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GENERALIZED TRANSLATION OPERATORS AND THEIR RELATED MARKOV PROCESSES

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Abstract. We study properties of generalized translation operators and their relations with the associated Markov processes. Some relations between a Levitan family of generalized translation operators and its associated random convolution (in the sense of Vol'kovich) are established. The generalized differential operator introduced by N. V. Thu (1994) is also investigated.

1. Preliminaries

Let $\mathcal{P}$ denote the class of all probability measures on Borel subsets of $R_+ = [0, \infty)$ and C_b the Banach space of all bounded continuous real valued functions on R_+. *A random convolution* (in the sense of Vol'kovich) of elements of $\mathcal{P}$ is a binary operation $\circ$ on $\mathcal{P}$ such that

a) $(\mathcal{P}, \circ)$ is a topological semigroup;

b) $(a\mu + b\nu) \circ \gamma = a(\mu \circ \gamma) + b(\nu \circ \gamma)$ for all $\mu, \nu, \gamma \in \mathcal{P}$, $a + b = 1$, $a \geq 0$, $b \geq 0$.

Let $\tau_\circ^x$, $x \in R_+$, denote the generalized translation operator defined on C_b by

$$\tau_\circ^x f(y) = \int f(u)\delta_x \circ \delta_y(du),$$

where δ_x is the Dirac measure and the symbol $\int$ denotes the integral over $[0, \infty)$.

For $\mu \in \mathcal{P}$, we put

$$\tau_\circ^\mu f(y) = \int \tau_\circ^u f(y)\, \mu(du),$$

where $y \in R_+$ and f is a continuous function on R_+.

In the case, when $\circ$ is a regular generalized convolution in the sense of Urbanik, we consider the following generalized differential operator

$$D^\circ f(x) = \lim_{y \to 0^+} \frac{\tau_\circ^x f(y) - f(x)}{\omega(y)}$$

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where $f \in C_0$ (C_0 is a subspace of C_b consisting of all functions vanishing at infinity) and $\omega(\cdot)$ is defined by

$$\omega(y) = \begin{cases} 1 - \Omega(y), & 0 \le y \le x_0, \\ 1 - \Omega(x_0), & y > x_0. \end{cases}$$

with x_0 being a number such that $0 < \Omega(y) < 1$ for $0 < y \le x_0$ and $\Omega(x)$ is the kernel of the characteristic function.

2. Weak uniformly continuity and weak convergence

Definition 2.1. Given $S = \{\mu_t,\ t \in I\}$, $I \subset R$ and $S \subset \mathcal{P}$, a function f in C_b is called weak uniformly continuous on S if for any $\varepsilon > 0$, there exists $\delta > 0$ such that for all $y, z \in I$ with $|y - z| < \delta$ we have

$$\Big| \int f(u)\mu_y(du) - \int f(u)\mu_z(du) \Big| < \varepsilon.$$

The following proposition is obvious.

Proposition 2.1. *For each $x \in R_+$, a function f in C_b is weak uniformly continuous on $\mathcal{P}_\circ \times \delta_x$ if and only if $\tau_\circ^x f$ is bounded uniformly continuous, where $\mathcal{P}_\circ$ is the set of all Dirac measures on $[0, \infty)$ and $\mathcal{P}_\circ \times \delta_x = \{\delta_x \circ \delta_t,\ t \in R_+\}$.*

It is easily seen that f is weak uniformly continuous on $\mathcal{P}_\circ$ if and only if f is uniformly continuous on R_+.

Now let μ be a set function on a σ-field $\mathcal{A}$. Then the total variation of the set function μ on A is the number

$$\mathrm{Var}\,(\mu, A) = \sup \sum_k |\mu(A_k)|,$$

where sup is taken over all the finite $\mathcal{A}$-measurable partitions $\{A_k\}$ of A.

We have the following theorem.

