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NONSMOOTH B-PREINVEX FUNCTIONS

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Abstract. Necessary and sufficient conditions under which a locally Lipschitz function is B-preinvex are established in terms of Clarke's generalized gradients.

1. Introduction

The convexity plays an important role in optimization theory. By weakening certain properties of convex functions, various generalizations of convex functions have been studied. The concept of convex functions was generalized to quasiconvex by Mangasarian [12]. Hanson [9] and Craven [6] introduced the class of invex functions, while Craven, Luu and Glover [7] studied the strengthened invexity together with Lagrangian sufficient conditions for minimax problems. Later, Bector and Singh [1] considered a class of functions called B-vex which are quite similar to the (α, λ)-convex functions introduced by Castagnoli and Mazzoleni [4]. The equivalence between the class of B-vex functions and that of quasiconvex functions has been shown by Li, Dong and Liu [11]. A class of functions called B-invex, pseudo B-vex, pseudo B-invex, quasi B-vex and quasi B-invex is introduced by Bector, Suneja and Lalitha [2] together with sufficient optimality conditions and duality results for mathematical programs involving B-vex and B-invex functions. Ben-Israel and Mond [3], Hanson and Mond [10] introduced a class of functions which were called preinvex by Weir and Jeyakumar (see [14]).

Recently, Suneja, Singh and Bector [13] introduced the concept of B-preinvex functions by relaxing the definitions of preinvex and B-vex functions. They studied some properties of B-preinvex functions and gave some examples to show that there exist functions which are B-preinvex but not preinvex or B-vex.

This paper develops further properties of nonsmooth B-preinvex functions.

The paper is organized as follows. Section 2 gives some preliminaries and a characteristic property of B-preinvex functions. In Section 3 necessary conditions and sufficient conditions for a locally Lipschitz function to be B-preinvex are established.

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2. A characteristic property of B-preinvex functions

Let D be a nonempty subset of a Banach space X, and let f be a real-valued function defined on D.

Assume that the set D is convex. Adapting Bector-Singh's definition [1], we shall say that f is B-vex at a point $x_0 \in D$ with respect to a function $b : D \times D \times [0,1] \to [0,1]$ if for every $x \in D$ and $\lambda \in [0,1]$,

$$f(\lambda x + (1-\lambda)x_0) \leq b(x, x_0, \lambda)f(x) + [1 - b(x, x_0, \lambda)]f(x_0).$$

The function f is said to be B-vex on D with respect to the function b if it is B-vex at each $x \in D$ with respect to same b.

Note that every convex function is B-vex with respect to the function $b(x, x_0, \lambda) = \lambda$ $(\forall x, x_0 \in D)$.

Following [13] the set D is said to be invex at $x_0 \in D$ with respect to a function $\omega : D \times D \to X$ if for every $x \in D$ and $\lambda \in [0,1]$, $x_0 + \lambda\omega(x, x_0) \in D$. The set D will be called invex with respect to the function ω if it is invex at each $x \in D$ with respect to the same function ω. It is clear that every convex set $C \subset X$ is invex at each point $x_0 \in C$ with respect to the function $\omega(x, x_0) = x - x_0$.

Assume now that the set D is invex at $x_0 \in D$ with respect to a function $\omega : D \times D \to X$. We say that the function f is preinvex at $x_0 \in D$ with respect to the function ω if for every $x \in D$ and $\lambda \in [0,1]$,

$$f(x_0 + \lambda\omega(x, x_0)) \leq \lambda f(x) + (1-\lambda)f(x_0).$$

The function f is said to be preinvex on D with respect to ω if it is preinvex at each $x \in D$ with respect to same ω.

It is obvious that every convex function is preinvex with respect to $\omega(x, x_0) = x - x_0$.

The function f will be called B-preinvex at $x_0 \in D$ with respect to some functions $\omega : D \times D \to X$ and $b : D \times D \times [0,1] \to [0,1]$ if for every $x \in D$ and, $\lambda \in [0,1]$,

$$f(x_0 + \lambda\omega(x, x_0)) \leq b(x, x_0, \lambda)f(x) + [1 - b(x, x_0, \lambda)]f(x_0).$$

We say that f is B-preinvex on D with respect to ω and b if f is B-preinvex at each $x_0 \in D$ with respect to ω and b.

