convex hulls of its finite subsets. A nonempty subset L of a convex space X is called a *c-compact set* if for each finite set $S \subset X$ there is a compact convex set $L_S \subset X$ such that $L \cup S \subset L_S$. Let $[x, L]$ denote the closed convex hull of $\{x\} \cup L$ in X, where $x \in X$.

Let E be a Hausdorff topological vector space (t.v.s.) and E^* its topological dual. A multimap or set-valued map (simply, map) $F : X \to 2^E \backslash \{\emptyset\}$ is said to be *upper hemicontinuous* (u.h.c.) if for each $h \in E^*$ and for any real α, the set $\{x \in X : \sup \operatorname{Re} h(Fx) < \alpha\}$ is open in X.

Let $cc(E)$ denote the set of nonempty closed convex subsets of E and $kc(E)$ the set of nonempty compact convex subsets of E. Bd, Int, and $\overline{}$ denote the boundary, interior, and closure, resp., with respect to E.

Let $X \subset E$ and $x \in E$. The *inward* and *outward sets* of X at x, $I_X(x)$ and $O_X(x)$, are defined as follows:

$$I_X(x) = x + \bigcup_{r>0} r(X - x), \quad O_X(x) = x + \bigcup_{r<0} r(X - x).$$

For $p \in \{\operatorname{Re} h : h \in E^*\}$ and $U, V \subset E$, let

$$d_p(U, V) = \inf\{|p(u - v)| \ : \ u \in U, \ v \in V\}.$$

Recall that a real function $g : X \to \mathbb{R}$ on a topological space X is *lower* [resp. *upper*] *semicontinuous* (l.s.c.) [resp. u.s.c.] if $\{x \in X : gx > r\}$ [resp. $\{x \in X : gx < r\}$] is open for each $r \in \mathbb{R}$. If X is a convex set, then g is *quasiconcave* [resp. *quasiconvex*] if $\{x \in X : gx > r\}$ [resp. $\{x \in X : gx < r\}$] is convex for each $r \in \mathbb{R}$.

In this paper all topological spaces are assumed to be Hausdorff.

We use the following form of the existence theorem of maximizable quasiconcave functions on convex spaces due to Park and Bae [PB].

Theorem 0. *Let X be a convex space and $\hat{X}$ the set of all u.s.c. quasiconcave real functions on X. Suppose that*

(0.1) *for each $x \in X$, Sx is a nonempty convex subset of $\hat{X}$;*

(0.2) *for each $g \in \hat{X}$, $S^{-1}g$ is compactly open in X; and*

(0.3) *there exists a c-compact set $L \subset X$ and a nonempty compact set $K \subset X$ such that for every $x \in X \backslash K$ and $g \in Sx$, $gx < \max g[x, L]$.*

Then there exist an $\overline{x} \in K$ and a $g \in S\overline{x}$ such that $g\overline{x} = \max g(X)$.

3. Main results

We begin with the following generalization of [P5, Corollary 3.1]:

Theorem 1. *Let X be a convex space, L a c-compact subset of X, K a nonempty subset of X, E a t.v.s. containing X as a subset, and $F : X \to 2^E\backslash\{\emptyset\}$. Suppose that, for each $p \in \{\mathrm{Re}\, h : h \in E^*\}$,*

(1.0) *$p|_X$ is continuous on X;*

(1.1) *$X_p = \{x \in X : \sup p(Fx) \geq p(x)\}$ is compactly closed in X;*

(1.2) *$x \in K$ and $p(x) = \max p(X)$ implies $x \in X_p$; and*

(1.3) *$x \in X\backslash K$ and $p(x) = \max p[x, L]$ implies $x \in X_p$.*

Then there exists an $x \in \bigcap\{X_p : p \in \{\mathrm{Re}\, h : h \in E^*\}\}$.

Proof. Note that $\{(\mathrm{Re}\, h)|_X : h \in E^*\} \subset \hat{X}$ by (1.0). For each $x \in X$, define

$$Sx = \{p|_X : p \in \{\mathrm{Re}\, h : h \in E^*\} \text{ and } \sup p(Fx) < p(x)\}.$$

Then Sx is a convex subset of $\hat{X}$. Suppose that $Sx \neq \emptyset$ for each $x \in X$; that is, for each $x \in X$, there exists a $p \in \{\mathrm{Re}\, h : h \in E^*\}$ such that $x \notin X_p$. Note that, for each $g \in \hat{X}$,

$$S^{-1}g = \{x \in X : \sup p(Fx) < p(x)\} = X\backslash X_p$$

if $g = p|_X$ for some $p \in \{\mathrm{Re}\, h : h \in E^*\}$ and

$$S^{-1}g = \emptyset \quad \text{if} \quad g \notin \{(\mathrm{Re}\, h)|_X : h \in E^*\}.$$

