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A NON-HOMOGENEOUS P-LAPLACE EQUATION IN BORDER CASE

PHAM XUAN DU AND DUONG MINH DUC

Abstract. Searching minimizers of functions on the convenient level set of the constraint function we obtain generalized solutions of a non-homogeneous p-Laplace equation in border case without using the regularity results of linear elliptic equations.

1. Introduction

Let Ω be a bounded domain in $\mathbb{R}^n$ having C^2-boundary $\partial\Omega$. Denote by $W^{1,n}(\Omega)$ the usual Sobolev space. In the present paper we study the existence of generalized solution of the following boundary equation.

$$\text{(P)} \qquad \begin{cases} \nabla(|\nabla u|^{n-2}\nabla u) \ + \ ke^u \ = \ h \text{ in } \Omega, \\ u\big|_{\partial\Omega} \ = \ \text{constant}. \end{cases}$$

If $n = 2$, the equation (P) is extensively studied (see [5] for references) and is related to the problem of constructing a metric γ with prescribed scalar curvature function k, which is conformal to γ_0, that is $\gamma = e^u\gamma_0$ for some function u on Ω. The Laplacian and the scalar curvature of γ_0 are denoted by Δ and h respectively, where we adopt the sign convention that $\Delta = \sum_{j=1}^{2} \dfrac{\partial^2}{\partial x_j^2}$ for the flat metric on $\mathbb{R}^2$.

For the case $n = 2$, Kazdan and Warner [6, 7] studied (P), where k and h belong to $L^s(\Omega)$ for some $s > 2$. If $k \geq 0$, using a generalized mountain pass theorem, we have solved (P) in [4] for non-integrable functions k and h. But the method in [4] could not be applied to the case in which k changes the sign. Vy and Schmitt in [8] have solved (P) if k is in $C(\overline{\Omega})$ and h is in $L^2(\Omega)$. In [1] we use the variational methods of [8] and a generalized Ljusternik theorem to get solutions of (P) with conditions on h and k in Remark 1.

For the case $n \geq 3$, Le and Schmitt [8] have announced the existence of generalized solutions to (P) for k is in $C(\overline{\Omega})$ and h is in $L^2(\Omega)$ without proofs.

We could not applied the methods of [1, 8] due to degeneracy and strong nonlinearity of (P). Our main idea to overcome the difficulties is as follows:

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we search minimizers of functions on the convenient level set of the constraint function and we obtain desired generalized solutions without using the regularity results of linear elliptic equations.

First we introduce following notations:

$$\|u\| = \Big(\int_\Omega (|u|^n + |\nabla u|^n)\,dx\Big)^{1/n} \qquad \forall\ u \in W^{1,n}(\Omega),$$

$$\|u\|_* = \Big(\int_\Omega |\nabla u|^n\,dx\Big)^{1/n} \qquad \forall\ u \in W^{1,n}(\Omega),$$

$$H = \Big\{\ u \in W^{1,n}(\Omega)\ : u\,|_{\partial\Omega} =\ 0\ \Big\},$$

$$V = \Big\{\ u \in W^{1,n}(\Omega)\ : u\,|_{\partial\Omega}\ \text{ is constant }\Big\}.$$

We consider H as a Banach space with the norm $\|.\|_*$. Let p be in $(1\ ,\ \infty)$. Denote by $\mathbf{F}_p(\Omega)$ the family of all measurable functions ϕ on Ω having the following properties:

(F) For any sequence $\{u_m\}$ weakly converging to u in H,

$$\lim_{m\to\infty}\int_\Omega |u_m - u|^p|\phi|dx \ = \ 0.$$

Denote by $\mathbf{G}(\Omega)$ the family of all measurable function ϕ on Ω such that

(G) There is an increasing function Φ on $[0,\infty)$ such that

$$\int_\Omega |\phi|e^u dx \ \le \ \Phi(\|u\|) \qquad \forall u \in H.$$

Our main result is the following theorem.

Theorem 1.1. *Let p be in $(1\ ,\ \infty)$, k in $\mathbf{G}(\Omega)\cap\mathbf{F}_p(\Omega)$ and h an integrable function in $\mathbf{F}_p(\Omega)$. Assume that the set $B = \Big\{\int_\Omega ke^w dx\ :\ w \in H\Big\}$ is unbounded from above in $\mathbb{R}$. Then there exists a generalized solution u of (P) in V.*

2. Proof of Theorem 1.1

We need the following notations and definitions.

