ACTA MATHEMATICA VIETNAMICA
Volume 27, Number 1, 2002, pp. 5-11

ON THE FIRST OCCURENCE OF IRREDUCIBLE REPRESENTATIONS OF THE MATRIX SEMIGROUP

TON THAT TRI

Abstract. Let $M_n = M(n, F_p)$ be the semigroup of all $n \times n$ matrices over the field F_p of p elements, p a prime number. As well known, each irreducible M_n-module appears as a composition factor of the space of homogeneous polynomials in some degree d. The purpose of the paper is to determine the lowest degree d for some irreducibles modules.

1. Introduction

Let $P = F_p[x_1, \dots, x_n]$ be the commutative polynomial algebra in n indeterminants $x_1, \dots, x_n$ over F_p. Let $f \in P$ and $\sigma = (a_{ij})$ be an element of M_n. We say that the polynomial f will change into σf if

$$(\sigma f)(x_1, \dots, x_n) = f(\sigma x_1, \dots, \sigma x_n),$$

where $\sigma x_1, \dots, \sigma x_n$ are defined by the following equations

$$\sigma x_j = \sum_{i=1}^{n} a_{ij} x_i, \quad 1 \leq j \leq n.$$

We have then $(\sigma\sigma')f = \sigma(\sigma' f)$ where σ, $\sigma' \in M_n$, and in this way M_n operates on P.

As well known, there are only p^n inequivalent irreducible modules for M_n and they all occur as composition factors in P. The irreducible modules are indexed by column p-regular partitions, i.e. sequences $(\alpha_1 \dots, \alpha_n)$, $0 \leq \alpha_i - \alpha_{i+1} \leq p-1$ for $i < n$, and $0 \leq \alpha_n \leq p-1$. The irreducible modules were originally constructed by using "Weyl modules" (see [5], [4]). An alternative construction using modular invariants was given in [9] and it was proved that each M_n-irreducible modules corresponding to the column p-regular partition $(\alpha_1, \dots, \alpha_n)$ is isomorphic to the irreducible modules, says $H_{(\alpha_1-\alpha_2,\dots,\alpha_{n-1}-\alpha_n,\alpha_n)}$, generated by

$$L^{(\alpha_1-\alpha_2,\dots,\alpha_{n-1}-\alpha_n,\alpha_n)} = L_1^{\alpha_1-\alpha_2} \dots L_{n-1}^{\alpha_{n-1}-\alpha_n} L_n^{\alpha_n},$$

Received Appril 5, 2000; in revised form January 10, 2001.

where L_i are the Dickson invariants

$$L_i = \begin{vmatrix} x_1 & \dots & x_i \\ \vdots & \ddots & \vdots \\ x_1^{p^{i-1}} & \dots & x_i^{p^{i-1}} \end{vmatrix}, \quad 1 \leq i \leq n.$$

Let P_d denote the subspace of P consisting of all homogeneous polynomials in degree d. For every irreducible H the lowest value d for which H appears as a composition factor in P_d was determined by Carlisle and Kuhn when $p = 2$.

For any prime p, the purpose of the paper is to provide another method in determining the lowest degree d for some irreducibles modules, which, when $p = 2$ implies the above result.

The main result of this paper is the following.

Let $\beta = (\beta_1, \dots, \beta_n)$, $\beta' = (\beta'_1, \dots, \beta'_n)$, $0 \leq \beta_i$, $\beta'_i \leq p-1$, $1 \leq i \leq n$ and denote $\beta * \beta' = (\alpha_1, \dots, \alpha_n)$ with

$$\alpha_i = \begin{cases} \beta_i + \beta'_i & \text{if } \beta_i + \beta'_i \leq p-1, \\ \beta_i + \beta'_i - (p-1) & \text{if } \beta_i + \beta'_i \geq p. \end{cases}$$

Theorem A. *Let $\beta = (\beta_1, \dots, \beta_n)$ and $\beta' = (\beta'_1, \dots, \beta'_n)$ with $0 \leq \beta_i$, $\beta'_i \leq p-1$, $1 \leq i \leq n$. Then $H_{\beta*\beta'}$ is a composition factor of $H_\beta \otimes H_{\beta'}$.*

