Volume 27, Number 1, 2002, pp. 77-85

EFFICIENCY EQUIVALENT POLYHEDRA FOR THE FEASIBLE SET OF MULTIPLE OBJECTIVE LINEAR PROGRAMMING

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Abstract. We propose an outer approximation algorithm for constructing a simple efficiency equivalent polyhedron for the feasible set of the multiple objective linear programming problem in the case where the ordering cone induced by the criteria functions is pointed and has a nonempty interior.

1. Introduction

In general, the efficient set of a multiple objective linear programming problem is a very complicated non-convex set. The complicated structure of the efficient set depends on the size of the problem, the size of data determining the feasible set and the criteria functions. That complication grows rapidly as the size of such data increases. An important problem arising is how one can reduce the complexity of the problem. In recent years, Benson [1]–[3], Dauer and Liu [4], Gallagher and Saleh [5] and some other authors have suggested the outcome set approach. This approach is effective when the number of the problem criteria is much smaller than that of the decision variables.

Another possibility to reduce the size of the problem is that instead of the feasible set one works with the so-called "efficiency equivalent" set, which has the same efficient set as the feasible set. The structure of the efficiency equivalent set is simpler than that of the feasible set of the original problem. Observe that, in general, not all of the data determining the feasible set must play a role in creating the efficient set.

In this paper we are concerned with the multiple objective linear programming problem in the special case where the ordering cone induced by criteria functions is pointed and has a nonempty interior. We propose here an outer approximation algorithm for finding all efficient extreme points and constructing efficiency equivalent polyhedron whose data size is, normally, smaller than that of the original feasible set.

Received November 13, 2000.

1991 *Mathematics Subject Classification.* 90 C29.

Key words and phrases. Multiple objective linear programming, efficiency equivalence, extreme efficient point, outer approximation.

This paper is partially supported by the Research Program of the National Center for Natural Sciences and Technology.

2. Efficiency and efficiency equivalent polyhedron

Throughout the paper $\mathbb{R}^k$ denotes an k-dimensional Euclidean space. For any two vectors $x' = (x'_1, \ldots, x'_k)$, $x = (x_1, \ldots, x_k) \in \mathbb{R}^k$ we write $x' \geq x$ if $x'_i \geq x_i$, for all $i = 1, \ldots, k$. We write $x' > x$ if $x' \geq x$ and $x' \neq x$. We always assume that C is a fixed $p \times n$-matrix with the row vectors $c^1, \ldots, c^p \in \mathbb{R}^n$, $p \geq 2$. Let

$$C_0 := \{x \in \mathbb{R}^n : Cx \geq 0\} \text{ and } \ell(C_0) := C_0 \cap (-C_0)$$

stand for the ordering cone induced by the matrix C and the lineality space of C_0, respectively. Recall that, for a given polyhedron Q, the efficient set Q_E of Q with respect to the ordering cone C_0 is defined by

$$Q_E := \{x^0 \in Q \ : \nexists x \in Q \text{ such that } x - x^0 \in C_0 \setminus \ell(C_0)\},$$

or, equivalently,

$$Q_E := \{x^0 \in Q \ : \nexists x \in Q \text{ such that } Cx > Cx^0\}.$$

It is well-known that Q_E is a connected set which is composed by some faces of the polyhedron Q.

Consider the multiple objective linear programming problem

$$\text{(VP)} \qquad \max Cx \ , \ x \in X,$$

where X is a polyhedron defined by a system of linear inequalities

$$(1) \qquad \langle a^i, x\rangle \geq b_i, \quad i = 1, \ldots, m,$$

where $a^1, \ldots, a^m$ are vectors from $\mathbb{R}^n$ and $b_1, \ldots, b_m$ are real numbers.

