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APPROXIMATING SOLUTIONS OF THE EQUATION $x = T(x,x)$

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ABSTRACT. Let D be a bounded closed convex subset of a Banach space, and let $T : D \times D \to D$ be a continuous mapping which satisfies for all $x, y, z, t \in D$,

$$\|T(x,y) - T(z,t)\| \le \max\{\|x - z\|, \|y - t\|\}$$

with strict inequality holding when $\|x - z\| \ne \|y - t\|$. Suppose T condensing in the sense that

$$\gamma(T(U,V)) < \max\{\gamma(U), \gamma(V)\}$$

for subsets U, V of D for which $\gamma(U \backslash V) > 0$ (where γ denotes the usual Kuratowski set-measure of noncompactness). A projection-iteration method is shown to converge to a solution of $x = T(x,x)$. The significance of this result is that it holds in arbitrary spaces.

The following is a Banach space version of an inequality proved in [6]. (The original version is proved in the more general context of a convex metric space of so-called 'hyperbolic type', but the following will be adequate for our purpose here.)

Proposition 1. *Let K be a convex subset of a Banach space and let $\{\alpha_n\}$ be a sequence of real numbers satisfying $0 \le \alpha_n < 1$. Suppose $\{x_n\}$, $\{y_n\} \subset K$ satisfy for all $n \ge 0$,*
(i) $x_{n+1} = (1 - \alpha_n) x_n + \alpha_n y_n$;
(ii) $\|y_{n+1} - y_n\| \le \|x_{n+1} - x_n\|$.
Then

$$\left(1 + \sum_{s=i}^{i+n-1} \alpha_s\right) \|x_i - y_i\| \le \|y_{i+n} - x_i\| + \left(\prod_{s=i}^{i+n-1} \frac{1}{1 - \alpha_s}\right) [\|x_i - y_i\| - \|x_{i+n} - y_{i+n}\|]$$

If, in addition, K is bounded, $\alpha_n \le b < 1$ for all $n \ge 0$, and $\sum_{n=1}^{\infty} \alpha_n = +\infty$, then

$$\lim_n \|x_n - y_n\| = 0.$$

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In this note we use the final conclusion of the above result to generalize the following theorem of [2] (Theorem 3). In this theorem γ denotes the usual Kuratowski set-measure of noncompactness.

Theorem 1. *Let D be a closed bounded convex subset of a uniformly convex Banach space X. Suppose $T : D \times D \to D$ is a continuous operator satisfying the conditions*

$$\|T(x,y) - T(z,t)\| \begin{cases} < \max\{\|x-z\|, \|y-t\|\}, & \text{if } \|x-z\| \neq \|y-t\| \\ \leq \|x-z\| = \|y-t\| \end{cases}$$

for all $x, y, z, t \in D$, and

$$\gamma(T(U,V)) < \max\{\gamma(U), \gamma(V)\}$$

for subsets U, V of D such that $\gamma(U \backslash V) > 0$. Then there exist numbers λ_n, $0 < a < \lambda_n < b < 1$, $n \geq 1$, where a, b are constants, such that the sequence $\{x_n\}$ defined by

$$x_n = \lambda_n x_{n-1} + (1-\lambda_n)\bar{x}_n,$$

where $\bar{x}_n = T(\bar{x}_n, x_{n-1})$, converges to a solution of $x = T(x,x)$ for any initial $x_0 \in D$.

Using Proposition 1 we show that the above theorem holds in an arbitrary Banach space.

Theorem 2. *Let D be a closed bounded convex subset of a Banach space X. Suppose $T : D \times D \to D$ is continuous and satisfies the conditions*

$$(*) \quad \|T(x,y) - T(z,t)\| \begin{cases} < \max\{\|x-z\|, \|y-t\|\}, & \text{if } \|x-z\| \neq \|y-t\| \\ \leq \|x-z\| = \|y-t\| \end{cases}$$

for all $x, y, z, t \in D$, and suppose

$$\gamma(T(U,V)) < \max\{\gamma(U), \gamma(V)\}$$

for subsets U, V of D such that $\gamma(U \backslash V) > 0$. Let $x_0 \in D$ and $b \in (0,1)$, and choose $\{\lambda_n\} \subset (b,1)$ such that $\sum_{n=1}^{\infty} (1-\lambda_n) = +\infty$. Then the sequence $\{x_n\}$ given by

$$x_n = \lambda_n x_{n-1} + (1-\lambda_n)\bar{x}_n, \quad n = 1, 2, \ldots,$$

where $\bar{x}_n = T(\bar{x}_n, x_{n-1})$, is well defined and converges to a solution of $x = T(x,x)$.