It should be noted here that every B-vex function at $x_0 \in D$ with respect to a function b is B-preinvex at $x_0 \in D$ with respect to b and $\omega(x, x_0) = x - x_0$. Moreover, every preinvex function at $x_0 \in D$ with respect to a function ω is B-preinvex at $x_0 \in D$ with respect to ω and $b(x, x_0, \lambda) = \lambda$ as well.

The following theorem gives a characteristic property of B-preinvex functions.

Theorem 2.1. *The function f is B-preinvex at $x_0 \in D$ with respect to some functions ω and b if and only if for every $x \in D$ and $\lambda \in [0,1]$,*

$$f(x_0 + \lambda\omega(x, x_0)) \leq \max\{f(x), f(x_0)\}. \tag{1}$$

Proof. Suppose that f is B-preinvex at x_0 with respect to some functions ω and b. So, for every $x \in D$ and $\lambda \in [0,1]$,

$$f(x_0 + \lambda\omega(x, x_0)) \le b(x, x_0, \lambda)f(x) + [1 - b(x, x_0, \lambda)]f(x_0). \tag{2}$$

Since $0 \le b(x, x_0, \lambda) \le 1 \quad (\forall x \in D,\ \forall\lambda \in [0,1])$, it follows that for every $x \in D$ and $\lambda \in [0,1]$,

$$\begin{aligned} &b(x, x_0, \lambda)f(x) + [1 - b(x, x_0, \lambda)]f(x_0)\\ &\le [b(x, x_0, \lambda) + (1 - b(x, x_0, \lambda))] \max\{f(x), f(x_0)\}\\ &= \max\{f(x), f(x_0)\}. \end{aligned} \tag{3}$$

Combining (2) and (3) yields (1).

Conversely, assume (1) holds. We define a function b on $D \times D \times [0,1]$ as follows. For $x \in D$ satisfying $f(x) \ge f(x_0)$ and $\lambda \in [0,1]$ we set $b(x, x_0, \lambda) = 1$, while for $x \in D$ satisfying $f(x) < f(x_0)$ and $\lambda \in [0,1]$ we set $b(x, x_0, \lambda) = 0$. Then, it is obvious that $0 \le b(x, x_0, \lambda) \le 1$ $(\forall x \in D,\ \forall\lambda \in [0,1])$ and

$$\begin{aligned} &b(x, x_0, \lambda)f(x) + [1 - b(x, x_0, \lambda)]f(x_0)\\ &= \max\{f(x), f(x_0)\}, \quad (\forall x \in D,\ \forall\lambda \in [0,1]). \end{aligned} \tag{4}$$

It follows from (1) and (4) that for every $x \in D$ and $\lambda \in [0,1]$,

$$\begin{aligned} f(x_0 + \lambda\omega(x, x_0)) &\le \max\{f(x), f(x_0)\}\\ &= b(x, x_0, \lambda)f(x) + [1 - b(x, x_0, \lambda)]f(x_0). \end{aligned}$$

□

Remark 2.1. From Theorem 2.1 we can see that the definition of B-preinvex function does not depend on b.

Recall that the function f is said to be quasiconvex at $x_0 \in D$ if for every $x \in D$ and $\lambda \in [0,1]$,

$$f(\lambda x + (1 - \lambda)x_0) \le \max\{f(x), f(x_0)\}.$$

Theorem 2.1 implies a result due to Li, Dong and Liu [11] as follows

Corollary 2.1. ([11]). *The function f is B-vex at $x_0 \in D$ with respect to some function b if and only if f is quasiconvex at x_0.*

Proof. Since f is B-vex at $x_0 \in D$ with respect to b, it is B-preinvex at x_0 with respect to b and $\omega(x, x_0) = x - x_0$. Applying Theorem 2.1 gives the assertion. □

Corollary 2.2. *The functions f and $-f$ are B-preinvex at $x_0 \in D$ with respect to the same function ω if and only if for every $x \in D$ and $\lambda \in [0,1]$,*

$$\min\{f(x), f(x_0)\} \le f(x_0 + \lambda\omega(x, x_0)) \le \max\{f(x), f(x_0)\}.$$