Then $S^{-1}g$ is compactly open in X for each $g \in \hat{X}$ by (1.1). Therefore, (0.1) and (0.2) are satisfied. Further, (1.3) implies (0.3). In fact, for every $x \in X\backslash K$ and $p \in \{\mathrm{Re}\, h : h \in E^*\}$ satisfying $\sup p(Fx) < p(x)$, we have $x \notin X_p$. Therefore, $p(x) < \max p[x, L]$ by (1.3). Now, by applying Theorem 0, there exist an $\overline{x} \in K$ and an $h \in E^*$ such that $p = \mathrm{Re}\, h$, $p|_X \in S\overline{x}$ and $p(\overline{x}) = \max p(X)$. Note that $p|_X \in S\overline{x}$ implies $\overline{x} \notin X_p$. This contradicts (1.2). □

Remarks. 1. As we noted in [P6], in Theorem 1, we do not require any concrete connection between topologies of X and E except

(1.0) $(\mathrm{Re}\, h)|_X \in \hat{X}$ (that is, $(\mathrm{Re}\, h)|_X$ is continuous on X) for all $h \in E^*$.

In order to assure the continuity of $(\mathrm{Re}\, h)|_X$ for all $h \in E^*$, it is sufficient to assume that

(i) as a convex space, X has any topology finer than the relative weak topology with respect to E, and

(ii) E has any topology finer than its weak topology.

This is why there have appeared fixed point theorems on maps whose domains and ranges have different topologies.

2. If F is u.h.c. on each nonempty compact subset C of X, then F satisfies the "continuity" condition (1.1) for all $p \in \{\mathrm{Re}\, h : h \in E^*\}$, but not conversely; see [P5]. Any map F satisfying (1.1) can be said to be *generalized u.h.c.*

3. The "boundary" condition (1.2) is equivalent to the following:

(1.2)$'$ $x \in K$ and $p(x) = \max p(\overline{I}_X(x))$ implies $x \in X_p$.

In fact, $p(x) = \max p(X)$ is equivalent to $p(x) = \max p(\overline{I}_X(x))$.

Let X be a nonempty convex subset of a vector space E. Following Fan [F], the *algebraic boundary* $\delta_E(X)$ of X in E is the set of all $x \in X$ for which there exists $y \in E$ such that $x + ry \notin X$ for all $r > 0$. If E is a t.v.s., the *topological boundary* Bd $X = \text{Bd}_E X$ of X is the complement of $\text{Int}_E X$ in $\overline{X}$. It is known that $\delta_E(X) \subset$ Bd X and in general $\delta_E(X) \neq$ Bd X; see [YSL].

Moreover, the "boundary" condition (1.2)$'$ is equivalent to the following:

(1.2)$''$ $x \in K \cap \delta_E(X)$ and $p(x) = \max p(\overline{I}_X(x))$ implies $x \in X_p$.

In fact, if $x \in K \backslash \delta_E(X)$ and $p(x) = \max p(\overline{I}_X(x))$, then for any $y \in E$, there exists an $r > 0$ such that $x + ry \in X$ and hence $p(x) \geq p(x + ry)$, which readily implies $p(y) \leq 0$ or $p = 0$. This contradicts the arbitrariness of p. Therefore, (1.2) is trivially satisfied.

4. The "coercivity" or "compactness" condition (1.3) is equivalent to the following:

(1.3)$'$ $x \in X \backslash K$ and $p(x) = \max p(\overline{I}_L(x))$ implies $x \in X_p$.

In fact, $p(x) = \max p[x, L]$ is equivalent to $p(x) = \max p(\overline{I}_L(x))$. Note that if X itself is compact (that is, if $X = K$), then (1.3)$'$ holds trivially.

From Theorem 1, we have the following basic fixed point theorem:

Theorem 2. *Under the hypothesis of Theorem 1, further suppose that either*

(A) *E^* separates points of E and $F : X \to kc(E)$; or*

(B) *E is locally convex and $F : X \to cc(E)$.*

Then there exists an $x \in X$ such that $x \in Fx$.

Proof. By Theorem 1, there exists an $x \in \bigcap\{X_p : p \in \{\text{Re}\, h : h \in E^*\}\}$. Suppose that $x \notin Fx$. Then under the assumptions (A) or (B), the standard separation theorems on a t.v.s. assure the existence of a $p \in \{\text{Re}\, h : h \in E^*\}$ satisfying $\inf p(Fx) > p(x)$; that is, $x \notin X_p$, which is a contradiction. □

Remarks. 1. Using the method in [P5], we can reformulate Theorem 2 to a coincidence theorem and an existence theorem for critical points or zeros of multimaps.