Let p be a prime and $(n_0, n_1, \dots, n_k)$, $(m_0, m_1, \dots, m_{k-1})$ be non-decreasing sequences of non-negative integers with $n_0 = m_0 = 0$ and $n_k + m_{k-1} \leq n$. Let

$$\beta = (\beta_1, \dots, \beta_{n_1}, \underbrace{0, \dots, 0,}_{m - m_0 \, times} \beta_{m_1+n_1+1}, \dots, \beta_{m_1+n_2}, \underbrace{0, \dots, 0,}_{m_2 - m_1 \, times} \dots,$$
$$\underbrace{0, \dots, 0,}_{m_{k-1} - m_{k-2} \, times} \beta_{m_{k-1}+n_{k-1}+1}, \dots, \beta_{m_{k-1}+n_k}, 0, \dots, 0)$$

with $0 \leq \beta_i \leq p-1$, $1 \leq i \leq n$. For every j, $1 \leq j \leq k$, and for any ℓ, $1 \leq \ell \leq n_j - n_{j-1}$, write

$$\sum_{i=1}^{\ell} \beta_{m_{j-1}+n_j-i+1} = (\ell-1)(p-1) + r_{j\ell}, \quad 0 \leq r_{j\ell} \leq p-1$$

and $r_{2,n_2-n_1}, \dots, r_{k,n_k-n_{k-1}}$ all are equal to zero or $p-1$. Put

$$s_j = \begin{cases} n_j - n_{j-1} & \text{if } r_{j,n_j-n_{j-1}} > 0, \\ n_j - n_{j-1} - 1 & \text{if } r_{j,n_j-n_{j-1}} = 0, \end{cases} \quad 1 \leq j \leq k.$$

Corrolary B. *With β defined as above the lowest value d in which H_β occurs as a composition factor in P_d is*

$$\begin{aligned}
d = {} & \big((m_{k-1}+n_k-1)(p-1)+r_{k1}\big)p^0+\dots \\
& + \big((m_{k-1}+n_k-s_k)(p-1)+r_{k_{s_k}}\big)p^{s_k-1} \\
& + \big((m_{k-2}+n_{k-1}-1)(p-1)+r_{k-1,1}\big)p^{s_k}+\dots \\
& + \big((m_{k-2}+n_{k-1}-s_{k-1})(p-1)+r_{k-1,s_k-1}\big)p^{s_k+s_{k-1}-1}+\dots \\
& + \big((n_1-1)(p-1)+r_{11}\big)p^{s_k+\dots+s_2}+\dots+ \\
& + \big((n_1-s_1)(p-1)+r_{1_{s_1}}\big)p^{s_k+\dots+s_2+s_1-1}.
\end{aligned}$$

Notice that by studying the action of a family of Steenrod operations on P the corollary is also proved by P. A. Minh and G. Walker (private communication).

When $p=2$ we have the following corollary.

Corollary C. (Carlisle and Kuhn [1, 1.1]) *Let $\beta=(\beta_1,\dots,\beta_n)$, $0\le\beta_i\le 1$, $1\le i\le n$ and $(\beta_{n_1},\dots,\beta_{n_k})$ the subsequence of non-zero elements. Then the lowest value d for which H_β occurs as a composition factor in P_d is*

$$d=n_k+2n_{k-1}+\dots+2^{k-1}n_1.$$

2. Proof of Theorem A

We first recall some facts on the coefficient space of a module V. Suppose that V is an M_n-module and $\dim V$ is finite. If $\{v_j : j\in I\}$ is a F_p basis of V we have equations

$$\sigma\cdot v_j=\sum_{i\in I} r_{ij}(\sigma)v_i$$

for $\sigma\in M_n$, $j\in I$, $r_{ij}(\sigma)\in F_p$. The functions $r_{ij}: M_n\longrightarrow F_p$ are called coefficient functions of V. Denote by $F_p^{M_n}$ the space of all mappings from M_n to F_p. Then the F_p-space of coefficient functions is a subspace of $F_p^{M_n}$, called the coefficient space of V. We denote this space by $cf(V)=\sum_{i,j}F_p r_{ij}$. It is independent of the choice of the basis $\{v_j\}$.

Suppose the functions $s_{k\ell}: M_n\longrightarrow F_p$ are coefficient functions of an M_n-module W then the functions $r_{ij}s_{k\ell}$ are coefficient functions of $V\otimes W$.

If $0\longrightarrow W\longrightarrow V\longrightarrow V/W\longrightarrow 0$ is a exact sequence of M_n-modules then $cf(W)$ and $cf(V/W)$ are subspaces of $cf(V)$.

The following lemma [3, 27.8] holds for an algebraically closed field. Moreover, it also holds for a splitting field for an algebra.

