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GEOMETRIC SOLUTIONS OF NONLINEAR SECOND ORDER HYPERBOLIC EQUATIONS

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Dedicated to Tran Duc Van on the occasion of his fiftieth birthday

Abstract. We will consider the Cauchy problem for nonlinear hyperbolic equations of second order with smooth data. It is well known that the Cauchy problem has a smooth solution in a neighbourhood of the initial curve. But it might fail to admit a smooth solution in the whole space. This means that singularities appear generally in finite time. We are interested in the global theory. Therefore our problem is how to extend the solution after the appearance of singularities. For this purpose, we will first lift the solution surface into cotangent space so that the singularities would disappear, and we will construct globally a geometric solution there. Next we will project it to the base space. In this procedure we will meet the singularities of smooth mappings.

1. Introduction

In this paper we will consider the Cauchy problem for nonlinear second order partial differential equations of hyperbolic type. It is well known that the Cauchy problem with smooth data has a smooth solution in a neighbourhood of the initial curve, and that singularities appear generally in finite time. But, even if singularities may appear in solutions, physical phenomena can exist with the singularities. Moreover it seems to us that the singularities might cause various kinds of interesting phenomena. We are interested in the global theory for the above Cauchy problem. Therefore we would like to extend the solutions beyond their singularities. The best method to solve this is to construct exact solutions in neighbourhoods of singularities. In §2, we will study the method of integration for second order partial differential equations. The principal idea of the method is to express the solution surface by a family of smooth curves. Historically, it is G. Darboux [3] and E. Goursat [5, 6] who investigated this subject for the first time. Their principal idea is to reduce the solvability of the above problem to the integration of first order partial differential equations. In [25], we have

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considered the meaning of "integrability in the sense of Darboux and Goursat" of second order partial differential equations "from our point of view". But, as the integrability condition of Darboux and Goursat is strong, many important examples are not contained in the class of equations which are integrable in that sense. The first purpose of this note is how to construct the solutions without their integrability condition. Then the family of characteristic strips is obtained as solutions of a certain nonlinear system of first order partial differential equations. This topic will be discussed in §2. In §3 we will apply that result to certain nonlinear hyperbolic equations, and we will construct geometric solutions of the equations in cotangent space. The second purpose is how to extend the solutions beyond their singularities. This will be discussed in §3 also. In §4 we will treat a certain system of conservation laws. The authors would like to express their sincere gratitude to Kazuhiko Aomoto for his fruitful comments and criticism.

2. Integration of Monge-Ampère equations

In this section we will study the method of integration of second order nonlinear partial differential equations, especially of Monge-Ampère type as follows:

$$F(x,y,z,p,q,r,s,t) = Ar + Bs + Ct + D(rt - s^2) - E = 0 \tag{2.1}$$

where $p = \partial z/\partial x$, $q = \partial z/\partial y$, $r = \partial^2 z/\partial x^2$, $s = \partial^2 z/\partial x\partial y$, and $t = \partial^2 z/\partial y^2$. Here we assume that A, B, C, D and E are real smooth functions of (x,y,z,p,q). Partial differential equations of second order which appear in physics and geometry are often written in the above form.

Before beginning our discussion, we will briefly explain some classical notions from our point of view so that our manuscript would become self-contained. Equation (2.1) is regarded as a smooth surface defined in eight dimensional space $\mathbb{R}^8 = \{(x,y,z,p,q,r,s,t)\}$. As p and q are first order derivatives of $z = z(x,y)$, we put the relation $dz = pdx + qdy$. Moreover, as r, s and t are second order derivatives of $z = z(x,y)$, we introduce the relations $dp = rdx + sdy$ and $dq = sdx + tdy$. Let us call $\{\, dz = pdx + qdy,\ dp = rdx + sdy,\ dq = sdx + tdy \,\}$ the "contact structure of second order". We define a solution of (2.1) as a maximal integral submanifold of the contact structure of second order in the surface $\{(x,y,z,p,q,r,s,t) \in \mathbb{R}^8; F(x,y,z,p,q,r,s,t) = 0\}$. We will use this geometric formulation to solve equation (2.1) in exact form. Let

$$\Gamma :\ (x,y,z,p,q) = \big(x(\xi), y(\xi), z(\xi), p(\xi), q(\xi)\big), \quad \xi \in \mathbb{R}^1,$$

be a smooth curve in $\mathbb{R}^5$. It is called a "strip" if it satisfies the following:

$$\frac{dz}{d\xi}(\xi) = p(\xi)\frac{dx}{d\xi}(\xi) + q(\xi)\frac{dy}{d\xi}(\xi). \tag{2.2}$$

Let Γ be any strip in $\mathbb{R}^5$, and consider equation (2.1) in an open neighbourhood of Γ. As a "characteristic" strip means that one can not determine the values of the second order derivatives of solution along the strip, we have

$$\det \begin{bmatrix} F_r & F_s & F_t \\ \dot{x} & \dot{y} & 0 \\ 0 & \dot{x} & \dot{y} \end{bmatrix} = F_t \dot{x}^2 - F_s \dot{x}\dot{y} + F_r \dot{y}^2 = 0 \tag{2.3}$$

where $F_t = \partial F/\partial t$, $F_s = \partial F/\partial s$, $F_r = \partial F/\partial r$, $\dot{x} = dx/d\xi$ and $\dot{y} = dy/d\xi$. Here we substitute the relations $\dot{p} = r\dot{x} + s\dot{y}$ and $\dot{q} = s\dot{x} + t\dot{y}$ into (2.3), then we get the following:

Definition 2.1. A curve Γ in $\mathbb{R}^5 = \{(x, y, z, p, q)\}$ is a "characteristic strip" if it satisfies (2.2) and

$$A\dot{y}^2 - B\dot{x}\dot{y} + C\dot{x}^2 + D(\dot{p}\dot{x} + \dot{q}\dot{y}) = 0. \tag{2.4}$$

A strip Γ is called "non-characteristic" when it does not satisfy (2.4). Next we will give the definition of "hyperbolicity". Denote the discriminant of (2.3) by Δ. Then it follows that

$$\Delta = F_s^2 - 4F_r F_t = B^2 - 4(AC + DE).$$

If $\Delta < 0$, equation (2.1) is called elliptic. If $\Delta > 0$, equation (2.1) is hyperbolic. In this note, we will treat the equations of hyperbolic type. Let λ_1 and λ_2 be the solutions of $\lambda^2 + B\lambda + (AC + DE) = 0$. Then, in the case where $D \neq 0$, a characteristic strip satisfies the following equations (see [3] and [6]):

$$\begin{cases} dz - pdx - qdy & = 0, \\ Ddp + Cdx + \lambda_1 dy & = 0, \\ Ddq + \lambda_2 dx + Ady & = 0, \end{cases} \tag{2.5}$$

or

$$\begin{cases} dz - pdx - qdy & = 0, \\ Ddp + Cdx + \lambda_2 dy & = 0, \\ Ddq + \lambda_1 dx + Ady & = 0. \end{cases} \tag{2.6}$$

Let us denote $\omega_0 = dz - pdx - qdy$, $\omega_1 = Ddp + Cdx + \lambda_1 dy$ and $\omega_2 = Ddq + \lambda_2 dx + Ady$. Exchanging λ_1 and λ_2 in ω_1 and ω_2, we define ϖ_1 and ϖ_2 by $\varpi_1 = Ddp + Cdx + \lambda_2 dy$ and $\varpi_2 = Ddq + \lambda_1 dx + Ady$. Take an exterior product of ω_1 and ω_2, and also of ϖ_1 and ϖ_2. Substitute into their product the relations of the contact structure $\omega_0 = 0$, $dp = rdx + sdy$ and $dq = sdx + tdy$. Then we get

$$\omega_1 \wedge \omega_2 = \varpi_1 \wedge \varpi_2 = D\left\{Ar + Bs + Ct + D(rt - s^2) - E\right\} dx \wedge dy. \tag{2.7}$$

In the above we have assumed $D \neq 0$, though it is not essential for our discussion. The key point is to represent equation (2.1) as a product of one forms. For example, we will consider in §3 and §4 a certain case where $D \equiv 0$. It will be shown that, though the above decomposition might be a small idea, it would effectively work to solve equation (2.1) in exact form. In a space whose dimension is greater than two, it is generally impossible to do so. Here we recall briefly the characteristic method developed principally by G. Darboux [3] and E. Goursat [5, 6] "from our point of view", because this would help us to explain our problem. The idea of Darboux and Goursat is how to reduce the solvability of (2.1) to the integration of first order partial differential equations.