Recall that a point $x^0 \in X$ is called an *efficient solution* for (VP) if there exists no $x \in X$ such that $Cx > Cx^0$. Thus, the set of all the efficient solutions for (VP) is just the efficient set X_E of X with respect to the ordering cone C_0. If every point of a face $F \subseteq X$ is an efficient solution, then F is said to be an efficient (solution) face.

Definition 2.1. A polyhedron $Q \subset \mathbb{R}^n$ is said to be an *efficiency equivalent polyhedron* for X if $Q_E = X_E$.

The just defined notion of efficiency equivalent polyhedron differs slightly from the one considered in [2], [5] and [8].

The following proposition can be derived from Proposition 2.2 in [8]. However, for the convenience of the reader, we give here a direct proof.

Proposition 2.1. *Suppose that the ordering cone C_0 is pointed and has a non-empty interior. Then, a polyhedron $Q \subset \mathbb{R}^n$ is an efficiency equivalent polyhedron for X if*

$$X \subseteq Q \subseteq X - C_0.$$

Proof. Since C_0 is pointed then $\ell(C_0) = \{0\}$. First, we show that $X_E \subseteq Q_E$. Let $x^* \in X_E$. Since $X \subset Q$, we have $x^* \in Q$. Assume to the contrary that $x^* \notin Q_E$. By the definition, there is $x^0 \in Q$ such that $x^0 - x^* \in C_0 \setminus \{0\}$, i.e.,

$$x^0 - x^* = k^0 \neq 0 \text{ with } k^0 \in C_0. \tag{2}$$

By the assumption, $Q \subset X - C_0$. Therefore $x^0 = x^1 - k^1$ for some $x^1 \in X$ and $k^1 \in C_0$. This fact and (2) imply that $x^1 - x^* = k^1 + k^0 = k^2 \in C_0$ for some $k^2 \in C_0$. As $\ell(C_0) = \{0\}$, we must have $k^2 \neq 0$. Thus $x^1 - x^* \in C_0 \setminus \{0\}$. This shows that $x^* \notin X_E$, a contradiction. Hence, $x^* \in Q_E$.

Now, we prove that $X_E \supseteq Q_E$. Let $x^0 \in Q_E$. Since $Q \subseteq X - C_0$, x^0 can be represented as $x^0 = x^1 - x^2$ for some $x^1 \in X$ and $x^2 \in C_0$. Then $x^1 - x^0 = x^2 \in C_0$. Since $x^0 \in Q_E$, we must have $x^1 = x^0$. This shows that $x^0 \in X$. As $X \subset Q$, the inclusion $x^0 \in Q_E$ implies that $x^0 \in X_E$. □

Now, in order to reduce the size of (VP), we are interested in the efficiency equivalent polyhedra determined by the subsystems of the system (1).

Definition 2.2. A subset $I \subset \{1, \dots, m\}$ is an *E-index set* of (VP) if the polyhedron defined by the system

$$\langle a^i, x\rangle \geq b_i,\ i \in I,$$

is an efficiency equivalent polyhedron for X.

Let us denote

$$I_E := \bigcup_{x \in X_E} I(x),$$

where, $I(x)$ stands for the set of the *active indices* at $x \in X$, i.e.,

$$I(x) := \{i \in \{1, \dots, m\} : \langle a^i, x\rangle = b_i\}.$$

Denote by $X^{reduced}$ the polyhedron determined by the system

$$\langle a^i, x\rangle \geq b_i,\ i \in I_E. \tag{3}$$

To $X^{reduced}$ we associate the reduced problem

$$\text{(RVP)} \qquad \max Cx\ ,\ x \in X^{reduced}.$$

For a point $x \in X^{reduced}$ we denote by $II(x)$ the set of all active indices at x with respect to the system (3). Denote by $X_E^{reduced}$ the set of all efficient solutions for (RVP).

As will be seen below, the polyhedron $X^{reduced}$ is a good efficiency equivalent polyhedron for X. Furthermore, the problem of determining $X^{reduced}$ is equivalent to that of finding the active index sets of the efficient extreme points for (VP).