Definition 2.1. Denote by G the space $C_c^\infty(\Omega)$. Let ν be a function from H into $\mathbb{R}$. We say ν is weakly continuously differentiable with respect to G on H if the following two conditions are satisfied

(i) For any $x \in H$ there exists a linear mapping $D\nu(x)$ from G into $\mathbb{R}$ such that

$$\lim_{t\to 0}\frac{\nu(x+th)\ -\ \nu(x)}{t} = D\nu(x)(h), \qquad \forall h \in G.$$

(ii) For any $h \in G$, the map $x \mapsto D\nu(x)(h)$ is continuous on H.

For any u in H we put

$$\varphi(u) = \frac{1}{n}\int_\Omega |\nabla u|^n \, dx,$$

$$j(u) = \int_\Omega hudx,$$

$$\psi(u) = \int_\Omega ke^u dx,$$

$$F(u) = \varphi(u) + j(u).$$

Under the conditions of Theorem 1.1 we have the following lemmas.

Lemma 2.1. (i) *φ and j are weakly lower-semi continuous on H.*

(ii) *ψ is weakly continuous on H and ψ, F are weakly continuously differentiable with respect to G on H, and for any u in H and v in G,*

$$D\psi(u)(v) = \int_\Omega kve^u dx,$$

$$DF(u)(v) = \int_\Omega |\nabla u|^{n-2} \nabla u \nabla v dx + \int_\Omega hvdx.$$

Proof. (i) It is clear that φ is weakly lower-semi continuous on H. Now we prove that j is weakly continuous on H. Let $\{u_m\}$ be a sequence weakly converging to u in H. By the Hölder's inequality, we get

$$\Big|\int_\Omega hu_m dx - \int_\Omega hudx\Big| \le \Big(\int_\Omega |h| \, dx\Big)^{\frac{1}{q}}\Big(\int_\Omega |u_m - u|^p |h| \, dx\Big)^{\frac{1}{p}}. \tag{2.1}$$

Thus j is weakly continuous on H by the condition (F).

(ii) Let $\{u_m\}$ be a sequence weakly converging to u in H. Let q be $\dfrac{p}{p-1}$. Using the Hölder's inequality we have

$$\begin{aligned}
&\Big|\int_\Omega (e^{u_m} - e^u)kdx\Big| \le \int_\Omega |e^{u_m} - e^u||k| \, dx \\
&= \int_\Omega e^u \, |e^{u_m - u} - 1||k| \, dx \le \int_\Omega e^u e^{|u_m - u|} \, |u_m - u||k| \, dx \\
&\le \Big(\int_\Omega e^{2qu} \, |k| \, dx\Big)^{\frac{1}{2q}}\Big(\int_\Omega e^{2q|u_m - u|} \, |k| \, dx\Big)^{\frac{1}{2q}}\Big(\int_\Omega |u_m - u|^p |k| \, dx\Big)^{\frac{1}{p}}.
\end{aligned}$$

Since $\{||u_m - u||\}_m$ is bounded in $\mathbb{R}$ and k is in $\mathbf{G}(\Omega)$, we see that

$$\sup_{m\in\mathbb{N}} \Big(\int_\Omega e^{2qu} \, |k| \, dx\Big)^{\frac{1}{2q}}\Big(\int_\Omega e^{2q|u_m - u|} \, |k| \, dx\Big)^{\frac{1}{2q}} < \infty.$$

Thus, ψ is weakly continuous on H by (F).

Let u be in H, v be in G and t be in $\mathbb{R}\setminus\{0\}$. We have

$$\frac{\psi(u+tv)-\psi(u)}{t} = \int_\Omega k.e^{u}\frac{e^{tv}-1}{t}dx.$$

Note that

$$\lim_{t\to 0} ke^{u}\frac{e^{tv}-1}{t} = ke^{u}v \text{ and } |k.e^{u}\frac{e^{tv}-1}{t}| \le |k|e^{u}|v|e^{|t||v|} \le C|k|e^{u}$$

for all $|t|$ sufficiently small, where C is a constant depending only on v. On the other hand, since k is in $\mathbf{G}(\Omega)$, ke^{u} is integrable on Ω. Thus, by the Lebesgue dominated convergence theorem we have

$$D\psi(u)(v) = \lim_{t\to\infty}\frac{\psi(u+tv)-\psi(u)}{t} = \int_\Omega kve^{u}dx, \qquad \forall u\in H,\ v\in G. \tag{2.2}$$