Theorem 2.1. *Let $\{\mu_t,\ t \in I\}$, $I \subset R$ be an $\circ$-semigroup of probability measures on R_+. Assume that for every Borel subset A of R_+ and $\varepsilon > 0$, there exists $\delta > 0$ such that for any $t, s \in I$ with $|t - s| < \delta$ we have* $\mathrm{Var}\,(\mu_t - \mu_s,\, A) \le \varepsilon$. *Then, every function f in C_b is weak uniformly continuous on $\{\mu_t \circ \alpha,\ t \in I,\ \alpha \in \mathcal{P}\}$.*

To prove this theorem we use the following result of Vol'kovich:

Lemma 2.1. [8] *For any $\mu_1, \mu_2 \in \mathcal{P}$ and $f \in C_b$ we have following relationship*

$$\int f(x)\, \mu_1 \circ \mu_2(dx) = \int \int \tau_\circ^{x_1} f(x_2) \mu_1(dx_1) \mu_2(dx_2).$$

Proof of Theorem 2.1. First, let f be a step function, i.e. $f = \sum\limits_{i=1}^{n} a_i I_{A_i}$, where $\{a_1, a_2, ..., a_n\} \subset R$ and $\{A_1, ..., A_n\}$ is a Borel partition of R_+. Then

$$\begin{aligned}\Big|\sum_{i=1}^{n} a_i[\mu_t(A_i) - \mu_s(A_i)]\Big| &\le \sup_{1\le j\le n} |a_j|. \sum_{i=1}^{n} \big|\mu_t(A_i) - \mu_s(A_i)\big| \\ &\le \sup_{1\le j\le n} |a_j|. \operatorname{Var}(\mu_t - \mu_s,\ R_+).\end{aligned}$$

Since $\sup\limits_{1\le j\le n} |a_j| < \infty$, the hypotheses of the theorem implies that for every $\varepsilon > 0$ there exists $\delta > 0$ such that for any $t, s \in I$ with $|t - s| < \delta$ we have

$$\Big|\sum_{i=1}^{n} a_i[\mu_t(A_i) - \mu_s(A_i)]\Big| < \varepsilon.$$

Hence

$$\Big|\int f(u)\mu_t(du) - \int f(u)\mu_s(du)\Big| < \varepsilon. \tag{2.1}$$

Now, we consider the general case, when f is an arbitrary element of C_b. Then there exists a sequence $\{f_n\}$ of step functions that converges to f in norm. This with (2.1) yields

$$\Big|\int f(u)\mu_t(du) - \int f(u)\mu_s(du)\Big| \le \varepsilon. \tag{2.2}$$

On the other hand, since $\tau_\circ^u f \in C_b$, $u \in R_+$ (cf. [5]) and $\int \alpha(du) = 1$, we have

$$\begin{aligned}&\Big|\int f(u)\mu_t \circ \alpha(du) - \int f(u)\mu_s \circ \alpha(du)\Big| \\ &= \Big|\iint \tau_\circ^u f(x)\mu_t(du)\alpha(dx) - \iint \tau_\circ^u f(x)\mu_s(du)\alpha(dx)\Big| \\ &\le \sup_u \Big|\int \tau_\circ^u f(x)\mu_t(du) - \int \tau_\circ^u f(x)\mu_s(du)\Big|.\end{aligned} \tag{2.3}$$

This with (2.2) and (2.3) proves the theorem. □

From the method used in the proof of Theorem 2.1 we easily obtain the following corollary.

Corollary 2.1. *Assume that the hypotheses of Theorem 2.1 hold. Then, every f in C_b is weak uniformly continuous on $\{\mu_t,\ t \in I\}$.*

Theorem 2.2. *Assume that $\tau_\circ^u f$, $u \in R_+$ are uniformly equicontinuous functions on R_+, i.e. $\forall\varepsilon > 0$, $\exists\delta > 0$, $\forall x, y \in R_+$, $|x - y| < \delta$ we have*

$$\sup_{u\in R_+} |\tau_\circ^u f(x) - \tau_\circ^u f(y)| < \varepsilon.$$

Then, for any $\mu \in \mathcal{P}$, the function $\tau_\circ^\mu f$ is uniformly continuous and bounded.