Lemma 2.1. [3, 27.8] *Let K be a splitting field for an algebra A, and $M_1,\dots,M_k$ a set of pairwise non-isomorphic irreducible A-modules, with $\dim_K M_r=n_r$, $1\le r\le k$. For each r, consider a matrix of coefficient functions $\big\{f_{ij}^{(r)} : 1\le i,j\le n_r\big\}$ of M_r. Then $\big\{f_{ij}^{(r)} : 1\le i,j\le n_r, 1\le r\le k\big\}$ are linearly independent over K.*

Proposition 2.2. [9, 1.1] *Let $\beta = (\beta_1, \dots, \beta_n)$, $0 \le \beta_i \le p-1$, $1 \le i \le n$ and H_β be the M_n-module generated by $L^\beta = L_1^{\beta_1} \dots L_n^{\beta_n}$. Then the set of H_β is a full set of pairwise non-isomorphic irreducible modules and F_p is the splitting for the algebra $F_p[M_n]$.*

We shall use the following notations. Let $\sigma \in M_n$ and $1 \le i \le n$. Denote by $\det_i \sigma$ the determinant of the $i \times i$ matrix consisting of elements of the rows $1, \dots, i$ and the columns $1, \dots, i$ of σ.

Lemma 2.3. *Let $\beta = (\beta_1, \dots, \beta_n)$, $0 \le \beta_i \le p-1$, $1 \le i \le n$ and $\sigma \in M_n$. Denote $\det_\beta(\sigma) = (\det_1 \sigma)^{\beta_1} \dots (det_n \sigma)^{\beta_n}$. Then $\det_\beta \in cf(H_\beta)$.*

Proof. Let $(\beta_{n_1}, \dots, \beta_{n_k})$ be the subsequence consisting of all non-zero elements of $(\beta_1, \dots, \beta_n)$. By convention, if f is a polynomial, W is a space of polynomials then $fW = \{fx : x \in W\}$ and M_i is identified as a subsemigroup of M_j for $i < j$ in the usual way. We denote by W'_k the M_{n_k}-module generated by $L_{n_1}^{\beta_{n_1}} \dots L_{n_{k-1}}^{\beta_{n_{k-1}}}$, i.e., the F_p-space with generators $\{\tau \cdot (L_{n_1}^{\beta_{n_1}} \dots L_{n_{k-1}}^{\beta_{n_{k-1}}}) : \tau \in M_{n_k}\}$; $W_k = L_{n_k}^{\beta_{n_k}} W'_k$; V_k the subspace of H_β generated by all polynomials having $L_{n_k}^\ell$ for any ℓ, $0 \le \ell < \beta_{n_k}$ as a factor. Then $H_\beta = W_k + V_k$ is a direct sum as F_p-spaces. For $1 \le i \le k-1$, let W'_i be the M_{n_i}-module generated by $L_{n_1}^{\beta_{n_1}}$, $W'_1 = F_p$ and let $W_i = L_{n_k}^{\beta_{n_k}} \dots L_{n_i}^{\beta_{n_i}} W'_i$. Let V'_i the subspace of W'_{i+1} generated by all polynomials having $L_{n_i}^\ell$ for any ℓ, $0 \le \ell < \beta_{n_i}$ as a factor, $V_i = L_{n_k}^{\beta_{n_k}} \dots L_{n_{i+1}}^{\beta_{n_{i+1}}}$. Then W_1 is a one-dimensional F_p-space generated L^β and $H_\beta = W_1 + V_1 + V_2 + \dots + V_k$ is a direct sum as F_p-spaces. For each $\sigma \in M_n$ and $1 \le i \le k$ let σ_i be the $n \times n$ matrix in which elements on the rows $1, \dots, n_i$ and the columns $1, \dots, n_i$ are same as in σ and elements in other positions are zero. We have

$$\begin{aligned}\sigma \cdot L^\beta &= \sigma_k \cdot (L_{n_1}^{\beta_{n_1}} \dots L_{n_k}^{\beta_{n_k}}) + v_k \\ &= (\det_{n_k} \sigma)^{\beta_{n_k}} L_{n_k}^{\beta_{n_k}} \sigma_k \cdot (L_{n_1}^{\beta_{n_1}} \dots L_{n_{k-1}}^{\beta_{n_{k-1}}}) + v_k\end{aligned}$$

where $\sigma_k \cdot (L_{n_1}^{\beta_{n_1}} \dots L_{n_{k-1}}^{\beta_{n_{k-1}}}) \in W'_k$, $v_k \in V_K$. For $3 \le i \le k$ we have

$$\begin{aligned}\sigma_i \cdot (L_{n_1}^{\beta_{n_1}} \dots L_{n_{i-1}}^{\beta_{n_{i-1}}}) &= \sigma_{i-1} \cdot (L_{n_1}^{\beta_{n_1}} \dots L_{n_{i-1}}^{\beta_{n_{i-1}}}) + v'_{i-1} \\ &= (\det_{n_{i-1}} \sigma)^{\beta_{n_{i-1}}} L_{n_{i-1}}^{\beta_{n_{i-1}}} \sigma_{i-1} \cdot (L_{n_1}^{\beta_{n_1}} \dots L_{n_{i-2}}^{\beta_{n_{i-2}}}) + v'_{i-1}\end{aligned}$$