Definition 2.2. A function $V = V(x, y, z, p, q)$ is called a "first integral" of $\{\omega_0, \omega_1, \omega_2\}$ if $dV \equiv 0 \mod \{\omega_0, \omega_1, \omega_2\}$.

Proposition 2.1. *Assume that $\lambda_1 \neq \lambda_2$ and $D \neq 0$, and that $\{\omega_0, \omega_1, \omega_2\}$, or $\{\omega_0, \varpi_1, \varpi_2\}$, has two independent first integrals $\{u, v\}$. Then there exists a function $k = k(x, y, z, p, q) \neq 0$ satisfying*

$$du \wedge dv = k\, \omega_1 \wedge \omega_2 = kD\{Ar + Bs + Ct + D(rt - s^2) - E\}dx \wedge dy. \tag{2.8}$$

If $\{\omega_0, \omega_1, \omega_2\}$, or $\{\omega_0, \varpi_1, \varpi_2\}$, has at least two independent first integrals, equation (2.1) is called *integrable in the sense of Monge*. But, if we may follow G. Darboux (p. 263 of [3]), it seems to us that we had better call it *integrable in the sense of Darboux*. Moreover, as E. Goursat had profoundly studied equations (2.1) satisfying the above condition, we would like to add the name of Goursat. By these reasons, we will call equations (2.1) with two independent first integrals *integrable in the sense of Darboux and Goursat*. Then the representation (2.8) gives the characterization of "Monge-Ampère equations which are integrable in the sense of Darboux and Goursat". Concerning the global existence of first integrals for certain Monge-Ampère equations, see [9].

Next we advance to the integration of the Cauchy problem for (2.1). Let $\{u, v\}$ be two independent first integrals of $\{\omega_0, \omega_1, \omega_2\}$. For any function g of two variables whose gradient does not vanish, $g(u, v) = 0$ is called an "intermediate integral" of (2.1). Let C_0 be an initial strip defined in $\mathbb{R}^5 = \{(x, y, z, p, q)\}$. If the strip C_0 is not characteristic, we can find an "intermediate integral" $g(u, v)$ which vanishes on C_0. Here we put $g(u, v) = f(x, y, z, p, q)$. The Cauchy problem for (2.1) satisfying the initial condition C_0 is to look for a solution $z = z(x, y)$ of (2.1) which contains the strip C_0, i.e., the two dimensional surface $\{(x, y, z(x, y), \partial z/\partial x(x, y), \partial z/\partial y(x, y))\}$ in $\mathbb{R}^5$ contains the strip C_0. The representation (2.8) assures that, as $du \wedge dv = 0$ on the surface $g(u, v) = 0$, a smooth solution of $f(x, y, z, \partial z/\partial x, \partial z/\partial y) = 0$ satisfies equation (2.1). Therefore we get the following:

Theorem 2.1. ([3], [5, 6]) *Assume that the initial strip C_0 is not characteristic. Then a function $z = z(x, y)$ is a solution of the Cauchy problem for* (2.1) *with the initial condition C_0 if and only if it is a solution of $f(x, y, z, \partial z/\partial x, \partial z/\partial y) = 0$ satisfying the same initial condition C_0.*

Now we will advance to the principal subject of this note which is to study the method of integration of (2.1) in the case where neither $\{\omega_0, \omega_1, \omega_2\}$ nor $\{\omega_0, \varpi_1, \varpi_2\}$ has not two independent first integrals. We start from the point at which equation (2.1) is represented as a product of one forms as (2.8). We suppose $D \neq 0$ for simplicity, though it is not indispensable for our study. The essential condition for our following discussion is $\Delta \neq 0$.

Let us pay attention to the property that the left hand side of (2.8) is a differential form of second order defined in $\mathbb{R}^5 = \{(x, y, z, p, q)\}$. This suggests us to introduce a notion of "geometric solution" as follows:

Definition 2.3 A regular geometric solution of (2.1) is a submanifold of dimension 2 defined in $\mathbb{R}^5 = \{(x, y, z, p, q)\}$ on which it holds that $dz = pdx + qdy$ and $\omega_1 \wedge \omega_2 = 0$.

Remark. In the above definition, we have added "regular" to "geometric solution". This means that we will soon introduce a "singular" geometric solution whose dimension may depend on each point. The problem to construct the geometric solution is similar to the Pfaffian problem. A difference between the classical Pfaffian problem and the above one is that we consider it in C^∞-space. Therefore we need some condition which is corresponding to "hyperbolicity". In [27], D. V. Tunitskii introduced the notion of "multi-valued solution" and proved the global existence of such a solution and its uniqueness. But we cannot follow some part of his discussion. For example, the author mentions in the first line of the first page that the value of the fixed number k belongs to the set $\{1, 2, ..., \infty\}$, and he develops his theory in C^k-space. If k is equal to 1, it seems to us that it would be difficult to construct multi-valued solutions.

Our problem is to find a "submanifold on which $\omega_0 = 0$ and $\omega_1 \wedge \omega_2 = 0$". First we will sum up the classsical method, though it is written in J. Hadamard [8], and also in R. Courant-D. Hilbert [2] a little. Let us consider the Cauchy problem for equation (2.1). The initial condition is given by a smooth strip C_0 which is defined in $\mathbb{R}^5 = \{(x, y, z, p, q)\}$ and written down as follows:

$$C_0 : (x, y, z, p, q) = \big(x_0(\xi), y_0(\xi), z_0(\xi), p_0(\xi), q_0(\xi)\big), \quad \xi \in \mathbb{R}^1.$$

The idea of the classical method is to represent the solution surface by a family of characteristic strips. Then they are determined as solutions of the following system of equations (see [16], [8], and [2]):

$$\begin{cases} \dfrac{\partial z}{\partial \alpha} - p\dfrac{\partial x}{\partial \alpha} - q\dfrac{\partial y}{\partial \alpha} = 0, \\ D\dfrac{\partial p}{\partial \alpha} + C\dfrac{\partial x}{\partial \alpha} + \lambda_1\dfrac{\partial y}{\partial \alpha} = 0, \\ D\dfrac{\partial q}{\partial \alpha} + \lambda_2\dfrac{\partial x}{\partial \alpha} + A\dfrac{\partial y}{\partial \alpha} = 0, \\ D\dfrac{\partial p}{\partial \beta} + C\dfrac{\partial x}{\partial \beta} + \lambda_2\dfrac{\partial y}{\partial \beta} = 0, \\ D\dfrac{\partial q}{\partial \beta} + \lambda_1\dfrac{\partial x}{\partial \beta} + A\dfrac{\partial y}{\partial \beta} = 0. \end{cases} \tag{2.9}$$

The initial condition for system (2.9) is given by

$$\begin{cases} x(\xi, \xi) &= x_0(\xi), \quad y(\xi, \xi) = y_0(\xi), \quad z(\xi, \xi) = z_0(\xi), \\ p(\xi, \xi) &= p_0(\xi), \quad q(\xi, \xi) = q_0(\xi), \quad \xi \in \mathbb{R}^1. \end{cases} \tag{2.10}$$