Recall (see [12]) that the system (1) has no redundant inequalities if X cannot be defined by a smaller number of inequalities of (1). A vertex x^0 of X is said to be *nondegenerate* if it satisfies as equalities exactly n inequalities (which must then be linearly independent) from (1). The polyhedron X is nondegenerate if every vertex of X is nondegenerate.

Proposition 2.2. *Suppose that X contains no lines and $X_E \neq \emptyset$. Then*

(i) *The set I_E is the union of the active index sets of the efficient extreme points for* (VP)*, i.e.,*

$$I_E = \bigcup_{x \in V(X_E)} I(x), \tag{4}$$

where $V(X_E)$ stands for the set of the efficient extreme points of (VP)*.*

(ii) *I_E is an E-index set for* (VP)*.*

(iii) *If the system* (1) *has no redundant inequalities and the polyhedron X is nondegenerate and has non-empty interior, then I_E is the unique smallest set among of the E-index sets for* (VP)*.*

Let us recall from [7] the condition for a point to be an efficient solution for problem (VP).

Proposition 2.3. (see [7, Corollary 5.4]) *A point $x^0 \in X$ is an efficient solution for* (VP) *if and only if the following system is consistent (has a solution)*

$$\sum_{i \in I(x^0)} \mu_i a^i = -\sum_{j=1}^{p} \lambda_j c^j$$

$$\mu_i \geq 0,\ i \in I(x^0), \quad \lambda_j > 0,\ j = 1, \ldots, p.$$

Proof of Proposition 2.2. (i) Since X contains no lines and $X_E \neq \emptyset$, the efficient set X_E is an union of some faces of X and every efficient face $F \subseteq X_E$ must contain some extreme point of X. In particular, $V(X_E) \neq \emptyset$.

To obtain the representation (4), we need only to show that

$$I(x^0) \subset \bigcup_{x \in V(X_E)} I(x) \tag{5}$$

for every point $x^0 \in X_E \setminus V(X_E)$.

Let $x_0 \in X_E \setminus V(X_E)$. Then x^0 must be a point in the relative interior of a face F of X_E. According to [9], F is determined by a system

$$\begin{aligned} \langle a^i, x\rangle &= b_i, \qquad i \in I_F, \\ \langle a^j, x\rangle &\geq b_j, \quad j \in \{1, \ldots, m\} \backslash I_F, \end{aligned}$$

for a nonempty index set $I_F \subseteq \{1, \ldots, m\}$. Since x^0 is a point in the relative interior of F, we must have $I(x^0) = I_F$. As mentioned in above, F must contain an efficient extreme point, say, $\bar{x} \in V(X_E)$. For such a vertex $\bar{x}$ we have that $I_F \subseteq I(\bar{x})$. Hence, $I(x^0) = I_F \subseteq I(\bar{x})$ and we get (5).

(ii) To show that I_E is an E-index set we have to verify that $X_E^{reduced} = X_E$.

First, we can show that $X_E \subseteq X_E^{reduced}$. Assume that $x^0 \in X_E$. Since $X \subset X^{reduced}$, we have that $x^0 \in X^{reduced}$. By definition of $X^{reduced}$, the index set I_E contains the set $I(x^0)$. Hence, $II(x^0) = I(x^0)$. Therefore, in view of Proposition 2.3 we can see that x^0 is an efficient solution for Problem (RVP), i.e. $x^0 \in X_E^{reduced}$. Thus, $X_E \subseteq X_E^{reduced}$.