Now fix a v in G. Note that v is in $L^\infty(\Omega)$. Arguing as above with kv instead of k, we see that the map $u\mapsto D\psi(u)(v)$ is continuous on H for any v in G. It is easy to see that

$$DF(u)(v) = \int_\Omega |\nabla u|^{n-2}\,\nabla u\nabla v dx + \int_\Omega hv dx \qquad \forall\ u\ \in H,\ v\ \in\ G.$$

Now let $\{u_m\}\subset H$ such that $u_m\to u$ in H. Since

$$\frac{1}{n}+\frac{n-2-i}{n}+\frac{i}{n}+\frac{1}{n}=1,$$

using Hölder's inequality we have

$$\begin{aligned}
&|\,DF\,(u_m)\,(v) - DF\,(u)\,(v)\,|\\
&\le \int_\Omega |\,|\nabla u_m|^{n-2}\,\nabla u_m - |\nabla u|^{n-2}\,\nabla u\ |\ .\ |\nabla v|\ dx\\
&\le \int_\Omega \big(|\nabla u_m|^{n-2}|\nabla u_m-\nabla u| + \big|\ |\nabla u_m|^{n-2}-|\nabla u|^{n-2}\big|\,|\nabla u|\big)\,|\nabla v|dx\\
&\le \sum_{i=0}^{n-2}\int_\Omega |\nabla u_m-\nabla u||\nabla u_m|^{n-2-i}|\nabla u|^{i}|\nabla v|dx\\
&\le \Big(\sum_{i=0}^{n-2}\|u_m\|_*^{n-2-i}\|u\|_*^{i}\,\|v\|_*\Big)\|u_m-u\|_*.
\end{aligned}$$

Thus the map $u\mapsto DF\,(u)\,(v)$ is continuous on H for any v in G and we get the lemma. □

Lemma 2.2. *Denote by A the family of all function u in H such that*

$$\int_\Omega |\nabla u|^{n-2}\,\nabla u\nabla v dx + \int_\Omega hv dx = 0 \qquad \forall v\in H. \tag{Q}$$

Then there exist u_1 in H and a real number α_1 such that

$$\int_\Omega ke^{u_1}dx = \alpha_1 > \int_\Omega ke^{u}dx \qquad \forall\ u\ \in\ A.$$

Proof. By (F) we see that j is a continuous linear function on $L^p(\Omega)$. Thus, by the embedding Sobolev theorem there is a positive real number $C_{h,p}$ such that

$$\Big|\int_\Omega hudx\Big| \leq C_{h,p}\Big(\int_\Omega |\nabla u|^n\, dx\Big)^{\frac{1}{n}} \qquad \forall\ u\ \in\ H. \tag{2.3}$$

Let u be in A. Choosing $v = u$ in (Q) and using (2.3), we get

$$\int_\Omega |\nabla u|^n\, dx = \Big|\int_\Omega hudx\Big| \leq C_{h,p}\Big(\int_\Omega |\nabla u|^n\, dx\Big)^{\frac{1}{n}}.$$

Consequently there is a real number M such that $\|u\|_H \leq M$ for any u in A. Therefore, by (G) we see that for any u in A

$$\int_\Omega |k|\, e^{u}dx \leq \Phi(M).$$

Since the set B is unbounded from above in $\mathbb{R}$, we get the lemma. $\square$

Lemma 2.3. *Denote by S the set $\{u \in H : \psi(u) \geq \alpha_1\}$, where α_1 is in Lemma 2.2. Then there exists $u_2 \in S$ such that*

$$F(u_2) = \min F(S).$$

Proof. By (2.3) we see that F is coercive. Thus we get the lemma by applying Theorem 1.2 of [9]. $\square$

Lemma 2.4. *Let u_2 be as in Lemma 2.3. Then there is a non-negative real number λ such that*

$$DF(u_2)(v) - \lambda D\psi(u_2)(v) = 0, \qquad \forall v \in G.$$

Proof. Using the Ljusternik theorem of [1] or [2] we can find a real number λ such that

$$DF(u_2)(v) - \lambda D\psi(u_2)(v) = 0, \qquad \forall v \in G.$$

Put $N = \{u \in H : \psi(u) \geq \beta\}$, where $\beta = \psi(u_2)$. We have $F(u_2) = \min F(N)$ and

$$\psi(u_2)\ \geq\ \alpha_1 \tag{2.4}$$

Since B is unbounded, we see that $k \not\equiv 0$ and there is a v in G such that

$$D\psi(u_2)(v)\ =\ \int_\Omega ke^{u_2}vdx\ >\ 0.$$

Because ψ is weakly continuously differentiable respect to G at u_2, there is a sufficiently small positive real number t_1 such that

$$\begin{aligned} \psi(u_2+tv) > \psi(u_2) \equiv \beta \qquad & \forall\ t \in (0,t_1) \quad \text{or} \\ u_2+tv \in N \qquad & \forall\ t \in (0,t_1) \end{aligned}$$