The local solvability of the Cauchy problem (2.9)-(2.10) is already proved first by H. Lewy [16] and afterward by J. Hadamard [8]. Let $\big(x(\alpha,\beta), y(\alpha,\beta), z(\alpha,\beta), p(\alpha,\beta), q(\alpha,\beta)\big)$ be a solution of (2.9)-(2.10). Then we can prove $\partial z/\partial\beta - p\partial x/\partial\beta - q\partial y/\partial\beta = 0$. Therefore we do not need to add this equation to system (2.9). This means that (2.9) is just a "determined" system. What we must do more is to represent $z = z(\alpha,\beta)$ as a function of (x, y). To do so, we calculate the Jacobian $D(x, y)/D(\alpha,\beta)$. As it holds along the initial strip C_0 that

$$
\text{(2.11)} \qquad \begin{cases} \dfrac{\partial x}{\partial\alpha}(\xi,\xi) &= \dfrac{1}{\lambda_1-\lambda_2}\big(D\dot{q_0}(\xi) + \lambda_1\dot{x_0}(\xi) + A\dot{y_0}(\xi)\big), \\ \dfrac{\partial x}{\partial\beta}(\xi,\xi) &= -\dfrac{1}{\lambda_1-\lambda_2}\big(D\dot{q_0}(\xi) + \lambda_2\dot{x_0}(\xi) + A\dot{y_0}(\xi)\big), \\ \dfrac{\partial y}{\partial\alpha}(\xi,\xi) &= -\dfrac{1}{\lambda_1-\lambda_2}\big(D\dot{p_0}(\xi) + C\dot{x_0}(\xi) + \lambda_2\dot{y_0}(\xi)\big), \\ \dfrac{\partial y}{\partial\beta}(\xi,\xi) &= \dfrac{1}{\lambda_1-\lambda_2}\big(D\dot{p_0}(\xi) + C\dot{x_0}(\xi) + \lambda_1\dot{y_0}(\xi)\big), \end{cases}
$$

it follows immediately that

$$
\frac{D(x,y)}{D(\alpha,\beta)} = \frac{1}{\lambda_1-\lambda_2}\big\{A\dot{y_0}^2 - B\dot{x_0}\dot{y_0} + C\dot{x_0}^2 + D(\dot{p_0}\dot{x_0} + \dot{q_0}\dot{y_0})\big\}.
$$

As we have assumed that the initial strip C_0 is not characteristic, we see by (2.4) that the Jacobian $D(x, y)/D(\alpha,\beta)$ does not vanish in a neighbourhood of C_0. Therefore we can uniquely solve the system of equations $x = x(\alpha,\beta)$, $y = y(\alpha,\beta)$ with respect to (α,β) in a neighbourhood of each point of C_0 and denote them by $\alpha = \alpha(x, y)$ and $\beta = \beta(x, y)$. Then we get the solution of the Cauchy problem for (2.1) by $z(x, y) = z(\alpha(x, y), \beta(x, y))$. Summing up the above discussion, we obtain the following:

Theorem 2.2. ([16], [8]) *Assume that the initial strip C_0 is not characteristic. Then the Cauchy problem for* (2.1) *with the initial condition C_0 uniquely admits a smooth solution in a neighbourhood of each point of C_0.*

Remark. If the equation and the solution are sufficiently differentiable, the solution is uniquely determined by the initial data. For example, the uniqueness of solution in C^∞-space is one of the classical known results (see H. Lewy [16] and J. Hadamard [8]). But the uniqueness of solution in C^2-space is a delicate problem. We will consider this subject in a forthcoming paper.

Before ending this section, we will give some remarks on the above characteristic method and the geometric solution in the sense of Definition 2.3. First we will give the meaning of (2.9) from our point of view. Suppose that a geometric solution is represented by two parameters as follows:

$$
\text{(2.12)} \qquad x = x(\alpha,\beta),\ y = y(\alpha,\beta),\ z = z(\alpha,\beta),\ p = p(\alpha,\beta),\ q = q(\alpha,\beta).
$$

Then ω_i and ϖ_i $(i = 1, 2)$ are written as

$$
\omega_i = c_{i1}d\alpha + c_{i2}d\beta, \quad \varpi_i = d_{i1}d\alpha + d_{i2}d\beta \quad (i = 1, 2).
$$

Hence we have $\omega_1 \wedge \omega_2 = (c_{11}c_{22} - c_{12}c_{21})\, d\alpha \wedge d\beta$ and $\varpi_1 \wedge \varpi_2 = (d_{11}d_{22} - d_{12}d_{21})\, d\alpha \wedge d\beta$. As it holds that $\omega_1 \wedge \omega_2 = \varpi_1 \wedge \varpi_2 = 0$ on the solution surface, a sufficient condition so that (2.12) be a geometric solution of (2.1) is given by

$$c_{11} = c_{21} = d_{12} = d_{22} = 0. \tag{2.13}$$

This is also a necessary condition. In fact, if $\omega_1 \wedge \omega_2 = \varpi_1 \wedge \varpi_2 = 0$, then we can choose the parameters (α, β) so that (2.13) is satisfied. On the other hand, from the contact relation $dz = pdx + qdy$, we get the following two equations:

$$\frac{\partial z}{\partial \alpha} - p\frac{\partial x}{\partial \alpha} - q\frac{\partial y}{\partial \alpha} = 0, \quad \frac{\partial z}{\partial \beta} - p\frac{\partial x}{\partial \beta} - q\frac{\partial y}{\partial \beta} = 0. \tag{2.14}$$

Therefore we get totally six equations from (2.13) and (2.14). As we are looking for a "determined" system, we add only one equation of (2.14) to (2.13). Then we get system (2.9). This is the meaning of system (2.9) from our point of view. Next let us start from Definition 2.3. Then the contact relation $dz = pdx + qdy$ is much more fundamental than (2.13). Therefore the characteristic system should contain two equations (2.14). Our problem is how to choose three equations from (2.13) so that a new system becomes equivalent to (2.9). Let us go back to the Cauchy problem for (2.1). As the initial strip C_0 is not characteristic, it follows from (2.5) and (2.6) that

$$(D\dot{p}_0(\xi) + C\dot{x}_0(\xi) + \lambda_2\dot{y}_0(\xi), D\dot{q}_0(\xi) + \lambda_1\dot{x}_0(\xi) + A\dot{y}_0(\xi)) \neq (0, 0),$$

and also

$$(D\dot{p}_0(\xi) + C\dot{x}_0(\xi) + \lambda_1\dot{y}_0(\xi), D\dot{q}_0(\xi) + \lambda_2\dot{x}_0(\xi) + A\dot{y}_0(\xi)) \neq (0, 0).$$

Here we assume $D\dot{p}_0(\xi) + C\dot{x}_0(\xi) + \lambda_2\dot{y}_0(\xi) \neq 0$. In this case we choose $c_{11} = c_{21} = d_{12} = 0$ as three equations which we add to (2.14). Then we get the following system of five equations:

$$\begin{cases} \dfrac{\partial z}{\partial \alpha} - p\dfrac{\partial x}{\partial \alpha} - q\dfrac{\partial y}{\partial \alpha} & = 0, \\ \dfrac{\partial z}{\partial \beta} - p\dfrac{\partial x}{\partial \beta} - q\dfrac{\partial y}{\partial \beta} & = 0, \\ D\dfrac{\partial p}{\partial \alpha} + C\dfrac{\partial x}{\partial \alpha} + \lambda_1\dfrac{\partial y}{\partial \alpha} & = 0, \\ D\dfrac{\partial q}{\partial \alpha} + \lambda_2\dfrac{\partial x}{\partial \alpha} + A\dfrac{\partial y}{\partial \alpha} & = 0, \\ D\dfrac{\partial p}{\partial \beta} + C\dfrac{\partial x}{\partial \beta} + \lambda_2\dfrac{\partial y}{\partial \beta} & = 0. \end{cases} \tag{2.15}$$

We see that system (2.15) does not satisfy the condition which is stated in p. 182 of H. Lewy [16] and in p. 489 of J. Hadamard [8]. But we can show that the Cauchy problem (2.15)-(2.10) has uniquely a classical solution in a neighbourhood of the initial strip C_0. In fact, substituting the first and second equations of (2.15) into $(\partial/\partial\beta)(\partial z/\partial\alpha) = (\partial/\partial\alpha)(\partial z/\partial\beta)$, we have

$$\frac{\partial p}{\partial \beta}\frac{\partial x}{\partial \alpha} + \frac{\partial q}{\partial \beta}\frac{\partial y}{\partial \alpha} = \frac{\partial p}{\partial \alpha}\frac{\partial x}{\partial \beta} + \frac{\partial q}{\partial \alpha}\frac{\partial y}{\partial \beta}.$$

This represents just $dp\wedge dx+dq\wedge dy=0$ obtained by $dz=pdx+qdy$. Substituting again $\partial p/\partial\alpha$, $\partial p/\partial\beta$ and $\partial q/\partial\alpha$ of (2.15) into the above, we get $(\partial y/\partial\alpha)(D\partial q/\partial\beta+\lambda_1\partial x/\partial\beta+A\partial y/\partial\beta)=0$. As we can prove in a similar way as (2.11) that $\partial y/\partial\alpha$ does not vanish along the initial strip C_0, we have $D\partial q/\partial\beta+\lambda_1\partial x/\partial\beta+A\partial y/\partial\beta=0$. This means that we can appropriately choose five equations from (2.13) and (2.14) so that a new system of five equations becomes equivalent to system (2.9).