Proof. We have

$$\begin{aligned}\left|\tau_\circ^\mu f(x)-\tau_\circ^\mu f(y)\right| &= \left|\int(\tau_\circ^x f-\tau_\circ^y f)(u)\mu(du)\right|\\ &\leq \int\left|(\tau_\circ^x f-\tau_\circ^y f)(u)\right|\mu(du)\\ &\leq \sup_{u\in R_+}\left|\tau_\circ^u f(x)-\tau_\circ^u f(y)\right|.\end{aligned}$$

□

From this inequality with the hypotheses on $\tau_\circ^u f$, $u\in R_+$, we deduce the following theorem.

Theorem 2.3. *Assume that the random convolution $\circ$ has unit element δ_{x_0}. Then $\mu_n \rightrightarrows \mu$ as $n\to\infty$ if and only if $\lim_{n\to\infty}\tau_\circ^{\mu_n}f(x)=\tau_\circ^\mu f(x)$, for each $x\in R_+$ and all bounded uniformly continuous f on R_+.*

Note that $\mu_n \rightrightarrows \mu$ as $n\to\infty$ denotes the weak convergence of μ_n to μ as $n\to\infty$.

Proof. First, assume that $\mu_n \rightrightarrows \mu$ as $n\to\infty$. Then by

$$\tau_\circ^x f(u)=\tau_\circ^u f(x),\quad \tau_\circ^u f\in C_b.$$

We obtain

$$\begin{aligned}\lim_{n\to\infty}\tau_\circ^{\mu_n}f(x) &= \lim_{n\to\infty}\int\tau_\circ^u f(x)\mu_n(du)\\ &= \lim_{n\to\infty}\int\tau_\circ^x f(u)\mu_n(du)=\int\tau_\circ^x f(u)\mu(du)\\ &= \int\tau_\circ^u f(x)\mu(du)=\tau_\circ^\mu f(x).\end{aligned}$$

Hence, for all $x\in R_+$ and bounded uniformly continuous f on R_+,

$$\lim_{n\to\infty}\tau_\circ^{\mu_n}f(x)=\tau_\circ^\mu f(x).$$

Conversely, suppose that $\lim_{n\to\infty}\tau_\circ^{\mu_n}f(x)=\tau_\circ^\mu f(x)$ for all bounded uniformly continuous f on R_+. Then we have

$$\lim_{n\to\infty}\int\tau_\circ^u f(x)\mu_n(du)=\int\tau_\circ^u f(x)\mu(du).$$

Therefore

$$\lim_{n\to\infty}\int\tau_\circ^x f(u)\mu_n(du)=\int\tau_\circ^x f(u)\mu(du). \tag{*}$$

In particular, substituting $x=x_0$ in (*) we obtain

$$\lim_{n\to\infty}\int f(u)\mu_n(du)=\int f(u)\mu(du),$$

because $\tau_\circ^{x_0}$ is the identity operator. Hence, $\mu_n \rightrightarrows \mu$ as $n \to \infty$. Thus, the theorem is proved. ☐

The following corollary is immediate.

Corollary 2.2. *Assume that the random convolution* $\circ$ *has an unit element* δ_{x_0} *and* $\{\mu_t,\ t \in R_+\}$ *is an* $\circ$*-semigroup. Then, for all bounded uniformly continuous* f *on* R_+

$$\lim_{t\to\infty} \tau_\circ^{\mu_t} f(x) = f(x), \quad \forall x \in R_+.$$

3. Random convolution and its translation operators

Proposition 3.1. *Assume that* $\circ$ *and* $\circ'$ *are two random convolutions on* $\mathcal{P}$. *Then* $\tau_\circ^x = \tau_{\circ'}^x$, *for all* $x \in R_+$ *if and only if* $\circ = \circ'$.