Proof. By virtue of Theorem 2.1, f and $-f$ are B-preinvex at $x_0 \in D$ with respect to b and ω if and only if for every $x \in D$ and $\lambda \in [0,1]$,

$$f(x_0 + \lambda\omega(x,x_0)) \leq \max\big\{f(x), f(x_0)\big\},$$
$$-f(x_0 + \lambda\omega(x,x_0)) \leq \max\big\{-f(x), -f(x_0)\big\}.$$

Observe that

$$\max\big\{-f(x), -f(x_0)\big\} = -\min\big\{f(x), f(x_0)\big\}.$$

Hence,

$$f(x_0 + \lambda\omega(x,x_0)) \geq \min\big\{f(x), f(x_0)\big\}.$$

This completes the proof. □

3. Lipschitz B-preinvex functions

Denote by $f'_-(x_0;d)$ and $f'_+(x_0;d)$ the lower and the upper Dini derivatives of f at x_0 in the direction d, respectively,

$$f'_-(x_0;d) = \liminf_{\lambda\downarrow 0} \frac{f(x_0+\lambda d) - f(x_0)}{\lambda},$$
$$f'_+(x_0;d) = \limsup_{\lambda\downarrow 0} \frac{f(x_0+\lambda d) - f(x_0)}{\lambda}.$$

Theorem 3.1. *Assume that the function f is B-preinvex at $x_0 \in D$ with respect to some functions b and ω. Then*

$$\underline{b}(x,x_0)[f(x) - f(x_0)] \geq f'_-(x_0;\omega(x,x_0)), \tag{5}$$
$$\overline{b}(x,x_0)[f(x) - f(x_0)] \geq f'_+(x_0;\omega(x,x_0)), \tag{6}$$

where

$$\underline{b}(x,x_0) = \liminf_{\lambda\downarrow 0} \lambda^{-1} b(x,x_0,\lambda),$$
$$\overline{b}(x,x_0) = \limsup_{\lambda\downarrow 0} \lambda^{-1} b(x,x_0,\lambda).$$

Proof. Since f is B-preinvex at x_0 with respect to b and ω, for every $x \in D$ and $\lambda \in [0,1]$,

$$f(x_0 + \lambda\omega(x,x_0)) \leq b(x,x_0,\lambda)f(x) + [1 - b(x,x_0,\lambda)]f(x_0),$$

which implies that for $\lambda > 0$,

$$\frac{f(x_0 + \lambda\omega(x,x_0)) - f(x_0)}{\lambda} \leq \frac{b(x,x_0,\lambda)}{\lambda}[f(x) - f(x_0)]. \tag{7}$$

In view of (7) one gets (5) and (6). □

Theorem 3.2. *Assume that f is locally Lipschitz at x_0 and B-preinvex at x_0 with respect to some functions b and ω. Suppose, furthermore, that f is regular at $x_0 \in D$ in the Clarke sence. Then, for every $x \in D$,*

$$\underline{b}(x,x_0)[f(x) - f(x_0)] \geq \langle \xi, \omega(x,x_0)\rangle \qquad (\forall \xi \in \partial f(x_0)),$$

where $\partial f(x_0)$ *is the Clarke generalized gradient of* f *at* x_0*.*

Proof. Since f is regular at x_0 in the Clarke sense at x_0, there exists the directional derivative $f'(x_0; .)$ of f at x_0 and $f'(x_0; d) = f^0(x_0; d)$ for all $d \in X$, where $f^0(x_0; .)$ is the Clarke generalized directional derivative of f at x_0.