3. Nonlinear hyperbolic equations

In this section we will consider the Cauchy problem for nonlinear hyperbolic equations as follows:

$$\begin{aligned} F(q,r,t) &= \frac{\partial^2 z}{\partial x^2} - \frac{\partial}{\partial y}\, f\Big(\frac{\partial z}{\partial y}\Big) \\ &= r - f'(q)t = 0 \quad \text{in} \quad \{x>0, y\in\mathbb{R}^1\}\equiv\mathbb{R}^2_+, \end{aligned} \tag{3.1}$$

$$z(0,y)=z_0(y), \quad \frac{\partial z}{\partial x}(0,y)=z_1(y) \quad \text{on} \quad \{x=0, y\in\mathbb{R}^1\} \tag{3.2}$$

where $f(q)$ is in $C^\infty(\mathbb{R}^1)$ and $f'(q)>0$. Here $z=z(x,y)$ is an unknown function of $(x,y)\in\mathbb{R}^2$. We assume that the initial functions $z_i(y)$ $(i=0,1)$ are sufficiently smooth, and that $z_0'(y)$ is bounded. Equation (3.1) is of Monge-Ampère type which we have studied in §2. In fact, if we may put $A=1$, $B=D=E=0$, and $C=-f'(q)$ in (2.1), then we get (3.1).

It is well known that the Cauchy problem (3.1)-(3.2) does not have a classical solution in the large. For example, see N. J. Zabusky [30] and P. D. Lax [14]. After them, many people have considered the life-span of classical solutions. As the number of papers on this subject is too many, we do not mention here on that subject.

The above phenomenon means that singularities generally appear in finite time. Our main problem is how to extend the solutions of (3.1) beyond the singularities. We can see that the solutions take many values after the appearance of singularities. If we may consider this problem from the physical point of view, we would be obliged to construct single-valued solutions of (3.1). To solve the problem of this kind, we recall what we have done for nonlinear first order partial differential equations. First we have lifted the solution surface into cotangent space so that its singularities would disappear. Then we could extend the lifted solution so that it would be defined in the whole space. Next we have projected it to the base space and gotten a multi-valued solution. Our final problem has been how to choose a single value from many values of the projected solution so that the new single-valued solution should satisfy some additional conditions attached to some physical phenomena.

Now we will construct a geometric solution of (3.1)-(3.2) by the method introduced in §2. The first step is to represent equation (3.1) as a product of one forms. Let us denote $\omega_1=dp\pm\lambda(q)dq$ and $\omega_2=\pm\lambda(q)dx+dy$ where $\lambda(q)=\sqrt{f'(q)}$.

Take an exterior product of ω_1 and ω_2, and substitute into their product the relations of the contact structure $\omega_0 = 0$, $dp = rdx + sdy$ and $dq = sdx + tdy$. Then we get

$$\omega_1 \wedge \omega_2 = \left\{r - f'(q)t\right\}\ dx \wedge dy. \tag{3.3}$$

Definition 2.3 and the decomposition (3.3) for equation (3.1) suggest us to consider the following system, which is similar to (2.9) for equation (2.1) in the general case:

$$\begin{cases} \dfrac{\partial p}{\partial \alpha} + \lambda(q)\dfrac{\partial q}{\partial \alpha} & = 0, \\ \lambda(q)\dfrac{\partial x}{\partial \alpha} + \dfrac{\partial y}{\partial \alpha} & = 0, \\ \dfrac{\partial p}{\partial \beta} - \lambda(q)\dfrac{\partial q}{\partial \beta} & = 0, \\ -\lambda(q)\dfrac{\partial x}{\partial \beta} + \dfrac{\partial y}{\partial \beta} & = 0. \end{cases} \tag{3.4}$$

The initial condition corresponding to (3.2) is given by

$$x(\xi,\xi) = 0, \quad y(\xi,\xi) = \xi, \quad p(\xi,\xi) = z_1(\xi), \quad q(\xi,\xi) = z_0'(\xi), \quad \xi \in \mathbb{R}^1. \tag{3.5}$$

Solving (3.4)-(3.5), we get the following:

Theorem 3.1. *The Cauchy problem* (3.1)-(3.2) *has globally a regular geometric solution.*

Proof. Integrating the first and the third equations of (3.4), we have

$$p + \Lambda(q) = \psi_1(\beta) \quad \text{ and } \quad p - \Lambda(q) = \psi_2(\alpha) \tag{3.6}$$

where $\Lambda'(q) = \lambda(q)$, $\psi_1(\beta) = z_1(\beta) + \Lambda(z_0'(\beta))$ and $\psi_2(\alpha) = z_1(\alpha) - \Lambda(z_0'(\alpha))$.

As $\Lambda'(q) > 0$, we see that an inverse function of $\Lambda(q)$ is smooth. Therefore p and q are obtained as smooth functions of (α, β) defined in the whole space $\mathbb{R}^2$. On the other hand, x and y are solutions of the following system:

$$\begin{cases} \lambda(q)\dfrac{\partial x}{\partial \alpha} + \dfrac{\partial y}{\partial \alpha} & = 0, \\ -\lambda(q)\dfrac{\partial x}{\partial \beta} + \dfrac{\partial y}{\partial \beta} & = 0. \end{cases} \tag{3.7}$$

As this is equivalent to a system of linear wave equations concerning x and y, we can get the solutions $x = x(\alpha, \beta)$ and $y = y(\alpha, \beta)$ as smooth functions of (α, β) defined in the whole space $\mathbb{R}^2 = \{(\alpha, \beta)\}$. The function $z = z(\alpha, \beta)$ is uniquely determined by the contact relation $dz = pdx + qdy$ and the initial conditions (3.2), that is to say,

$$\begin{cases} \dfrac{\partial z}{\partial \alpha} & = p\dfrac{\partial x}{\partial \alpha} + q\dfrac{\partial y}{\partial \alpha}, \\ \dfrac{\partial z}{\partial \beta} & = p\dfrac{\partial x}{\partial \beta} + q\dfrac{\partial y}{\partial \beta}, \\ z(\xi,\xi) & = z_0(\xi), \quad \xi \in \mathbb{R}^1. \end{cases}$$

We can easily see that $z = z(\alpha,\beta)$ is a smooth function defined in the whole space $\mathbb{R}^2 = \{(\alpha,\beta)\}$. For the existence of the regular geometric solution, we must prove that it is regular, that is to say,

$$\text{rank}\begin{pmatrix} \partial x/\partial\alpha & \partial y/\partial\alpha & \partial z/\partial\alpha & \partial p/\partial\alpha & \partial q/\partial\alpha \\ \partial x/\partial\beta & \partial y/\partial\beta & \partial z/\partial\beta & \partial p/\partial\beta & \partial q/\partial\beta \end{pmatrix} = 2. \tag{3.8}$$

Taking the derivatives of (3.6) with respect to α and β, we get

$$2\frac{\partial p}{\partial \alpha} = \psi_2'(\alpha), \;\; 2\frac{\partial p}{\partial \beta} = \psi_1'(\beta), \;\; 2\lambda(q)\frac{\partial q}{\partial \alpha} = -\psi_2'(\alpha), \;\; 2\lambda(q)\frac{\partial q}{\partial \beta} = \psi_1'(\beta).$$

Therefore, for the proof of (3.8), we must consider whether $\psi_1'(\beta)$ and $\psi_2'(\alpha)$ are 0 or not. Concerning this problem, we give the following two lemmas.