with $\sigma_{i-1} \cdot (L_{n_1}^{\beta_{n_1}} \dots L_{n_{i-2}}^{\beta_{n_{i-2}}}) \in W'_{i-1}$, $v'_{i-1} \in V'_{i-1}$, and

$$\begin{aligned}\sigma_2 \cdot (L_{n_1}^{\beta_{n_1}}) &= \sigma_1 \cdot (L_{n_1}^{\beta_{n_1}}) + v'_1 \\ &= (\det_{n_1} \sigma)^{\beta_{n_1}} L_{n_1}^{\beta_{n_1}} + v'_1\end{aligned}$$

with $v'_1 \in V'_1$. Therefore

$$\begin{aligned}\sigma \cdot L^\beta &= (\det_{n_1} \sigma)^{\beta_{n_1}} \dots (\det_{n_k} \sigma)^{\beta_{n_k}} L_{n_1}^{\beta_{n_1}} \dots L_{n_k}^{\beta_{n_k}} + v_1 + \dots + v_k, \\ &= \det_\beta(\sigma) L^\beta + v_1 + \dots + v_k,\end{aligned}$$

$v_i \in V_i$ and the lemma follows. □

Proof of Theorem A. We have $\det_\beta \in cf(H_\beta)$ and $\det_{\beta'} \in cf(H_{\beta'})$. For each $\sigma \in M_n$ then $\det_{\beta*\beta'}(\sigma) = \det_\beta(\sigma)\det_{\beta'}(\sigma)$ therefore $\det_{\beta*\beta'} \in cf(H_\beta \otimes H_{\beta'})$ and the theorem follows from Lemma 2.3 and Lemma 2.1. □

We recall that the p-connectivity (see [2]) of a sequence $\alpha = (\alpha_1, \dots, \alpha_n)$ of non-negative integers is the non-negative integer

$$w(\alpha) = \sum_{i=1}^{n}((\ell_i + 1)p^{k_i} - 1),$$

where $\alpha_i = k_i(p-1) + \ell_i$, $0 \le \ell_i < p-1$.

Lemma 2.4. ([2, 2.13]) *Let $\alpha = (\alpha_1, \dots, \alpha_n)$ be a column p-regular partition and $F(\alpha)$ an irreducible M_n-module corresponding to α, then $F(\alpha)$ does not occur as a composition factor in P_d when $d < w(\alpha)$.*

Proof of Corrolary B. We consider β as in Corollary B. For each j, $1 \le j \le k$ put $\overline{\beta}_j = (0, \dots, 0, \beta_{m_{j-1}+n_{j-1}+1}, \dots, \beta_{m_{j-1}+n_j}, 0, \dots, 0)$. Then we have

$$\overline{\beta}_j = \overline{\beta}_{j,s_j} * \cdots * \overline{\beta}_{j1},$$

where

$$\overline{\beta}_{j\ell} = (0, \dots, 0, p-1-r_{j\ell}, \underbrace{r_{j\ell}}_{m_{j-1}+n_j-\ell+1}, 0, \dots, 0)$$

for $1 \le \ell \le s_j$ if $(j, \ell) \ne (1, s_1)$ and

$$\overline{\beta}_{1,s_1} = \begin{cases} (p-1-r_{1s_1}, r_{1s_1}, 0, \dots, 0), & \text{if } s_1 = n_1 - 1 \\ (r_{1s_1}, 0, \dots, 0), & \text{if } s_1 = n_1. \end{cases}$$

By Theorem A, H_β is a composition factor of $\bigotimes_{j=1}^{k} H_{\overline{\beta}_j}$ and $H_{\overline{\beta}_j}$ is a composition factor of $\bigotimes_{i=1}^{s_j} H_{\overline{\beta}_{ji}}$, hence H_β is a composition factor of $\bigotimes_{j=1}^{k}\bigotimes_{i=1}^{s_j} H_{\overline{\beta}_{ji}}$.