Our point of departure is the following fact which is essentially proved in [9].

Proposition 2. *Let (M,d) be a metric space and K a bounded closed convex subset of a Banach space X. Let $\{T_\alpha\}_{\alpha \in M}$ be a family of mappings of K into K which has the property that for some $A > 0$,*

$$(**) \qquad \|T_\alpha x - T_\beta y\| \leq \max\{Ad(\alpha,\beta), \|x-y\|\}$$

for all $\alpha, \beta \in M$, $x, y \in K$. Suppose also that T_α is condensing for each $\alpha \in M$. Then there exists a mapping $f : M \to K$ for which $T_\alpha f(\alpha) = f(\alpha)$ and for which $\|f(\alpha) - f(\beta)\| \le Ad(\alpha, \beta)$ for all $\alpha, \beta \in M$.

Proof. This result is proved in [9] under the assumption $T_\alpha(K)$ is precompact for each $\alpha \in M$. We need only modify a portion of that argument. Following [9] for each $\alpha \in M$ define the mapping f_α by setting for each $x \in K$,

$$f_\alpha(x) = (1/2)(x + T_\alpha x).$$

Then by a result of Ishikawa [7] (or by Proposition 1), for each $x_0 \in K$,

$$\left\| f_\alpha^{n+1}(x_0) - f_\alpha^n(x_0) \right\| \to 0 \text{ as } n \to \infty.$$

Also, by well-known properties of γ, (for example, see [1], p. 19)

$$\begin{aligned}
\gamma\left(\left\{ f_\alpha^{n+1}(x_0) \right\}\right) &= \gamma\left(\left\{ (1/2)\left(f_\alpha^n(x_0) + T_\alpha f_\alpha^n(x_0) \right\}\right)\right. \\
&= (1/2)\,\gamma\left(\left\{ \left(f_\alpha^n(x_0) + T_\alpha f_\alpha^n(x_0) \right\}\right)\right. \\
&\le (1/2)\left(\gamma\left(\left\{ f_\alpha^n(x_0) \right\}\right) + \gamma\left(\left\{ T_\alpha f_\alpha^n(x_0) \right\}\right) \right))
\end{aligned}$$

from which follows (since $\gamma\left(\left\{ f_\alpha^{n+1}(x_0) \right\}\right) = \gamma\left(\left\{ f_\alpha^n(x_0) \right\}\right)$),

$$\gamma\left(\left\{ f_\alpha^n(x_0) \right\}\right) \le \gamma\left(\left\{ T_\alpha f_\alpha^n(x_0) \right\}\right).$$

Since T_α is condensing $\gamma\left(\left\{ f_\alpha^n(x_0) \right\}\right) = 0$, so $\left\{ f_\alpha^n(x_0) \right\}$ has a subsequence which converges (by continuity) to a fixed point $f(\alpha)$ off_α. The fixed points of f_α and T_α coincide, so $T_\alpha f(\alpha) = f(\alpha)$.

The proof is now completed precisely as in [9]. For convenience we include the details.

Notice in particular that condition (**) implies that the mapping T_α, hence f_α, is nonexpansive, so $f_\alpha^n(x_0) \to f(\alpha)$ as $n \to \infty$.

Now fixe $\alpha, \beta \in M$. Then

$$\| f_\alpha(x_0) - f_\beta(x_0) \| = (1/2)\, \| T_\alpha(x_0) - T_\beta(x_0) \| \le (1/2)\, Ad(\alpha, \beta).$$