Lemma 3.1. *Assume that there exists α_0 satisfying $\psi_2'(\alpha_0) = 0$. Then $(\partial x/\partial\alpha)(\alpha_0,\beta) \neq 0$ for any $\beta \in \mathbb{R}^1$.*

Lemma 3.2. *Assume that there exists β_0 satisfying $\psi_1'(\beta_0) = 0$. Then $(\partial x/\partial\beta)(\alpha,\beta_0) \neq 0$ for any $\alpha \in \mathbb{R}^1$.*

As we will prove the above two lemmas after the proof of Theorem 3.1, we will now continue it. Lemmas 3.1 and 3.2 suggest us to consider the following four cases:

(i) In the case where $\psi_2'(\alpha_0)\cdot\psi_1'(\beta_0) \neq 0$, we get

$$\begin{aligned} \det\begin{pmatrix} \partial p/\partial\alpha & \partial q/\partial\alpha \\ \partial p/\partial\beta & \partial q/\partial\beta \end{pmatrix} &= \det\begin{pmatrix} \psi_2'(\alpha_0)/2 & -\psi_2'(\alpha_0)/2\lambda(q) \\ \psi_1'(\beta_0)/2 & \psi_1'(\beta_0)/2\lambda(q) \end{pmatrix} \\ &= \psi_2'(\alpha_0)\psi_1'(\beta_0)/2\lambda(q) \neq 0. \end{aligned}$$

(ii) In the case where $\psi_2'(\alpha_0) = 0$ and $\psi_1'(\beta_0) \neq 0$, we get by Lemma 3.1

$$\begin{aligned} \det\begin{pmatrix} \partial x/\partial\alpha & \partial p/\partial\alpha \\ \partial x/\partial\beta & \partial p/\partial\beta \end{pmatrix} &= \det\begin{pmatrix} \partial x/\partial\alpha & \psi_2'(\alpha_0)/2 \\ \partial x/\partial\beta & \psi_1'(\beta_0)/2 \end{pmatrix} \\ &= \frac{1}{2}\frac{\partial x}{\partial\alpha}\psi_1'(\beta_0) \neq 0. \end{aligned}$$

(iii) In the case where $\psi_2'(\alpha_0) \neq 0$ and $\psi_1'(\beta_0) = 0$, we get by Lemma 3.2

$$\det\begin{pmatrix} \partial x/\partial\alpha & \partial p/\partial\alpha \\ \partial x/\partial\beta & \partial p/\partial\beta \end{pmatrix} = -\frac{1}{2}\frac{\partial x}{\partial\beta}\psi_2'(\alpha_0) \neq 0.$$

(iv) In the case where $\psi_2'(\alpha_0) = \psi_1'(\beta_0) = 0$, we get by (3.7) and Lemmas 3.1-3.2

$$\det\begin{pmatrix} \partial x/\partial\alpha & \partial y/\partial\alpha \\ \partial x/\partial\beta & \partial y/\partial\beta \end{pmatrix} = \det\begin{pmatrix} \partial x/\partial\alpha & -\lambda(q)\partial x/\partial\alpha \\ \partial x/\partial\beta & \lambda(q)\partial x/\partial\beta \end{pmatrix}$$
$$= 2\lambda(q)\frac{\partial x}{\partial\alpha}\frac{\partial x}{\partial\beta} \neq 0.$$

Hence we see that (3.8) holds at any point $(\alpha, \beta) = (\alpha_0, \beta_0)$. □

Proof of Lemmas 3.1 and 3.2. Eliminating $y = y(\alpha, \beta)$ from the two equations of (3.7), we get

$$\frac{\partial}{\partial\beta}(\lambda(q)\frac{\partial x}{\partial\alpha}) + \frac{\partial}{\partial\alpha}(\lambda(q)\frac{\partial x}{\partial\beta}) = 0.$$

As $2\lambda(q)(\partial q/\partial\alpha) = -\psi_2'(\alpha)$, it follows that $(\partial q/\partial\alpha)(\alpha_0, \beta) = 0$. Hence we get

$$2\lambda(q)\frac{\partial^2 x}{\partial\alpha\partial\beta}(\alpha_0, \beta) + \lambda'(q)\frac{\partial q}{\partial\beta}(\alpha_0, \beta)\frac{\partial x}{\partial\alpha}(\alpha_0, \beta) = 0.$$

As this is just a first order linear ordinary differential equation with respect to $(\partial x/\partial\alpha)(\alpha_0, \beta)$, we can solve it exactly. As $(\partial x/\partial\alpha)(\alpha_0, \alpha_0) = -1/2\lambda(q(\alpha_0, \alpha_0)) \neq 0$, (3.4)-(3.5) implies that $(\partial x/\partial\alpha)(\alpha_0, \beta)$ is not identically zero. Hence we have Lemma 3.1. Exchanging α and β in the above, we obtain Lemma 3.2. □

In a domain where the Jacobian $D(x, y)/D(\alpha, \beta)$ does not vanish, we can uniquely solve $x = x(\alpha, \beta)$ and $y = y(\alpha, \beta)$ with respect to (α, β). We write $\alpha = \alpha(x, y)$ and $\beta = \beta(x, y)$. Then $z(x, y) = z(\alpha(x, y), \beta(x, y))$ is a classical solution of (3.1)-(3.2), because $(\partial z/\partial x)(x, y) = p(\alpha(x, y), \beta(x, y))$ and $(\partial z/\partial y)(x, y) = q(\alpha(x, y), \beta(x, y))$. Next we will prove the explosion of classical solutions at points where the Jacobian vanishes.

Theorem 3.2. *Assume*

$$\frac{D(x, y)}{D(\alpha, \beta)}(\alpha^0, \beta^0) = 0.$$

If a point (x, y) goes to $(x(\alpha^0, \beta^0), y(\alpha^0, \beta^0))$ along a curve in the existence domain of the classical solution $z = z(x, y)$, then (r, s, t) tends to ∞.

Proof. In a domain where there exists a classical solution, we have

$$\begin{cases} \dfrac{\partial p}{\partial\alpha} & = r\dfrac{\partial x}{\partial\alpha} + s\dfrac{\partial y}{\partial\alpha}\,, \\ \dfrac{\partial p}{\partial\beta} & = r\dfrac{\partial x}{\partial\beta} + s\dfrac{\partial y}{\partial\beta}\,, \\ \dfrac{\partial q}{\partial\alpha} & = s\dfrac{\partial x}{\partial\alpha} + t\dfrac{\partial y}{\partial\alpha}\,, \\ \dfrac{\partial q}{\partial\beta} & = s\dfrac{\partial x}{\partial\beta} + t\dfrac{\partial y}{\partial\beta}\,, \end{cases} \tag{3.9}$$

where r, s and t are the second order derivatives of $z = z(x,y)$ introduced in (2.1). For (3.1)-(3.2), the Jacobian is written down by $D(x,y)/D(\alpha,\beta) = 2\lambda(q)(\partial x/\partial\alpha)(\partial x/\partial\beta)$. If the Jacobian does not vanish, then it follows from (3.9) that

$$
\begin{cases}
r &= \dfrac{1}{4}\Big\{\dfrac{\psi_1{}'(\beta)}{\dfrac{\partial x}{\partial\beta}(\alpha,\beta)} + \dfrac{\psi_2{}'(\alpha)}{\dfrac{\partial x}{\partial\alpha}(\alpha,\beta)}\Big\}, \\
s &= \dfrac{1}{4\lambda(q)}\Big\{\dfrac{\psi_1{}'(\beta)}{\dfrac{\partial x}{\partial\beta}(\alpha,\beta)} - \dfrac{\psi_2{}'(\alpha)}{\dfrac{\partial x}{\partial\alpha}(\alpha,\beta)}\Big\}, \\
t &= \dfrac{1}{\lambda(q)^2}r\,.
\end{cases}
\tag{3.10}
$$

Applying Lemma 3.1 and Lemma 3.2 to (3.10), we can get the above result. □

Remark. We explain the meaning of Theorem 3.2. Sometimes, even if the Jacobian may vanish, there exists a classical solution. This phenomenon happens even for nonlinear first order partial differential equations (see M. Tsuji [22]). What Theorem 3.2 insists is that, for the Cauchy problem (3.1)-(3.2), a classical solution always blows up at a point where the Jacobian vanishes. Therefore we can exactly determine the life-span of the classical solution by the information on zeros of the Jacobian. Concerning the life-span of classical solutions, various kinds of results have been published, for example [30], [14], [26], etc, etc.