Proof. Using Lemma 2.1, we have

$$\begin{aligned} \tau_\circ^u = \tau_{\circ'}^u, \quad \forall u \in R_+ &\Leftrightarrow \int f(u)\mu_\circ\nu(du) = \int f(u)\mu_{\circ'}\nu(du),\ \forall f \in C_b \\ &\Leftrightarrow \mu_\circ\nu = \mu_{\circ'}\nu, \quad \forall \mu,\nu \in \mathcal{P} \\ &\Leftrightarrow \circ = \circ'. \end{aligned}$$

☐

By the proposition, one can easily prove the following

Corollary 3.1. *Let* $\circ$ *and* $\circ'$ *be two random convolutions on* $\mathcal{P}$. *Then*

(a) $\circ = \circ'$ *if and only if* $\tau_\circ^{\delta_x} = \tau_{\circ'}^{\delta_x}$, $x \in R_+$,

(b) $\circ = \circ'$ *on* $\mathcal{P}$, *provide* $\circ = \circ'$ *on* $\mathcal{P}_0$, *i.e.* $\delta_x \circ \delta_y = \delta_x \circ' \delta_y$, $x, y \in R_+$.

From now on, we assume $\circ$ to be a regular generalized convolution in the sense of Urbanik. Let $\{\mu_t\}_{t\geq 0}$ be a semigroup in the generalized convolution algebra $(\mathcal{P}, \circ)$ and $\{X_t\}$ an $\circ$-Lévy process generated by $\{\mu_t\}_{t\geq 0}$ (cf. [5]). We put $S_t^\circ = \tau_\circ^{\mu_t}$. It is clear that $\{S_t^\circ\}$ is also a semigroup.

An interesting problem is how to find relations between generalized differential operator D° and the random convolution $\circ$? For this problem we have the following result.

Theorem 3.1. *Let* $\circ$ *and* $\circ'$ *be two regular random convolutions,* $\{X_t\}$ *an* $\circ$*-Lévy process and* D° *its generalized differential operator. Suppose that the* $\mathcal{P}^\circ$*-distribution of* X_1 *is equal to* σ_κ *and*

$$V^{-1} = \int x^\kappa \sigma_\kappa(dx) < \infty,$$

where κ *is the characteristic exponent of the convolution* $\circ$, *taken in the sense of Urbanik and* σ_κ *the characteristic measure of* $(\mathcal{P}, \circ)$. *Then* $D^\circ = D^{\circ'}$ *if and only if*

$$S_t^\circ = S_t^{\circ'},\ t \in R_+.$$

Proof. First, we prove the “if” part. Let A° be the infinitesimal generator for the $\circ$-Lévy process $\{X_t\}$ such that $\mathcal{P}^\circ$-distribution of X_1 is equal to σ_κ. By the regularity of the operator $\circ$, it follows from Theorem 3.5 in [5] that

$$A^\circ f = D^\circ f, \quad f \in \mathcal{D}(D^\circ) \tag{3.1}$$

where $\mathcal{D}(D^\circ)$ be the domain of D°. Since $D^\circ = D^{\circ'}$ by the hypotheses and (3.1), we have

$$A^\circ = A^{\circ'}.$$

Therefore,

$$S_t^\circ = S_t^{\circ'}, \quad t \geq 0.$$

Conversely, suppose that $S_t^\circ = S_t^{\circ'}$ for all $t \geq 0$. By taking $t = 1$ we obtain

$$S_1^\circ = S_1^{\circ'}.$$

Let $\sigma_\kappa(\circ)$ and $\sigma_{\kappa'}(\circ')$ be the characteristic measures of the convolution algebras $(\mathcal{P}, \circ)$, $(\mathcal{P}, \circ')$, respectively. We have

$$\int \tau_\circ^a f(x)\sigma_\kappa(dx) = \int \tau_{\circ'}^a f(x)\sigma_{\kappa'}(dx). \tag{3.2}$$