By Theorem 3.1 it follows that

$$\begin{aligned} \underline{b}(x, x_0)[f(x) - f(x_0)] &\geq f'(x_0; \omega(x, x_0)) \\ &= f^0(x_0; \omega(x, x_0)). \end{aligned} \tag{8}$$

Taking into account Proposition 2.1.5 [5], one gets

$$f^0(x_0; \omega(x, x_0)) = \max\left\{\langle \xi, \omega(x, x_0)\rangle : \xi \in \partial f(x_0)\right\}. \tag{9}$$

Combining (8) and (9) yields that

$$\underline{b}(x, x_0)[f(x) - f(x_0)] \geq \langle \xi, \omega(x, x_0)\rangle \qquad (\forall \xi \in \partial f(x_0)).$$

□

The following corollary follows immediately from Theorem 3.2.

Corollary 3.1. *Assume that all the hypothese of Theorem 3.2 are fulfilled. Suppose, in addition, that for each* $x \in D$ *there exists the limit*

$$\lim_{\lambda \downarrow 0} \lambda^{-1} b(x, x_0, \lambda) = b^*(x, x_0).$$

Then

$$b^*(x, x_0)[f(x) - f(x_0)] \geq \langle \xi, \omega(x, x_0)\rangle \qquad (\forall \xi \in \partial f(x_0)).$$

Remark 3.1. If f is preinvex at x_0 with respect to some function ω, we can choose $b(x, x_0, \lambda) = \lambda$ $(\forall x \in D)$ so that f is B-preinvex with respect to the function b and ω. Hence

$$\lim_{\lambda \downarrow 0} \lambda^{-1} b(x, x_0, \lambda) = 1 \qquad (\forall x \in D).$$

A sufficient condition for a locally Lipschitz function to be B-preinvex can be stated as follows.

Theorem 3.3. *Assume that the function* f *is locally Lipschitz on* D*, and* ω *is a mapping from* $D \times D$ *into* X *satisfying* $\omega(x_0, x_0) = 0$ *and*

$$f(x_0 + \omega(x, x_0)) \leq f(x) \qquad (\forall x \in D). \tag{10}$$

Suppose, furthermore, that the set D *is invex at* x_0 *and the following condition holds*

$$\begin{bmatrix} y, z \in D, \lambda \in [0,1] \\ f(x_0 + \lambda\omega(y, x_0)) > f(z) \end{bmatrix} \Rightarrow \begin{bmatrix} \langle x^*, \omega(z, x_0) - \lambda\omega(y, x_0)\rangle \leq 0 \\ \forall x^* \in \partial f(x_0 + \lambda\omega(y, x_0)) \end{bmatrix}. \tag{11}$$

Then f *is B-preinvex at* x_0 *with respect to* ω*.*

Proof. Assume to the contrary that f is not B-preinvex at x_0 with respect to ω. By Theorem 2.1 it follows that there exists $x_1 \in D$ and $\overline{\lambda} \in [0,1]$ such that

$$f(x_0 + \overline{\lambda}\omega(x_1, x_0) > \max\{f(x_0), f(x_1)\}. \tag{12}$$

It is obvious that $\overline{\lambda} \neq 0$. In view of (10) it follows that $\overline{\lambda} \neq 1$ because if $\overline{\lambda} = 1$, by virtue of (10) and (12) one has

$$f(x_1) \geq f(x_0 + \omega(x_1, x_0)) > \max\{f(x_0), f(x_1)\},$$

which gives a contradiction. Hence, $\overline{\lambda} \in (0,1)$. Since $\omega(x_0, x_0) = 0$, it follows from (12) that $x_1 \neq x_0$.

We now consider the following function

$$\varphi(\lambda) = f(x_0 + \lambda\omega(x_1, x_0)) \quad (\lambda \in [0,1]).$$

Observe that for $\lambda \in [0,1]$, $x_0 + \lambda\omega(x_1, x_0) \in D$ because D is invex at x_0. Since f is locally Lipschitz on D, $\varphi(\lambda)$ is locally Lipschitz on $(0,1)$. We set

$$\alpha = \max\{f(x_0), f(x_1)\},$$
$$A = \{\lambda \in (0,1) : \varphi(\lambda) > \alpha\}.$$

Due to the continuity of φ, it follows that A is open since it is open in $(0,1)$. It follows from (12) that $A \neq \emptyset$.

We now show that there exists $\lambda_0 \in A$ such that $\partial\varphi(\lambda_0) \neq \{0\}$.