Next, we prove the reverse inclusion $X_E^{reduced} \subseteq X_E$. Assume to the contrary that $X_E^{reduced} \not\subseteq X_E$. Recall that the efficient sets X_E and $X_E^{reduced}$ consist of some faces of X and $X^{reduced}$, respectively. Furthermore, they are closed connected sets and $X \subseteq X^{reduced}$. Hence, if $X_E^{reduced} \not\subseteq X_E$, then there exists an efficient face $F \subset X_E^{reduced}$ such that $F \not\subseteq X_E$ and $F \cap X_E \neq \emptyset$. For such a face F there exists a point $x_0 \in ri(F) \setminus X_E$, since F is a face of $X_E^{reduced}$ and X_E is closed. Therefore, we can choose a point $x^* \in X_E \cap F$ such that $[x^0, x^*] \subseteq F$ and $[x^0, x^*] \cap X_E = \{x^*\}$. Note from definition that $X_E^{reduced} \cap X \subseteq X_E$. So, we have also $[x^0, x^*] \cap X = \{x^*\}$. Let $J := \{1, \dots, m\} \setminus I_E$ and consider the sets

$$L_i := \{x \in [x^0, x^*] : \langle a_i, x\rangle \geq b_i\}$$

for $i \in J$. We have

$$\bigcap_{i \in J} L_i = [x^0, x^*] \cap X = \{x^*\}.$$

Since $x^* \in X_E \subseteq X$, it is clear that $x^* \in L_i$ for all $i \in J$. Then, each of the sets L_i is of the form $[x^i, x^*]$ for a point $x^i \in [x^0, x^*]$. Hence, there exists an index $k \in J$ such that $L_k = \{x^*\}$. This ensures that $\langle a_k, x^*\rangle = b_k$, i.e. $k \in I(x^*)$. This means that $J \cap I(x^*) \neq \emptyset$ that contradicts the definition of I_E. Thus, $X_E^{reduced} \subseteq X_E$.

(iii) Let $I \subset \{1, \dots, m\}$ be an E-index set of (VP). To prove that I_E is the smallest E-index set, we have to show only that $I(x^0) \subseteq I$ for all $x^0 \in V(X_E)$. This fact together with (i) implies that $I_E \subseteq I$. Let Q be the polyhedron determined by the system

$$\langle a^i, x\rangle \geq b_i, \quad i \in I,$$

which is an efficiency equivalent polyhedron for X. By definition, $X \subset Q$, $X_E = Q_E$ and $V(X_E) = V(Q_E)$, where $V(Q_E)$ stands for the set of the efficient extreme points of Q_E. Let $x^0 \in V(X_E)$ and denote $J := \{i \in I : \langle a_i, x^0\rangle = b_i\}$. By definition, $J \subseteq I(x^0)$. Since $V(X_E) = V(Q_E)$, x^0 is a vertex of Q. Therefore, the set J consists of at least n distinct indices. On the other hand, by assumption, X is a nondegenerate polyhedron with non-empty interior and the system (1) has no redundant inequalities. Then, x^0 is a non-degenerate vertex of X and the active index set $I(x^0)$ has exactly n distinct elements. Hence, $J = I(x^0)$. Thus, the active index set $I(x^0)$ is a subset of I. □

3. Algorithm for constructing efficiency equivalent polyhedra

We consider Problem (VP) with the assumption that the ordering cone C_0 is pointed. We shall restrict ourselves to the case where the system (1) has no redundant inequalities and the feasible set X has non-empty interior and contains no lines. To avoid trivial cases, we also assume that the efficient set X_E is nonempty. Note that there are several method can be used to check whether the efficient solution set X_E is nonempty (see, for example, Corollary 5.5 in [7]).

Denote by Rec$Y$ the recession cone of a polyhedral convex set Y consisting of all vectors $v \in R^n$ such that $x + tv \in Y$ for all $x \in Y$ and $t \geq 0$.

Our algorithm consists of two phases:

1) Construct an efficiency equivalent polyhedron Q for X. As a result, we also obtain the vertex set $\mathrm{V}(Q)$ and the extreme direction set $\mathrm{R}(Q)$ of Q .

2) Determine the set $V(X_E)$ of all efficient extreme points of X and the E-index set I_E.