2. Note that $x \in Fx$ if and only if $x \in \bigcap\{X_p : p \in \{\text{Re}\, h : h \in E^*\}\}$. This is a useful information on the location of a fixed point.

From Theorem 2, we obtain the following more visualizable geometric form of a fixed point and surjectivity theorem, which generalizes [P5, Theorem 6] and refines [P6, Theorem 2]:

Theorem 3. *Let X be a convex space, L a c-compact subset of X, K a nonempty compact subset of X, E a t.v.s. containing X as a subset, and F a map satisfying either*

(A) *E^* separates points of E and $F : X \to kc(E)$, or*

(B) *E is locally convex and $F : X \to cc(E)$.*

(I) *Suppose that for each $p \in \{\operatorname{Re} h : h \in E^*\}$,*

(1.0) *$p|_X$ is continuous on X;*

(3.1) *$X_p = \{x \in X : \inf p(Fx) \leq p(x)\}$ is compactly closed in X;*

(3.2) *$d_p(Fx, \overline{I}_X(x)) = 0$ for every $x \in K \cap \delta_E(X)$; and*

(3.3) *$d_p(Fx, \overline{I}_L(x)) = 0$ for every $x \in X \backslash K$.*

Then there exists an $x \in X$ such that $x \in Fx$.

(II) *Suppose that for each $p \in \{\operatorname{Re} h : h \in E^*\}$,*

(1.0) *$p|_X$ is continuous on X;*

(3.1)$'$ *$X_p = \{x \in X : \sup p(Fx) \geq p(x)\}$ is compactly closed in X;*

(3.2)$'$ *$d_p(Fx, \overline{O}_X(x)) = 0$ for every $x \in K \cap \delta_E(X)$; and*

(3.3)$'$ *$d_p(Fx, \overline{O}_L(x)) = 0$ for every $x \in X \backslash K$.*

Then there exists an $x \in X$ such that $x \in Fx$. Further, if F is u.h.c., then $F(X) \supset X$.

Proof. In order to use Theorem 2, we first show that (3.2) $\Longrightarrow$ (1.2). Let $x \in K \cap \delta_E(X)$ such that $p(x) = \max p(X)$. Suppose that $\inf p(Fx) > p(x)$. Then for any $v \in Fx$, $u \in X$, $z = x + r(u - x) \in I_X(x)$, and $r > 0$, we have

$$|p(v - z)| = p(v - x) + rp(x - u) \geq p(v - x) = p(v) - p(x)$$

and hence

$$d_p(Fx, \overline{I}_X(x)) = d_p(Fx, I_X(x)) \geq \inf p(Fx) - p(x) > 0.$$

This contradicts (3.2). Therefore, we should have $\inf p(Fx) \leq p(x)$ or $x \in X_p$. Hence, (1.2)$''$ holds.

Similarly, we can show that (3.3) $\Longrightarrow$ (1.3). Note that "(3.1) holds for all p" is equivalent to "(1.1) = (3.1)$'$ holds for all p". Therefore, all of the requirements of Theorem 2 are satisfied. Now by Theorem 2, Case (I) follows.

For (II) consider $2x - Fx$ instead of Fx in (I) as in [P5], we can conclude that F has a fixed point. For the surjectivity result, let $y \in X$. Consider $x \mapsto Fx + x - y$ instead of Fx and $[y, L]$ instead of L in Case (II). Then there exists an $x \in X$ such that $x \in Fx + x - y$; that is, $y \in Fx$. This completes our proof. $\square$

Remarks. 1. (3.1) and (3.1)$'$ are actually the same.

2. Note that the map $x \mapsto Fx + x - y$ in the proof of Case (II) is u.h.c.

3. Note that if K is a weakly compact convex subset of a t.v.s. (E,τ) on which E^* separates points, then a continuous map $f:(K,\tau)\to(K,\tau)$ may have no fixed point. See Kakutani [K, Theorem 1]. In this case, $K_p=\{x\in K : p(fx)\le p(x)\}$ in Theorem 3(I) may not be closed for some $p\in\{\operatorname{Re} h : h\in E^*\}$.

We give some of the simplest examples of Theorem 3.