As we have stated in the above, our principal interest is to extend a solution beyond the singularities. Therefore we introduce the notion of "solution with singularities" which is called "weak solution", though we do not yet arrive at the final decision on the definition of weak solutions. Let us introduce the most typical definition which is corresponding to P. D. Lax's one [15] introduced for systems of conservation laws.

Definition 3.1. A function $z = z(x,y)$ is called a weak solution of (3.1)-(3.2) if the following conditions (i) and (ii) are satisfied:

(i) The function $z = z(x,y)$ is continuous with $z(0,y) = z_0(y)$; and its derivatives in the sense of distributions, $(\partial z/\partial x)(x,y)$ and $(\partial z/\partial y)(x,y)$, are bounded and measurable,

(ii) The function $z = z(x,y)$ satisfies the Cauchy problem (3.1)-(3.2) in the weak sense, that is to say, it holds for any $\varphi(x,y) \in C_0^\infty(\mathbb{R}^2)$ that

$$
\int_{\mathbb{R}^2_+} \Big\{\frac{\partial z}{\partial x}\frac{\partial\varphi}{\partial x} - f\Big(\frac{\partial z}{\partial y}\Big)\frac{\partial\varphi}{\partial y}\Big\}dxdy + \int_{\mathbb{R}^1} z_1(y)\varphi(0,y)dy = 0. \tag{3.11}
$$

Let $z = z(x,y)$ be a weak solution of (3.1)-(3.2) in the sense of Definition 3.1. Assume that $(\partial z/\partial x)(x,y) \equiv p(x,y)$ and $(\partial z/\partial y)(x,y) \equiv q(x,y)$ have jump discontinuities along a smooth curve $y = \gamma(x)$. As $z = z(x,y)$ is continuous, it holds that $z(x, \gamma(x)+0) = z(x, \gamma(x)-0)$. Differentiating this with respect to x, we get

$$[p] + [q]\dot{\gamma} = 0. \tag{3.12}$$

where [ ] means the quantity of difference, i.e., $[p] = p(x, \gamma(x)+0) - p(x, \gamma(x)-0)$. Since $z = z(x,y)$ satisfies (3.11), we obtain

$$[p]\dot{\gamma} + [f(q)] = 0. \tag{3.13}$$

This means that the curve $y = \gamma(x)$ must satisfy two kinds of differential equations, (3.12) and (3.13). This suggests us the following:

Theorem 3.3. *Assume that $f'(q) > 0$ and $f''(q) \neq 0$. Then we can not generally construct a weak solution of the Cauchy problem* (3.1)-(3.2) *in the sense of Definition* 3.1 *by projecting the above geometric solution to the base space and cutting off some part of the multi-valued projected solution so that it would become a single-valued solution of* (3.1)-(3.2)*.*

Remark. In this theorem, we have used the word "generally" to state that there exists the case in which we cannot construct a weak solution by the method explained in the theorem. Therefore we do not deny the possibility that there may exist a case where we can do so.

Proof. Supposing that we could construct a weak solution of the Cauchy problem (3.1)-(3.2) by the geometric method stated in the theorem, we will show that we would be led to a contradiction. We consider the case where the initial data satisfy $\psi_1(\beta) = 0$ or $\psi_2(\alpha) = 0$. Here we assume $\psi_2(\alpha) = 0$. Then it holds that $p - \Lambda(q) = 0$ for all $(\alpha, \beta) \in \mathbb{R}^2$. Moreover, solving (3.6) and (3.7), we get

$$y = \beta - \lambda(q)x, \quad p = z_1(\beta), \quad q = z_0'(\beta), \quad \text{and}$$

$$x(\alpha, \beta) = \frac{1}{2\sqrt{\lambda(z_0'(\beta))}} \int_\alpha^\beta \frac{d\tau}{\sqrt{\lambda(z_0'(\tau))}}.$$

As z_0' is bounded, $x(\alpha, \beta)$ tends to $\pm\infty$ when $\beta - \alpha$ goes to $\pm\infty$ respectively. Therefore, when (α, β) moves in the whole space $\mathbb{R}^2$, so does $(x(\alpha,\beta), y(\alpha,\beta))$. As $z = z(\alpha, \beta)$ is defined by $dz = pdx + qdy$, we have $\partial z/\partial x = p$ and $\partial z/\partial y = q$ in the domain $\{(x(\alpha,\beta), y(\alpha,\beta));\ D(x,y)/D(\alpha,\beta) \neq 0\}$. As, by Sard's theorem, the measure of the set $\{(x(\alpha,\beta), y(\alpha,\beta));\ D(x,y)/D(\alpha,\beta) = 0\}$ is zero, we define the values of $\partial z/\partial x$ and $\partial z/\partial y$ at a point where the Jacobian vanishes by the limits of $p = p(\alpha, \beta)$ and $q = q(\alpha, \beta)$, respectively. Then $(\partial z/\partial x)(x,y)$ and $(\partial z/\partial y)(x,y)$ become multi-valued functions defined in the whole space $\mathbb{R}^2$, and $z = z(x,y)$ turns out to satisfy the following Cauchy problem:

$$\frac{\partial z}{\partial x} - \Lambda\Big(\frac{\partial z}{\partial y}\Big) = 0 \quad \text{in} \quad \{x > 0, y \in \mathbb{R}^1\} \equiv \mathbb{R}^2_+, \tag{3.14}$$

$$z(0, y) = z_0(y) \quad \text{in} \quad \{x = 0, y \in \mathbb{R}^1\}. \tag{3.15}$$

The characteristic differential equations for (3.14)-(3.15) are written by

$$\left\{\begin{array}{l} \dfrac{dy}{dx} = -\lambda(q), \quad \dfrac{dz}{dx} = p - \lambda(q)q, \quad \dfrac{dp}{dx} = \dfrac{dq}{dx} = 0, \\ \\ y(0) = \xi, \quad z(0) = z_0(\xi), \quad p(0) = \Lambda(z_0'(\xi)), \quad q(0) = z_0'(\xi). \end{array}\right. \tag{3.16}$$

We can immediately solve (3.16). Moreover, in the case where $z_1(\xi) - \Lambda(z_0'(\xi)) = 0$, we can easily show that the surface $\{(x, y, p, q); y = \xi - \lambda(z_0'(\xi))x, p = z_1(\xi), q = z_0'(\xi)\}$, i.e. the (x, y, p, q)-components of the solution of (3.16), is the same as the solution surface of (3.4)-(3.5). Here we recall the result of [21, 22]. As $(\partial y/\partial\xi)(x, \xi) = 1 - \lambda'(z_0'(\xi))z_0''(\xi)x$ does not vanish in a neighbourhood of $x = 0$, we can uniquely solve the equation $y = y(x, \xi)$ with respect to ξ and denote it by $\xi = \xi(x, y)$. Then we can get a classical solution of (3.1)-(3.2) by $z = z(x, \xi(x, y))$ where $z = z(x, \xi)$ is the z-component of the solution of (3.16). Here we put $h(\xi) = \lambda'(z_0'(\xi))z_0''(\xi)$ and assume that $h = h(\xi)$ takes its positive maximum at $\xi = \xi_0$ with $h''(\xi_0) > 0$. We write $M = h(\xi_0)$, $x_0 = 1/M$ and $y_0 = y(x_0, \xi_0)$. Then the function $\xi = \xi(x, y)$ takes three values for $x > x_0$ in a neighbourhood of (x_0, y_0), and so does the solution $z = z(x, y)$. In [21] we have proved that we can choose only one from these three values of $z = z(x, y)$ so that the solution becomes a single-valued continuous solution of (3.14)-(3.15), and that it automatically satisfies the conditions for generalized solutions introduced in the case of Hamilton-Jacobi equations. Therefore, if a continuous solution of (3.1)-(3.2) may be obtained by the geometric method, it must coincide with the continuous solution of (3.14)-(3.15) constructed as above. In the situation under consideration, its derivatives have jump discontinuity along some smooth curve $y = \gamma(x)$ whose starting point is (x_0, y_0) (see [21]). Combining (3.12) and (3.13), we get

$$\frac{[f(q)]}{[p]} = \frac{[p]}{[q]}, \qquad \text{i.e.,} \qquad \frac{[f(q)]}{[q]} = \left(\frac{[\Lambda(q)]}{[q]}\right)^2. \tag{3.17}$$

The following Lemma 3.3 assures us that (3.17) does not hold. This is a contradiction. Hence we get Theorem 3.3. □

Lemma 3.3. *Assume that $f'(q) > 0$ and $f''(q) \neq 0$. Then, if $q_1 \neq q_2$ and $q_2 - q_1$ is sufficiently small, we have*

$$\frac{f(q_2) - f(q_1)}{q_2 - q_1} \neq \Big(\frac{\Lambda(q_2) - \Lambda(q_1)}{q_2 - q_1}\Big)^2.$$

Proof. Keeping q_1 fixed, we expand the both sides in the Taylor series with respect to q_2 about $q_2 = q_1$. Then the coefficient of $(q_2 - q_1)^2$ of the left-hand side is $f^{(3)}(q_1)/6$, while that of the right-hand side is equal to $f^{(3)}(q_1)/6 - (f''(q_1))^2/48f'(q_1)$. This means the above conclusion. □

If we may change the definition of weak solution, we shall be able to get various results of another type. For example, the following definition of "weak solution" is also possible.