Phase I

Initialization Step.

a) X is bounded: Construct a simple polytope Q^0 (for example a suitable simplex or a box) such that $X \subseteq Q^0$. (This is possible because X is a bounded set.) Store the vertex set $\mathrm{V}(Q^0)$. Set $k := 0$ and go to Step 2.

b) X is unbounded: Construct a simple polyhedral convex set $Q^0 \supset X$. For example,

$$Q^0 := \{x \in R^n : \langle a^i, x\rangle \geq b_i,\ i \in I(x^0)\},$$

where x^0 is a vertex of X. Store the vertex set $\mathrm{V}(Q^0)$ and the extreme direction set $\mathrm{R}(Q^0)$. Set $\ell := 0$ and go to Step 1.

Step 1. Iteration ℓ, $\ell \geq 0$.

If $\mathrm{R}(Q^\ell) \nsubseteq \mathrm{Rec}(X - C_0)$ **Then**

find a $v^0 \in \mathrm{R}(Q^\ell) \setminus \mathrm{Rec}X$ and an index $i_0 \in \{1, \dots, m\}$ such that $\langle a^{i_0}, v^0\rangle < 0$. Set

$$Q^{\ell+1} := \{x \in Q^\ell \ :\ \langle a^{i_0}, x\rangle \geq b_{i_0}\}.$$

Using $\mathrm{V}(Q^\ell)$, $\mathrm{R}(Q^\ell)$ and the definition of $Q^{\ell+1}$, determine $\mathrm{V}(Q^{\ell+1})$ and $\mathrm{R}(Q^{\ell+1})$. Set $\ell := \ell + 1$ and go to Iteration ℓ.

Else Set $Q^0 := Q^\ell$. Go to Step 2

Step 2. Iteration k, $k \geq 0$.

If $\mathrm{V}(Q^k) \nsubseteq X - C_0$ **Then**

find a $x^0 \in V(Q^k) \setminus X$ and an index $i_0 \in \{1, \dots, m\}$ such that $\langle a^{i_0}, x^0\rangle < b_{i_0}$. Set

$$Q^{k+1} := \{x \in Q^k \ :\ \langle a^{i_0}, x\rangle \geq b_{i_0}\}.$$

Using $\mathrm{V}(Q^k)$, $\mathrm{R}(Q^k)$ and the definition of Q^{k+1}, determine $\mathrm{V}(Q^{k+1})$ and $\mathrm{R}(Q^{k+1})$. Set $k = k + 1$ and go to Iteration k.

Else Set $Q := Q^k$. STOP.

The iteration process is terminated after a finite steps. Note that $Q^k \subseteq X - C_0$ if and only if $\mathrm{V}(Q^k) \subseteq X - C_0$ and $\mathrm{R}(Q^k) \subseteq \mathrm{Rec}(X - C_0)$. So, after Phase I we obtain a polyhedron Q together with the sets $\mathrm{V}(Q)$ and $\mathrm{R}(Q)$ such that

$$X \subseteq Q \subseteq X - C_0.$$

Hence, by Proposition 2.1 Q is an efficiency equivalent polyhedron for X and

$$V(X_E) = V(Q) \cap X_E. \tag{6}$$

Phase II

Let $I := \emptyset$ and $V := \emptyset$.

For every point $x \in V(Q)$ **do**

If $x \in X$ **Then**

If the system

$$\begin{aligned} \sum_{i \in I(x)} \mu_i a^i &= -\sum_{j=1}^{p} \lambda_j c^j \\ \mu_i \geq 0, &\quad i \in I(x), \\ \lambda_j > 0, &\quad j = 1, \ldots, p, \end{aligned} \tag{7}$$

has a solution

Then add x into V and $I(x)$ into I.

By Proposition 2.3, if the system (7) has a solution then $x \in X_E$. It follows from (6) that $x \in V(X_E)$. Hence, according to (6), after Phase II one obtains two sets V and I such that $V = V(X_E)$ and $I = I_E$.