Examples. 1. [P2, Example 1]: Let $X=K=[0,1]$ in $E=\mathbb{R}$, $fx=x$ for $x\in X\backslash(1/3,2/3)$, and $fx=1$ for $x\in(1/3,2/3)$. Then the set $\{x\in X : p(x)\le p(fx)\}$ is closed for all $p\in E^*$. Note that $f:X\to X$ is not continuous, but has a fixed point by Theorem 3(I).

2. [P2, Example 2]: Let $X=K=[0,1]$ and $E=\mathbb{R}$. For a given $c\in(0,1)$, let $f:X\to X$ be a function such that $fx>x$ for $x<c$ and $fx<x$ for $x>c$. Then c is the only fixed point of f if and only if the set $\{x\in X : p(x)\le p(fx)\}$ is closed for all $p\in E^*$.

3. Let $X=(0,1]$, $K=L=[1/2,1]$, $E=\mathbb{R}$, and $f:X\to X$ be given by $fx=(x+1)/2$. Then $\{x\in X : p(x)\le p(fx)\}$ is closed for all $p\in E^*$. Note that

$$fx=\frac{x+1}{2}\in\overline{I}_L(x)=[x,\infty)\quad\text{for all}\quad x\in(0,\frac{1}{2})=X\backslash K.$$

Therefore, Theorem 3(I) works.

4. Particular results

(1) A particular form of Theorem 3 for the real case is given in [P5, Theorem 6], which unifies, improves, and generalizes historically well-known fixed point theorems published in nearly 40 papers; see the diagram in [P5, p.205].

Now we add some more known consequences of Theorem 3 as follows:

(2) Knaster, Kuratowski, and Mazurkiewicz [KKM, p.136]: If $f:B^n\to\mathbb{R}^n$ is a continuous map such that f maps $S^{n-1}=\operatorname{Bd}B^n$ back into B^n, then f has a fixed point.

This is the origin of the so-called Rothe boundary condition.

(3) Sehgal and Singh [SS, Corollary 2]: Let K be a convex and weakly compact subset of a real locally convex t.v.s. E and $f:K\to E$ a strongly continuous map such that $f(\operatorname{Bd}K)\subset K$. Then f has a fixed point.

Note that f satisfies (3.1).

(4) Deimling [D, p.93]: Let X be a nonempty closed bounded convex subset of a reflexive Banach space E, and $f:X\to X$ a weakly sequentially continuous map. Then f has a fixed point.

Equip E with the weak topology.

(5) Arino, Gautier, and Penot [AGP, Theorem 1]: Let X be a nonempty weakly compact convex subset of a metrizable locally convex t.v.s. E, and $f:X\to X$ a weakly sequentially continuous. Then f has a fixed point.

Note that f is weakly continuous.

(6) Roux and Singh [RS, Theorem 5]: Let (E, τ) be a t.v.s. on which E^* separates points, w the weak topology of E, K a nonempty τ-compact convex subset of E, and $f : (K, \tau) \to (E, w)$ a continuous inward map. Then f has a fixed point.

Here, inward means $fx \in I_K(x)$ for all $x \in K$.

(7) Roux and Singh [RS, Theorem 6]: Let (E, τ) be a t.v.s. on which E^* separates points, w the weak topology of E, K a nonempty w-compact convex subset of E, and $f : (K, w) \to (E, \tau)$ a continuous inward map. Then f has a fixed point.

This contains some results in Sehgal, Singh, and Whitfield [SSW].

(8) Park [P2, Theorem]: Let X be a nonempty compact convex subset of a t.v.s. E on which E^* separates points, and $f : X \to E$ a weakly inward [outward] map such that

$$\{x \in X : \operatorname{Re} h(x) < \operatorname{Re} h(fx)\}$$

is open for all $h \in E^*$. Then f has a fixed point.

Here, weakly inward means $fx \in \overline{I}_X(x)$ for all $x \in X$.

(9) Kim and Tan [KT, Theorem 2]: Let X be a nonempty paracompact bounded convex subset of a locally convex t.v.s. E, K a nonempty compact subset of X, and $F : X \to cc(E)$ an u.h.c. map satisfying the following:

(a) for each $x \in X$, $Fx \cap \overline{I}_X(x) \neq \emptyset$; and

(b) for each $x \in X \backslash K$, $y \in X$ and $h \in E^*$, if $\operatorname{Re} h(y) > \inf \operatorname{Re} h(Fy)$, then $\operatorname{Re} h(y) \leq \operatorname{Re} h(x)$.

Then there exists an $\hat{x} \in X$ such that $\hat{x} \in F\hat{x}$.

Choose a point $y \in X$ and let $L = \{y\}$. If we replace (b) by

(b)$'$ for each $x \in X \backslash K$ and $h \in E^*$, $\operatorname{Re} h(x) < \inf \operatorname{Re} h(Fx)$ implies $\operatorname{Re} h(y) > \operatorname{Re} h(x)$.