As in [1], if we write $d = i_1 + pi_2 + \cdots + p^{r-1}i_r$ then the composition factors of $T(1)_{i_0} \otimes \dots T(1)_{i_r}$ are also compositions factors of P_d where $T(1) = P/(x_1^p, \dots, x_n^p)$ and $T(1)_i = T(1) \cap P_i$, $0 \le i \le n(p-1)$. Further, according to [1, 6.1], each $T(1)_i$ is an irreducible M_n-module; if we write $i = q(p-1) + \ell$ with $0 \le q \le n$ and $0 \le \ell < p-1$, then $T(1)_i$ is isomorphic to an irreducible M_n-module, says $F(\gamma_i)$, where γ_i is the column p-regular partition consisting of q terms equal to $p-1$ followed by a term equal to ℓ, and hence $T(1)_i$ is isomorphic to $H_{(0,\dots,\underbrace{p-1-\ell}_{q},\ell,0,\dots,0)}$

by Theorem 1.3 in [9]. For

$$\begin{aligned}
&r = s_1 + \cdots + s_k,\\
&i_1 = (m_{k-1} + n_k - 1)(p-1) + r_{k1},\\
&i_2 = (m_{k-1} + n_k - 2)(p-1) + r_{k2},\\
&\dots\ \dots\ \dots\\
&i_{s_k} = (m_{k-1} + n_k - s_k)(p-1) + r_{k,s_k},\\
&i_{s_k+1} = (m_{k-2} + n_{k-1} - 1)(p-1) + r_{k-1,1},\\
&i_{s_k+2} = (m_{k-2} + n_{k-1} - 2)(p-1) + r_{k-1,2},\\
&\dots\ \dots\ \dots\\
&i_{s_k+s_{k-1}} = (m_{k-2} + n_{k-1} - s_{k-1})(p-1) + r_{k-1,s_{k-1}},\\
&\dots\ \dots\ \dots,\\
&i_{s_k+\cdots+s_2+1} = (n_1 - 1)(p-1) + r_{11},\\
&i_{s_k+\cdots+s_2+2} = (n_1 - 2)(p-1) + r_{12},\\
&\dots\ \dots\ \dots\\
&i_{s_k+\cdots+s_2+s_1} = (n_1 - s_1)(p-1) + r_{1,s_1},
\end{aligned}$$

then H_β is a composition factor of P_d. On the other hand, the column p-regular partition α such that $H_\beta \cong F(\alpha)$ is

$$\begin{aligned}
\alpha = \big(&(n_1 - 1 + s_2 + \cdots + s_k)(p-1) + r_{1n_1},\\
&(n_1 - 2 + s_2 + \cdots + s_k)(p-1) + r_{1,n_1-1},\\
&\dots (s_2 + \cdots + s_k)(p-1) + r_{11},\\
&\underbrace{(s_2 + \cdots + s_k)(p-1), \dots, (s_2 + \cdots + s_k)(p-1)}_{m_1 - m_0\, times},\\
&\dots, \underbrace{s_k(p-1), \dots, s_k(p-1)}_{m_{k-1}-m_{k-2}\, times}, \underbrace{s_k(p-1)}_{m_{k-1}+n_{k-1}+1},\\
&\dots, p-1+r_{k2}, \underbrace{r_{k1},}_{m_{k-1}+n_k} 0, \dots, 0\big)
\end{aligned}$$

and $w(\alpha) = d$. Hence the corollary follows from Lemma 2.4. □

References

[1] D. P. Carlisle and N. L. Kuhn, *Subalgebras of the Steenrod algebra and action of matrices on truncated polynomial algebras*, J. Algebra **121** (1989), 370-387.

[2] D. P. Carlisle and G. Walker, *Poincaré series for the occurence of certain modular representations of $GL(n,p)$ in the symmetric algebra*, Proceedings of the Royal Society of Edinburgh, **113A** (1989), 27-41.

[3] C. W. Curtis and I. Reiner, *Representation Theory of Finite Groups and Associative Algebras*, Interscience, New York, 1962.

[4] J. C. Harris and N. J. Kuhn, *Stable decompositions of classifying spaces of finite abelian p-groups*, Math. Proc. Camb. Phil. Soc. (1988). 103, 427.

[5] G. James and A. Kerber, *The representation theory of the symmetric group*, Encyclopedia of Mathematics and its Application **16**, Addison-Wesley, 1981.
[6] Pham Anh Minh, *Irreducible modules of general linear groups and composition factors in the polynomial algebra*, preprint.
[7] Pham Anh Minh and G. Walker, *Private communication.*
[8] Huynh Mui, *Modular invariant theory and the cohomology algebras of symmetric groups*, J. Fac. Sci. Univ. Tokyo Sec. IA Math. **22** (1975), 319-369.
[9] Ton That Tri, *The irreducible modular representations of semigroups of all matrices*, Acta Math. Vietnam. **20** (1995), 43-53.

Department of Mathematics
College of Sciences,
University of Hue