Let us conclude this section with some remarks on the implementation of the computational modules in the above algorithm.

Remark.

(i) One can verify by definition that $\mathrm{Rec}X \subseteq \mathrm{Rec}(X - C_0)$. Hence, if $v^0 \notin \mathrm{Rec}(X - C_0)$ then $v^0 \notin \mathrm{Rec}X$. When X is given by the system (1), $\mathrm{Rec}X$ is the solution set to the homogeneous system

$$\mathrm{Rec}X = \{x \in R^n \ : \ \langle a^i, x \rangle \geq 0, \ i = 1, \ldots, m\},$$

(see, for example, [12, Prop. 1.26]). So, if $v^0 \notin \mathrm{Rec}X$ then there is an index $i_0 \in \{1, \ldots, m\}$ such that $\langle a^{i_0}, v^0 \rangle < 0$.

(ii) To check the inclusions $\mathrm{R}(Q^\ell) \not\subseteq \mathrm{Rec}(X - C_0)$ and $\mathrm{V}(Q^k) \not\subseteq X - C_0$ in Phase I, we can use the following simple observation, which can be easily verified by definition.

Proposition 3.1. *A point $x^0 \in X - C_0$ if and only if the following system is consistent*

$$\begin{aligned} \langle a^i, x\rangle &\geq b_i, \quad i = 1, \dots, m,\\ Cx &\geq Cx^0. \end{aligned}$$

By Lemma 1.1 from [12], $v \in \mathrm{Rec}(X - C_0)$ if and only if $x^0 + tv \in X - C_0$ for some $x^0 \in X$ and $t \geq 0$.

(iii) The most expensive computational cost in the algorithm is in determining vertices and extreme directions of each generated polyhedron Q^{k+1} in Phase I. Since Q^{k+1} is obtained from Q^k by adding a new constraint inequality, the vertices and the extreme directions of Q^{k+1} can be calculated from those of Q^k by using some existing efficient methods (see, for example, [6], [10] and [11]).

4. An example

To illustrate the notion of efficiency equivalent polyhedron and the algorithm, we consider the (VP) where C is 3×3-unit matrix, and X is defined by 10 linear inequalities

$$\begin{aligned} -x - y - z &\geq -2,9\\ -x &\geq -1\\ -y &\geq -1\\ -z &\geq -1\\ x &\geq 0\\ y &\geq 0\\ z &\geq 0\\ 4x + 4y + z &\geq 0.8\\ x + 4y + 4z &\geq 0.8\\ x + y + z &\geq 0.5 \end{aligned} \tag{8}$$

In this example, the ordering cone C_0 is the positive orthant cone of $\mathbb{R}^3$ and the efficient set X_E consists of a 2-dimensional face with 3 efficient vertices:

$$\begin{aligned} v^1 &= (1.000, 0.900, 1.000)\\ v^2 &= (1.000, 1.000, 0.900)\\ v^3 &= (0.900, 1.000, 1.000) \end{aligned}$$

We now show how the algorithm works with this example.

Suppose that in Phase I we start with Q^0 to be the box $[0,1] \times [0,1] \times [0,1]$ in $\mathbb{R}^3$. Q^0 has 8 vertices and $(1,1,1)$ is the unique vertex of Q^0 not belonging to $X - C_0$. The process of iteration is terminated after only one step and we obtain

$$V(Q) = (V(Q^0) \setminus \{(1,1,1)\}) \cup \{v^1, v^2, v^3\}.$$

After Phase II we obtain $V(X_E) = \{v^1, v^2, v^3\}$. The efficiency equivalent polyhedron $X^{reduced}$ is determined just by the first four inequalities in (9).

Acknowledgments

The author would like to express the deepest gratitude to her advisers, Prof. Dinh The Luc and Prof. Le Dung Muu, for their help and encouragement. The author is greatly indebted to the referee for his valuable comments and suggestions that helped her very much in revising the paper.

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