Definition 3.2. Let $z = z(x, y)$, $(\partial z/\partial x)(x, y)$, and $(\partial z/\partial y)(x, y)$ be bounded and measurable. Moreover $z = z(x, y)$ is continuous as a function of x with values in $L^1_{loc}(\mathbb{R}^1)$. The function $z = z(x, y)$ is a weak solution of (3.1)-(3.2) if it satisfies equation (3.1) in the following integral form:

$$\int_{\mathbb{R}^2_+} \Big\{ \frac{\partial z}{\partial x}\frac{\partial \varphi}{\partial x} - f\Big(\frac{\partial z}{\partial y}\Big)\frac{\partial \varphi}{\partial y}\Big\} dxdy + \int_{\mathbb{R}^1} z_1(y)\varphi(0, y)\ dy = 0$$

for all $\varphi(x, y) \in C_0^\infty(\mathbb{R}^2)$, and $z = z(x, y)$ tends to $z_0(y)$ in $L^1_{loc}(\mathbb{R}^1)$ when x goes to $+0$.

Then we get the following:

Theorem 3.4. *In the same situation as Theorem* 3.3*, we can get a weak solution of the Cauchy problem* (3.1)-(3.2) *in the sense of Definition* 3.2 *by projecting the above geometric solution to the base space and cutting off appropriate parts of that projected surface.*

Proof. Let us begin our discussion from (3.14)-(3.15). As it holds that $p - \Lambda(q) = 0$, $p(x, y) = \partial z/\partial x$ satisfies the following Cauchy problem:

$$\begin{cases} \dfrac{\partial p}{\partial x} - \dfrac{\partial}{\partial y} g(p) = 0, \\ \quad p(0) = z_1(y). \end{cases} \tag{3.18}$$

where $g(p) = f(\Lambda^{-1}(p))$. We consider this Cauchy problem in the same situation as in the proof of Theorem 3.2. We can easily show that the family of characteristic curves of (3.18) is written by $\{(x, y, p); y = \xi - g'(z_1(\xi))x, p = z_1(\xi)\}$. This is just the (x, y, p)-components of the solution of (3.16). Then we can uniquely construct a weak solution $p = p(x, y)$ of (3.18) by cutting off some part of this characteristic surface so that $p = p(x, y)$ becomes single-valued and satisfies (3.18) in the weak sense (for example, refer to [22]). In this case $p = p(x, y)$ has a jump discontinuity across a curve $y = \gamma(x)$ which starts from the point (x_0, y_0). As it satisfies (3.18) in the weak sense, the jump condition for $p = p(x, y)$ is the same as (3.13). Then we can show that $z = z(x, y)$, i.e. the corresponding part of the z-component of the solution of (3.16), is not continuous across the curve $y = \gamma(x)$. In fact, if so, it follows that $z(x, \gamma(x) - 0) = z(x, \gamma(x) + 0)$. Hence we get (3.12). As we have already proved in Lemma 3.3 that the conditions (3.12) and (3.13) are not compatible, we see that the above solution $z = z(x, y)$ can not become continuous across the curve $y = \gamma(x)$. Therefore it is not a weak solution in the sense of Definition 3.1, but it is so in the sense of Definition 3.2. □

Remark. In Definition 3.2, $\partial z/\partial x$ and $\partial z/\partial y$ are the derivatives of $z = z(x, y)$ in the classical sense, not in the sense of distribution theory. Hence, even if $z = z(x, y)$ may have jump discontinuities, Dirac's measure does not appear in $\partial z/\partial x$ and $\partial z/\partial y$. But we insist that equation (3.1) is satisfied in the above integral form. Therefore we guess that Definition 3.2 would not be accepted by many people. But we have shown in [26] that it works well in some case. As an example in [26], we have considered the well-known equation appeared in N. J. Zabusky [30].

As we will write in §4, equation (3.1) can be transformed into a system of conservation laws (4.1). For systems of conservation laws, P. D. Lax [15] introduced the notion of weak solutions. Then we can show in Proposition 4.1 that weak solutions of (3.1) in the sense of Definition 3.1 can be transformed to "weak solutions in the sense of Lax" for systems of conservation laws (4.1). Here we recall the method used to solve the problem of singularities. The most traditional and typical method is the "resolution of singularities" whose idea is to lift a surface with singularities into a space of higher dimension so that the singularities would disappear. After solving our problems in higher dimensional space, we project it to the base space. This method has been well developed especially in algebraic geometry. If we might follow this approach, we would be obliged to change the definition of weak solution in the sense of Lax. For example, R. Thom [20] originated "Catastrophe theory", and he has applied his theory to understand various kinds of phenomena caused by "singularities". If we might accept his idea, we might be led to a definition in which a weak solution would be constructed by the projection of a geometric solution to the base space.

4. Systems of conservation laws

In this section we will consider a certain hyperbolic system of conservation laws which is related with equation (3.1). Let $z = z(x, y)$ be a solution of (3.1), and write $p = \partial z/\partial x$, $q = \partial z/\partial y$, $U(x, y) = (p, q)$, $F(U) = (f(q), p)$ and $U_0(y) = (z_1(y), z_0'(y)) \equiv (p_0(y), q_0(y))$. Then we get

$$\frac{\partial}{\partial x}U - \frac{\partial}{\partial y}F(U) = 0 \quad \text{in} \quad \{x > 0, y \in \mathbb{R}^1\}, \tag{4.1}$$

$$U(0, y) = U_0(y) \quad \text{on} \quad \{x = 0, y \in \mathbb{R}^1\}. \tag{4.2}$$

The system of the form (4.1) is called "p-system". It is well known that, even if the initial data are sufficiently smooth, singularities generally appear in the solution of (4.1)-(4.2). Therefore we will construct a "solution with singularities" called "weak solution". As we have done for hyperbolic Monge-Ampère equations in §3, we will introduce a notion of "geometric solution" for (4.1)-(4.2). To do so, we will rewrite the equations by using differential forms. Then system (4.1) is represented by

$$\begin{cases} dp \wedge dy + df(q) \wedge dx & = 0, \\ dq \wedge dy + dp \wedge dx & = 0. \end{cases} \tag{4.3}$$

Definition 4.1. A regular geometric solution of (4.1) is a submanifold of dimension 2 defined in $\mathbb{R}^4 = \{(x, y, p, q)\}$ on which system (4.3) is satisfied.