Substituting $a = 0$ into (3.2) and noticing that $\tau_\circ^0 = \tau_{\circ'}^0 = I$ (where I is an identity operator), we get

$$\int f(x)\sigma_\kappa(dx) = \int f(x)\sigma_{\kappa'}(dx), \ \forall f \in C_b.$$

Therefore

$$\sigma_\kappa(\circ) = \sigma_{\kappa'}(\circ').$$

But if $\sigma_\kappa(\circ) = \sigma_{\kappa'}(\circ')$ then $\kappa = \kappa'$ and $\circ = \circ'$ (cf. [4]). It follows that $D^\circ = D^{\circ'}$. This completes the proof. ☐

By the same method, we get the following corollaries.

Corollary 3.2. *Suppose that the operations $\circ$ and $\circ'$ are regular. Then the equality $D^\circ = D^{\circ'}$ holds if and only if $\circ = \circ'$.*

Corollary 3.3. *The following equalities are equivalent*

(i) $\circ = \circ'$

(ii) $D^\circ = D^{\circ'}$

(iii) $S_t^\circ = S_t^{\circ'}, \ \forall t \geq 0$

(iv) $\tau_\circ^x = \tau_{\circ'}^x, \ \forall x \geq 0.$

4. Markov processes

In this section we assume $\circ$ to be a regular generalized convolution (in the sense of Urbanik). Let $\{X_t\}$ be an $\circ$-Lévy process, i.e. $\{X_t\}$ is generated by an $\circ$-semigroup $\{\mu_t,\ t \geq 0\}$, (cf. [5]) and A its infinitesimal operator.

In the case the $\mathcal{P}^\circ$-distribution of X_1 is equal to σ_κ, together with N. V. Thu, we get the following special properties of the generalized differential operator.

Theorem 4.1. *Let D° be the generalized differential operator for the Lévy process $\{X_t\}$ such that $\mathcal{P}^\circ$ distribution of X_1 is equal to σ_κ and $f \in \mathcal{D}(D^\circ)$. Suppose that $V^{-1} = \int x^\kappa \sigma_\kappa(dx) < \infty$. Then $u(t) = S_t f$ $(S_t \overset{def}{\to} = S_t^\circ)$ is the unique solution of the following differential equation:*

$$\frac{du}{dt} = D^\circ u,$$

subject to the following conditions

(a) *$u(t)$ is continuous differenftiable for $t > 0$,*

(b) *$\|u(t)\| \leq c.e^{mt}$ for some $c, m < \infty$,*

(c) *$u(t) \to f$ as $t \to 0^+$.*

Proof. Applying Theorem 3.5 in [5] we have

$$Af = D^\circ f, \quad f \in \mathcal{D}(D^\circ),$$

where A is an infinitesimal operator for $\circ$-Lévy process $\{X_t\}$. So, the differential equation $\dfrac{du}{dt} = D^\circ u$ is equivalent to following equation

$$\frac{du}{dt} = Au.$$

It is easily seen that $u(t) = S_t(f)$ is a solution that satisfies all the above mentioned conditions. It remains to show only the uniqueness.

Suppose that u_1 and u_2 are two solutions satisfying (a), (b), (c). We put $v(t) = u_1(t) - u_2(t)$. Then, $v(t)$ is a solution satisfying a) and b) and $v(t) \to 0$ as $t \to 0^+$. Let $w(t) = e^{-\lambda t}.v(t)$, where $\lambda > \max(m_1, m_2)$, (m_1, m_2 are the constants of the condition b) with respect to u_1, u_2). Since $v(t)$ is a solution of the equation $\dfrac{du}{dt} = Au$, we have

$$\frac{d}{dt} w(t) = -\lambda w(t) + e^{-\lambda t} A v(t) = -R_\lambda^{-1} w(t),$$

where $R_\lambda f(y) = \displaystyle\int_0^\infty e^{-\lambda t} S_t f(y) dt.$

We know that $\{S_t\}$ is a contraction semigroup with generator A. For each $\lambda > 0$, $(\lambda I - A)$ is an one-to-one map of $\mathcal{D}(A)$ onto C_b and the inverse map taking C_b onto $\mathcal{D}(A)$ is R_λ. Hence

$$w(t) = -R_\lambda \frac{dw(t)}{dt}.$$

Integrating both sides from 0 to s, we have

$$\int\limits_0^s w(t)dt = -R_\lambda \int\limits_0^s \frac{dw(t)}{dt}dt = -R_\lambda w(s).$$