Put $\varphi(t) = \dfrac{F(u_2+tv)-F(u_2)}{t} - DF(u_2)(v)$ for any t in $(0,t_1)$. We have $\lim_{t\to 0}\varphi(t) = 0$ and for any t in $(0,t_1)$ we have

$$0 \le F(u_2+tv)-F(u_2) = tDF(u_2)(v) + t\varphi(t) = t\lambda D\psi(u_2)(v) + t\varphi(t).$$

Thus λ should be nonnegative. □

Proof of Theorem 1.1. Let u_2 and λ be as in Lemma 2.4. We have

$$\int_\Omega (|\nabla u_2|^{n-2}\nabla u_2 \nabla v + hv)dx = \lambda \int_\Omega ke^{u_0} v dx \qquad \forall v \in G. \tag{2.5}$$

Let $\{v_m\}$ be a sequence in G which converges to v in H. We have

$$\begin{aligned} &\lim_{m\to\infty}\Big(\int_\Omega |\nabla u_2|^{n-2}\nabla u_2\nabla v_m dx + \int_\Omega hv_m dx) \\ &= \int_\Omega |\nabla u_2|^{n-2}\nabla u_2 \nabla v dx + \int_\Omega hvdx, \end{aligned} \tag{2.6}$$

$$\lim_{m\to\infty}\lambda\int_\Omega ke^{u_2}v_m dx = \lambda\int_\Omega ke^{u_2}vdx. \tag{2.7}$$

Combining (2.5), (2.6) and (2.7) we obtain

$$\int_\Omega (|\nabla u_2|^{n-2}\nabla u_2\nabla v + hv)dx = \lambda\int_\Omega ke^{u_2}vdx \qquad \forall v \in H.$$

We will prove that $\lambda > 0$. In fact, suppose by contradiction that $\lambda = 0$. Then we get

$$\int_\Omega (|\nabla u_2|^{n-2}\nabla u_2 \nabla v + hv)dx = 0 \qquad \forall v \ \in \ H.$$

Thus u_2 is in A and $\psi(u_2) < \alpha_1$ by Lemma 2.2, which contradicts to (2.4). Therefore λ is positive.

Put $u_0 = u_2 + \ln\lambda$. It is obvious that u_0 is in V and

$$\int_\Omega (|\nabla(u_0-\ln\lambda)|^{n-2}\nabla(u_0-\ln\lambda))\nabla v dx + \int_\Omega hvdx = \lambda\int_\Omega ke^{u_0 - \ln\lambda}vdx \qquad \forall v \in G.$$

Therefore

$$\int_\Omega (|\nabla u_0|^{n-2}\nabla u_0)\nabla vdx + \int_\Omega hvdx = \int_\Omega ke^{u_0}vdx \qquad \forall v \in G$$

and u_0 is a generalized solution of (P) in V. $\square$

Remark 1. Put $d_\Omega(x) = \inf\{|x-y| : y \in \partial\Omega\}$ for any $x \in \mathbb{R}^n$. Let s_1 and s_2 be in $(1, \infty]$, γ_1 and γ_2 be in $(\frac{1}{s_i} - 2\ ,\ 0]$, $i = 1, 2$, and ϕ_1 and ϕ_2 be nonnegative measurable functions on Ω such that $d_\Omega^{-\gamma_i}\phi_i \in L^{s_i}(\Omega)$ for $i = 1, 2$. Let k and h be functions on Ω having the following properties:

(C_1) $h,\ k$ are integrable on Ω such that $\left\{\int_\Omega ke^w dx \ :\ w \in H\right\}$ is unbounded from above,

(C_2) $$|h| \le \phi_1 \text{ and } |k| \le \phi_2.$$

Using the results in [4] we can prove that $h,\ k$ satisfy the conditions of Theorem 1.1 with $p = 2$.

Remark 2. If k is in $C(\overline{\Omega})$ and h is in $L^{\frac{p}{p-1}}$ with $p \in [n, \infty)$, Theorem 1.1 is proved in [8].

Remark 3. If $n = 2$ Theorem 1.1 was proved in [1].

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Department of Mathematics-Informatics,
Faculty of Natural Sciences
National University of Hochiminh City
227 Nguyen Van Cu, District 5
Hochiminh City, Vietnam.

E-mail address: `dmduc@netnam2.org.vn`