For getting a geometric solution in the above sense, we will decompose the equations of differential forms as a product of one forms just as in §3. Then we can easily see that system (4.3) is equivalent to the following system:

$$\begin{cases} (dp+\lambda(q)dq)\wedge(\lambda(q)dx+dy) & =0, \\ (dp-\lambda(q)dq)\wedge(-\lambda(q)dx+dy) & =0, \end{cases} \tag{4.4}$$

where $\lambda(q) = \sqrt{f'(q)}$. We will repeat the same discussion as in §3 for solving system (4.4). Let us represent a geometric solution by

$$x = x(\alpha,\beta), \quad y = y(\alpha,\beta), \quad p = p(\alpha,\beta), \quad q = q(\alpha,\beta).$$

A sufficient condition so that it is a geometric solution of (4.1)-(4.2) is given by

$$\begin{cases} \dfrac{\partial p}{\partial \alpha} + \lambda(q)\dfrac{\partial q}{\partial \alpha} & = 0, \\ \lambda(q)\dfrac{\partial x}{\partial \alpha} + \dfrac{\partial y}{\partial \alpha} & = 0, \\ \dfrac{\partial p}{\partial \beta} - \lambda(q)\dfrac{\partial q}{\partial \beta} & = 0, \\ -\lambda(q)\dfrac{\partial x}{\partial \beta} + \dfrac{\partial y}{\partial \beta} & = 0. \end{cases} \tag{4.5}$$

The above system is just the same as (3.4). The initial condition corresponding to (4.2) is

$$x(\xi,\xi) = 0, \quad y(\xi,\xi) = \xi, \quad p(\xi,\xi) = p_0(\xi), \quad q(\xi,\xi) = q_0(\xi), \quad \xi \in \mathbb{R}^1. \tag{4.6}$$

As discussed in §3, we can prove that the Cauchy problem (4.5)-(4.6) has a unique solution defined for all $(\alpha,\beta) \in \mathbb{R}^2$, and it satisfies

$$p + \Lambda(q) = \psi_1(\beta) \quad \text{and} \quad p - \Lambda(q) = \psi_2(\alpha)$$

where $\Lambda'(q) \equiv \lambda(q)$ and $\psi_1(\beta) = p_0(\beta)+\Lambda(q_0(\beta))$, and $\psi_2(\alpha) = p_0(\alpha)-\Lambda(q_0(\alpha))$. This solution determines a regular geometric solution of (4.1)-(4.2) in the large. Let us recall here the definition of weak solutions of (4.1)-(4.2) introduced by P. D. Lax [15].

Definition 4.2. A bounded and measurable 2-vector function $U = U(x,y)$ is a weak solution of (4.1)-(4.2) if it satisfies (4.1)-(4.2) in the weak sense, i.e.,

$$\int_{\mathbb{R}^2_+} \left\{ U(x,y)\frac{\partial \Phi}{\partial x}(x,y) - F(U)\frac{\partial \Phi}{\partial y}(x,y) \right\} dxdy + \int_{\mathbb{R}^1} U_0(y)\Phi(0,y)dy = 0 \tag{4.7}$$

for any 2-vector function $\Phi(x,y) \in C_0^\infty(\mathbb{R}^2)$.

Remark. Putting $p = \partial z/\partial x$ and $q = \partial z/\partial y$, we arrive at Definition 4.2 from Definition 3.1. Contrarily, the following shows that we can get Definition 3.1 from Definition 4.2.

Proposition 4.1. *Assume* $U = (p(x,y), q(x,y))$ *to be a weak solution of* (4.1)-(4.2) *in the sense of Definition* 4.2. *Then the Cauchy problem* (3.1)-(3.2) *has a weak solution* $z = z(x,y)$ *in the sense of Definition* 3.1 *for which* $\partial z/\partial x = p$ *and* $\partial z/\partial y = q$.

Proof. Define a function $z = z(x,y)$ by

$$z(x,y) = z_0(y) + \int_0^x p(\tau, y)d\tau. \tag{4.8}$$

Then $z = z(x,y)$ is measurable and locally bounded in $\mathbb{R}^2_+$. Moreover, it becomes then a Lipschitz continuous function of $x \in (0,\infty)$ for almost all $y \in \mathbb{R}^1$, and admits $p = p(x,y)$ as its derivative with respect to x in the distribution sense in $\mathbb{R}^2_+$. Next we will show that $q = q(x,y)$ is its derivative with respect to y in the distribution sense. For this, let $\varphi(x,y) \in C_0^\infty(\mathbb{R}^2_+)$ be arbitrarily given. We extend the definition domain of $\varphi(x,y)$ to $\mathbb{R}^2$ by setting $\varphi(-x,y) = -\varphi(x,y)$, and let $\psi(x,y) = \int_{-\infty}^x \varphi(\tau,y)d\tau$. We thus get a new function $\psi(x,y) \in C_0^\infty(\mathbb{R}^2)$ with $\partial\psi/\partial x = \varphi$. In view of (4.8), we see from the integration by parts formula that

$$\begin{aligned} &\int_{\mathbb{R}^2_+} z\frac{\partial\varphi}{\partial y}dxdy \\ &= \int_0^\infty\int_{-\infty}^\infty z_0(y)\frac{\partial\varphi}{\partial y}(x,y)dydx + \int_{-\infty}^\infty\int_0^\infty \left[\int_0^x p(\tau,y)d\tau\right]\frac{\partial}{\partial x}\frac{\partial\psi}{\partial y}(x,y)dxdy \\ &= -\int_0^\infty\int_{-\infty}^\infty z_0'(y)\varphi(x,y)dydx - \int_{-\infty}^\infty\int_0^\infty p(x,y)\frac{\partial\psi}{\partial y}(x,y)dxdy. \end{aligned} \tag{4.9}$$

But the second identity of (4.7) implies

$$\begin{aligned} \int_{\mathbb{R}^2_+} p(x,y)\frac{\partial\psi}{\partial y}(x,y)dxdy &= \int_{\mathbb{R}^2_+} q(x,y)\frac{\partial\psi}{\partial x}(x,y)dxdy + \int_{-\infty}^\infty z_0'(y)\psi(0,y)dy \\ &= \int_{\mathbb{R}^2_+} q(x,y)\varphi(x,y)dxdy - \int_{-\infty}^\infty z_0'(y)\int_0^\infty \varphi(x,y)dxdy. \end{aligned}$$

Hence, it follows from (4.9) that

$$\int_{\mathbb{R}^2_+} z(x,y)\frac{\partial\varphi}{\partial y}(x,y)dxdy = -\int_{\mathbb{R}^2_+} q(x,y)\varphi(x,y)dxdy.$$

This means that $(\partial z/\partial y)(x,y) = q(x,y)$ in the sense of distributions in the domain $\mathbb{R}^2_+$. Therefore, $z = z(x,y)$ can be regarded as a Lipschitz continuous function in $\mathbb{R}^2_+$. Finally we extend the definition domain of $z = z(x,y)$ to the boundary $\{x = 0\}$ by using (4.8). Then we see that $z(0,y) = z_0(y)$, and that (3.11) is just the first identity of (4.7). □

If $U = U(x,y)$ is a weak solution of (4.1) which has jump discontinuity along a smooth curve $y = \gamma(x)$, we get jump conditions of Rankine-Hugoniot as follows:

$$\begin{aligned}[p]\dot{\gamma} + [f(q)] &= 0,\\ [q]\dot{\gamma} + [p] &= 0.\end{aligned}$$

Therefore the jump discontinuity must satisfy two kinds of differential equations. Using this property, we can show that Theorem 3.3 is rewritten in the following form, though we do not write the proof because it is almost similar to that of Theorem 3.3.

Theorem 4.1. *Assume that $f'(q) > 0$ and $f''(q) \neq 0$. Then we can not generally construct a weak solution of the Cauchy problem* (4.1)-(4.2) *in the sense of Definition* 4.2 *by cutting off some part of the above geometric solution so that it would become a single-valued solution of* (4.1)-(4.2).

Remark. For Riemann's problem to (4.1), the same result has been already proved in [1]. But, in the case where the initial data are smooth, the above result has been announced as a conjecture (see p. 59 in [1]). The above result is also written in [26] which is the Proceeding of a meeting held in September 1996. At that time we did not know the paper [1]. We thank Y. Machida that he informed us the existence of [1].

Concerning single first order partial differential equations, we could construct weak solutions by the above method. For example, see M. Tsuji [21, 22], S. Nakane [18, 19], S. Izumiya [10], S. Izumiya and G. T. Kossioris [11, 12], etc.

For second order hyperbolic equations or hyperbolic systems of conservation laws, we could not construct weak solutions by the same method. But, if we may change the definition of weak solutions, we can show that the above method would still work well. For example, if we may introduce a new definition of weak solutions of (4.1)-(4.2) corresponding to Definition 3.2, we can get an affirmative answer which is corresponding to Theorem 3.4. In a forthcoming paper, we will discuss more precisely what kind of solutions we can get by projecting the geometric solutions into the base space.

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