Assume to the contrary that for every $\lambda \in A$, $\partial\varphi(\lambda) = \{0\}$. Note that the connected component of $\overline{\lambda}$ in A is an open subset of A, so it is of the form (λ_1, λ_2) $(\lambda_1, \lambda_2 \in [0,1])$. According to the Lebourg mean-valued theorem [5], for $\lambda \in (\lambda_1, \lambda_2)$ there exists $\xi \in (\lambda, \overline{\lambda})$ such that

$$\varphi(\lambda) - \varphi(\overline{\lambda}) \in \langle \partial\varphi(\xi), \lambda - \overline{\lambda} \rangle.$$

Since $\partial\varphi(\xi) = 0$, it follows that $\varphi(\lambda) = \varphi(\overline{\lambda})$ $(\forall\lambda \in (\lambda_1, \lambda_2))$. Observing that $\lambda_1 \notin A$, we get

$$\varphi(\lambda_1) \leq \alpha < \varphi(\lambda) = \varphi(\overline{\lambda}) \qquad (\forall\lambda \in (\lambda_1, \lambda)).$$

Hence φ is not continuous at λ_1. This contradicts the continuity of φ on $[0,1]$.

Consequently, there exists $\lambda_0 \in (0,1)$ such that $\varphi(\lambda_0) > \alpha$ and $\partial\varphi(\lambda_0) \neq \{0\}$. Taking into account of Theorem 2.3.10 [5] we get

$$\partial\varphi(\lambda_0) \subset \langle \partial f(x_0 + \lambda_0\omega(x_1, x_0)), \omega(x_1, x_0) \rangle,$$

which implies that there exists $x^* \in \partial f(x_0 + \lambda_0\omega(x_1, x_0))$ such that

$$\langle x^*, \omega(x_1, x_0) \rangle \neq 0. \tag{13}$$

On the other hand, since $\varphi(\lambda_0) > \alpha$ it follows that

$$f(x_0 + \lambda_0\omega(x_1, x_0)) > f(x_0),$$
$$f(x_0 + \lambda_0\omega(x_1, x_0)) > f(x_1).$$

Making use of assumption (11) yields that

$$\langle x^*, \omega(x_0, x_0) - \lambda_0 \omega(x_1, x_0)\rangle \leq 0,$$
$$\langle x^*, \omega(x_1, x_0) - \lambda_0 \omega(x_1, x_0)\rangle \leq 0.$$

Observing that $\omega(x_0, x_0) = 0$ and $\lambda_0 > 0$ we get

$$\langle x^*, \omega(x_1, x_0)\rangle \geq 0,$$
$$(1 - \lambda_0)\langle x^*, \omega(x_1, x_0)\rangle \leq 0.$$

Since $\lambda_0 < 1$, this implies that

$$\langle x^*, \omega(x_1, x_0)\rangle = 0, \tag{1}$$

which contradicts (13). Consequently,

$$f(x_0 + \lambda\omega(x, x_0)) \leq \max\{f(x_0), f(x)\}, \quad (\forall x \in D, \forall \lambda \in [0, 1]). \tag{2}$$

By Theorem 2.1, f is B-preinvex at x_0 with respect to ω. The proof is complete. □

From Theorem 3.3 we obtain a result of [8] for B-vex function as a special case.

Corollary 3.2. *Assume that the set D is convex and the function f is locally Lipschitz on D. Suppose, in addition, that the following property holds*

$$\begin{bmatrix} x, x_0 \in D \\ f(x_0) > f(x) \end{bmatrix} \Rightarrow \begin{bmatrix} \langle x^*, x - x_0\rangle \leq 0 \\ \forall x^* \in \partial f(x_0) \end{bmatrix}. \tag{14}$$

Then f is B-vex at x_0.

Proof. Choosing $\omega(x, x_0) = x - x_0$, the condition (10) of Theorem 3.3 is automatically satisfied. Taking $y = x_0$ it follows from (14) that the condition (11) of Theorem 3.3 is fulfilled. Applying Theorem 3.3 we can conclude that f is B-preinvex at x_0 with respect to the function $\omega(x, x_0) = x - x_0$. This means that f is B-vex at x_0. □

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