Then (b)$'$ implies (1.3), which is equivalent to (3.3). See Jiang [J]. Therefore, in this case, the result follows from Theorem 3(B) for Case (I).

Actually, Kim and Tan based their argument on [KT, Corollary 2], which is quite different from our results.

(10) Kim and Tan [KT, Theorem 4]: Let X be a nonempty convex subset of a normed vector space E, K a nonempty compact subset of X, and $F : X \to cc(E)$ an u.h.c. map satisfying (a) and (b) in (9). Then there exists an $\hat{x} \in X$ such that $\hat{x} \in F\hat{x}$.

In this result, if we replace (b) by (b)$'$, then it also follows from Theorem 3(B) for Case (I). Further, note that in a normed vector space E, for any $A, B \in cc(E)$, $d(A, B) = 0$ if and only if $d_p(A, B) = 0$ for all $p \in \{\operatorname{Re} h : h \in E^*\}$, where d denotes the induced metric. See Jiang [J].

(11) Ding and Tan [DT2, Corollary 3]: Let X be a nonempty convex subset of a normed vector space E, and $G : X \to kc(E)$ continuous on each nonempty compact subset C of X. Suppose that there exist a nonempty compact convex subset L of X and a nonempty compact subset K of X such that

(i) for each $y \in K$, $Gy \cap \overline{I}_X(y) \neq \emptyset$ [resp. $Gy \cap \overline{O}_X(y) \neq \emptyset$];

(ii) for each $y \in X \backslash K$, $Gy \cap \overline{I}_L(y) \neq \emptyset$ [resp. $Gy \cap \overline{O}_L(y) \neq \emptyset$].

Then G has a fixed point.

This result contains Browder [Br, Corollaries 2 and 2′] and Shih and Tan [ST, Corollary 1].

(12) Yuan, Smith, and Lou [YSL]: Using Theorem 0 due to Park and Bae [PB], they proved some coincidence theorems for u.h.c. multimaps in t.v.s., and, as applications, coincidence theorems and several matching theorems for closed coverings of convex sets were derived.

Most of results in [YSL] are consequences or slight variations of earlier works of Park [P3-6]. Interested readers may compare these results. Especially, Theorems 3, 3′ and Corollaries 4, 4′, 6, 6′ in [YSL] are consequences of Theorem 3 of this paper.

Final Remark. The major particular forms of Theorem 3 can be adequately summarized by the following enlarged version of the diagrams given in [P1,4,5]. For the references which are not appeared in the end of this paper, see [P5,7].

In the diagram, the class I stands for that of Euclidean spaces, II for normed vector spaces, III for locally convex Hausdorff topological vector spaces, and IV for topological vector spaces having sufficiently many linear functionals. Moreover, f stands for single-valued maps and F for set-valued maps; and K stands for a nonempty compact convex subset of a space E, and X for a nonempty convex subset of E satisfying certain coercivity conditions with respect to $F : X \to 2^E$ with certain boundary conditions.

In fact, Theorem 3 contains all of the fixed point theorems in the diagram. Note that, in the diagram, Bohl's theorem [Bo] in 1904 was well-known to be equivalent to Brouwer's theorem in 1912.

E	$F : K \longrightarrow K$		$F : K \longrightarrow 2^K$	
I	Brouwer	1912	Kakutani	1941
II	Schauder	1927, 1930	Bohnenblust and Karlin	1950
III	Tychonoff	1935	Fan Glicksberg	1952 1952
IV	Fan	1964	Granas and Liu	1986

	$f : K \longrightarrow E$		$F : K \longrightarrow 2^E$	
I	Bohl	1904		
	Knaster, Kuratowski			
	and Mazurkiewicz	1929		
II	Rothe	1938		
	Halpern	1965	Browder	1968
	Fan	1969	Fan	1969
	Reich	1972	Glebov	1969
	Sehgal and Singh	1983	Halpern	1970
III		Cellina	1970	
			Reich	1972, 1978
			Cornet	1975
			Lasry and Robert	1975
			Simons	1986
	Halpern and Bergman	1968	Granas and Liu	1986
IV	Kaczynski	1983	Park	1988, 1991
	Roux and Singh	1989		
	Sehgal, Singh			
	and Whitfield	1990		

		$F : X \longrightarrow 2^E$	
II		Ding and Tan	1992
		Fan	1984
III		Shih and Tan	1987, 1988
		Jiang	1988
IV		Park	1992, 1993
		Yuan, Smith, and Lou	1998

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