When $s \to +\infty$, the left side tends to the Laplace transform of v and the right side tends to 0 because of assumption (b) and the choice of λ. Hence

$$\int\limits_0^\infty e^{-\lambda t} v(t)dt = 0$$

for each $\lambda > m$. We deduce $v(t) = 0$. Thus, the theorem is proved. $\square$

Finally, we give an application of the generalized differential operator to the ordinary differential equation.

Theorem 4.2. *The ordinary differential equation of Bessel type*

$$f''(x) + \frac{2s+1}{x} f'(x) - \lambda f(x) = g(x), \tag{4.1}$$

where $\lambda > 0$, $2(s+1) > 1$ *and* $g \in C^1$ *with* $g \neq 0$, *has a unique solution* $f \in \mathcal{D}(D^\circ)$ *with* $\|f\| \leq \|g\|$, *where* D° *is the infinitesimal operator in the Kingman convolution algebra. Moreover, this solution is*

$$f(x) = -\int\limits_0^\infty e^{-\lambda t} S_t g(x)dt.$$

Proof. In the case of Kingman convolution $*_{1,\beta}$ $(\beta = 2(s+1) > 1)$ we have the following formula (cf. N. V. Thu [5], p. 166):

$$D^\circ f(x) = f''(x) + \frac{(2s+1)}{x} f'(x).$$

So, the equation (4.1) is equivalent to the following equation

$$D^\circ f(x) - \lambda f(x) = g(x). \tag{4.2}$$

Let A denote the infinitesimal operator of the $\circ$-Lévy process $\{X_t\}$ in Theorem 3.1. By Theorem 3.5 (cf. N. V. Thu [5], p. 166) we see that equation (4.2) is equivalent to

$$Af(x) - \lambda f(x) = g(x) \quad (\text{because } Af = D^\circ f,\ \forall f \in \mathcal{D}(D^\circ))$$

$$\lambda f(x) - Af(x) = g(x). \tag{4.3}$$

By the theorem of Hille-Yosida, equation (4.3) has a unique solution belonging to $\mathcal{D}(D^\circ)$ such that $\|f\| \leq \|g\|$ and

$$f(x) = -\int_0^\infty e^{-\lambda t} S_t g(x) dt.$$

Thus, the theorem is proved. □

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REFERENCES

[1] N. H. Bingham, *On a theorem of Klosowska about generalized convolutions*, Coll. Math. **48** (1984), 117-125.

[2] M. Klosowska, *On the domain of attraction for generalized convolution algebras*. Re. Roumaine Math. Pures Appl. **22** (1977), 669-677.

[3] K. Urbanik, *Generalized convolutions*, Studia Math. **23** (1964), 217-245.

[4] K. Urbanik, *Generalized convolution IV*, Studia Math. **83** (1986), 57-95.

[5] Nguyen Van Thu, *Generalized independent increments processes*, Nagoya Math. J. **133** (1994), 155-175.

[6] Tran Hung Thao and K. S. Kuan, *On generalized independent increments processes*, Journal of Physical Science **8** (1977), 35-44.

[7] Nguyen Van Thu and Cao Van Nuoi, *The generalized differential operation and Markov processes*, International conference on probability and their applications, June 9-11, 1999, Hanoi, Vietnam.

[8] V. E. Vol'kovich, *Quasiregular stochastic convolution*, Soviet Math. J. **47** (1980), No 5, 2685-2698.

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