Moreover, if $\left\| f_\alpha^n(x_0) - f_\beta^n(x_0) \right\| \le Ad(\alpha, \beta)$, then

$$\begin{aligned}
&\left\| f_\alpha^{n+1}(x_0) - f_\beta^{n+1}(x_0) \right\| \\
&= \left\| f_\alpha\left(f_\alpha^n(x_0) \right) - f_\beta\left(f_\beta^n(x_0) \right) \right\| \\
&= (1/2) \left\| f_\alpha^n(x_0) + T_\alpha\left(f_\alpha^n(x_0) \right) - \left(f_\beta^n(x_0) + T_\beta\left(f_\beta^n(x_0) \right) \right) \right\| \\
&\le (1/2) \left\| f_\alpha^n(x_0) - f_\beta^n(x_0) \right\| + (1/2) \left\| T_\alpha\left(f_\alpha^n(x_0) \right) - T_\beta\left(f_\beta^n(x_0) \right) \right\| \\
&\le (1/2)\, Ad(\alpha, \beta) + (1/2) \max\left\{ Ad(\alpha, \beta), \left\| f_\alpha^n(x_0) - f_\beta^n(x_0) \right\| \right\} \\
&= Ad(\alpha, \beta).
\end{aligned}$$

So by induction $\left\| f_\alpha^n(x_0) - f_\beta^n(x_0) \right\| \le Ad(\alpha, \beta)$ for all $n \ge 1$. Therefore

$$\| f(\alpha) - f(\beta) \| = \lim_n \left\| f_\alpha^n(x_0) - f_\beta^n(x_0) \right\| \le Ad(\alpha, \beta).$$

□

Proof of Theorem 2. By taking $M = K = D$ and defining $T_\alpha = T(\cdot, \alpha)$ for $\alpha \in D$, we see immediately that

$$\|T_\alpha x - T_\beta y\| \leq \max\{d(\alpha, \beta), \|x - y\|\}.$$

By Proposition 2 we know that there exists a mapping $f : D \to D$ such that $T_\alpha f(\alpha) = f(\alpha)$ and $\|f(\alpha) - f(\beta)\| \leq \|\alpha - \beta\|$ for all $\alpha, \beta \in D$. Select $x_0 \in D$, let $\bar{x}_1 = f(x_0)$, and define $\bar{x}_n = f(x_{n-1})$, $n \geq 1$. This assures that the sequence $\{\bar{x}_n\}$ is well defined. Clearly

$$\bar{x}_n = f(x_{n-1}) = T_{x_{n-1}} f(x_{n-1}) = T(\bar{x}_n, x_{n-1}).$$

The convergence part of the theorem (which in this instance also proves existence) is now a direct consequence of Proposition 1 upon taking $\alpha_n = 1 - \lambda_n$ and $y_n = \bar{x}_{n+1}$. Thus $\lim_{n\to\infty} \|x_n - \bar{x}_{n+1}\| = 0$. Also, as in [2], $\gamma(\{\bar{x}_n\}) = \gamma(\{x_n\}) = 0$. Thus $x_{n_i} \to u \in D$ as $i \to \infty$ for some subsequence $\{x_{n_i}\}$ of $\{x_n\}$. By Proposition 1 $\bar{x}_{n_i+1} \to u$. Since T is continuous we conclude that $u = T(u, u)$. Also

$$\begin{aligned}\|u - \bar{x}_n\| &= \|T(u, u) - T(\bar{x}_n, x_{n-1})\| \\ &\leq \max\{\|u - \bar{x}_n\|, \|u - x_{n-1}\|\}.\end{aligned}$$

The alternatives of condition (*) require that either $\|u - \bar{x}_n\| = \|u - x_{n-1}\|$ or $\|u - \bar{x}_n\| < \|u - x_{n-1}\|$. In either case we have for $n \geq 1$,

$$\begin{aligned}\|u - x_n\| &\leq \lambda_n \|u - x_{n-1}\| + (1 - \lambda_n)\|u - \bar{x}_n\| \\ &\leq \|u - x_{n-1}\|.\end{aligned}$$

This implies that $\{\|u - x_n\|\}$ is monotone decreasing and, since $\lim_i \|u - x_{n_i}\| = 0$, $\lim_n \|u - x_n\| = 0$. $\square$

Remark. The case $\alpha_n = \lambda_n \equiv 1/2$ would appear to be of more practical interest. This case of Proposition 1 is dealt with in more detail in the forthcoming paper [8]. Also see [4].

Several other conditions are listed in [9] under which the conclusion of Proposition 2 holds. These should lead in fairly direct ways to additional extensions of the existence part of Theorem 2.

Other applications of the inequality of Proposition 1 are found in [3].

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