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AN EXTENSION OF RENYI'S CHARACTERISTIC THEOREM TO TWO-SIDED EXPONENTIAL DISTRIBUTIONS

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ABSTRACT. In studying geometric composed random variable, A. Rényi showed a characteristic property of the exponential distribution. In this paper, using the geometric compounding we extend Rényi's idea to the case of two-sided exponential distributions and get one characteristic property of the set of distributions.

1. INTRODUCTION

Let Z be a geometric composed random variable of $X_j's$, with parameter p $(0 < p < 1)$. That means

$$Z = X_1 + \cdots + X_N,$$

where $X_1, X_2, \ldots$ are independent identically distributed random variables and N is independent of $X_j's$ with geometric distribution, i.e.

$$P(N = k) = pq^{k-1}(q = 1 - p), \quad k = 1, 2, \ldots$$

Rényi [2] characterized the exponential distribution proving the following assertion

$$\overline{G}_p = \overline{F} \Leftrightarrow \overline{F}(x) = e^{-x},$$

where $X_j \geq 0$, $EX_j = 1$, $\overline{F}(x) = P(X_j > x)$ and $\overline{G}_p(x) = P(pZ > x)$. In [5], we estimated the stable degree of this characteristic property. In [4], under the assumption $E|X_j| < +\infty$, we showed that for the distribution of $\sqrt{p}Z$ to be two-sided exponential, it is sufficient and necessary that the distribution of X_j is two-sided exponential. In this paper, we asserted that the property that $\sqrt{p}Z$ and X_j are identically distributed random variables is a characteristic property of the set of all of two-sided exponential distributions under certain conditions.

2. MAIN RESULT

Throughout this paper, we denote by Z a geometric composed random variable of $X_1, X_2, \ldots$, with parameter p $(0 < p < 1)$. The notation $G_a(x)$ means $P(aZ < x)$ and $g_a(t)$ means Ee^{itaZ}. Moreover, $F(x)$, $f(t)$ will denote the distribution

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and characteristic function, respectively, of $X_j's$. Then our main result can be formulated as follows.

Theorem 2.1. *Under the stated assumption and the additional assumption that $X_j's$ are symmetric random variables with $EX_j = 0$, $\mu = E|X_j|$, $0 < \sigma^2 = EX_j^2 < +\infty$, for $\sqrt{p}Z$ and X_j to be identically distributed random variables, i.e. $G_{\sqrt{p}} = F$, it is necessary and sufficient that F is a two-sided exponential distribution function.*

For the proof of the theorem we need the following two lemmas which are presented in [4] and [3].

Lemma 2.1. *Assume that $E|X_j| < +\infty$. Then*

$$g_a(t) = \frac{pf(at)}{1 - qf(at)} \tag{1}$$

for any constant a.

Relation (1) has been proved by applying property of conditional expectation.

Lemma 2.2. (see [3, Theorem 2.3.2]) *Assume that f_1, f_2 are two characteristic functions, and $f(z)$ is a function which is analytic in the strip $\alpha < \operatorname{Im} z < \beta$ $(\alpha > 0,\ \beta > 0)$ and $f(-t) = \overline{f(t)}\ \forall t \in R$. Assume, furthermore, that there exists an infinite sequence of real numbers $\{t_k\}$ tending to zero such that*

$$f(t_k) = f_1(t_k)f_2(t_k) \quad \forall k.$$

Then f_1, f_2 are analytic in the same strip and the relation

$$f(t) = f_1(t)f_2(t)$$

holds for any real number t and because of that f is a characteristic function.

Proof of Theorem 2.1. We shall begin with the necessity. It follows readily from the assumption that $f'(t)$ is a continuous function and $f(0) = 1$, $f'(0) = 0$, $|f''(t)| \le \sigma^2$. Furthermore, there exists a real number $\delta > 0$ such that

$$|f(t)| \ge \frac{1}{2} \quad \forall t : |t| \le \delta.$$

Let us consider the function $u(t) = \dfrac{1}{f(t)}$, $t \in [-\delta, \delta]$. It can be varified that

$$u'(0) = 0, \quad |u''(t)| \le 8\big[2\mu^2 + \sigma^2\big] \quad \forall t \in [-\delta, \delta].$$

Hence, $u''(t)$ is continuous and bounded in $[-\delta, \delta]$. It follows from relation (1) that the condition $G_{\sqrt{p}} = F$ is equivalent to the following relation

$$f(t) = \frac{pf(\sqrt{p}t)}{1 - qf(\sqrt{p}t)} \quad \forall t \in R. \tag{2}$$

Therefore,

$$pu(t) = u(\sqrt{p}t) - q \quad \forall t \in [-\delta, \delta]. \tag{3}$$

Clearly, as for the characteristic function $f_0(t) = \dfrac{1}{1+t^2}$, the function $u_0(t) = \dfrac{1}{f_0(t)} = 1 + t^2$ satisfies (3), i.e.

$$pu_0(t) = u_0(\sqrt{p}t) - q \quad \forall t \in R, \tag{4}$$

and $u_0''(t)$ is bounded, continuous. Consequently, for $h(t) = u(t) - u_0(t)$, there exists $h''(t)$ which is continuous, bounded and $h(0) = h'(0) = 0$. From (3) and (4) we have

$$ph(t) = h(\sqrt{p}t) \quad \forall t \in [-\delta, \delta]. \tag{5}$$

Hence

$$\frac{h(t)}{t^2} = \frac{h(\sqrt{p}t)}{(\sqrt{p}t)^2} \quad \forall t \in [-\delta, \delta],\ t \neq 0. \tag{6}$$

The Taylor's formula (Maclaurin's formula) for $h(t)$ is written as

$$h(t) = \frac{h''(s)}{2} \cdot t^2 \quad \text{(for some } s \text{ between 0 and } t\text{)}.$$

Therefore, if we set

$$k(t) = \begin{cases} \dfrac{h(t)}{t^2}, & \text{for } 0 < |t| \leq \delta \\ \dfrac{h''(0)}{2}, & \text{for } t = 0, \end{cases}$$

then $k(t)$ is continuous in $[-\delta, \delta]$ and from (6) we obtain

$$k(t) = k(\sqrt{p}t), \quad \forall t \in [-\delta, \delta]. \tag{7}$$

Note that if $|t| \leq \delta$, then $|\sqrt{p}t| \leq \delta$.

Hence, in (7) we can replace t by $\sqrt{p}t$. Repeating this process n times, we obtain the following relation

$$k(t) = k(p^{\frac{n}{2}}t) \quad \forall t \in [-\delta, \delta] \tag{8}$$

for any natural numbers n. From (8), by letting $n \to +\infty$, we get

$$k(t) = k(0) = \frac{h''(0)}{2} = c \text{ (const)}, \quad \forall t \in [-\delta, \delta].$$

That means

$$\frac{h(t)}{t^2} = c, \quad \forall t \in [-\delta, \delta].$$

Hence,

$$u(t) = ct^2 + u_0(t) = at^2 + 1 \ (a = c + 1), \quad \forall t \in [-\delta, \delta].$$

It follows readily from this discussion that

$$f(t) = \frac{1}{at^2 + 1}, \quad \forall t \in [-\delta, \delta],$$

for a certain constant a. Since X_j is symmetric with $EX_j = 0$, it follows that $f(t)$ is a real value function and hence a is a real constant. Furthermore, since

$|f(t)| \le 1\ \forall t$ we get $a \ge 0$. But we have $a \ne 0$, because $\sigma^2 = EX_j^2 > 0$. Hence $a > 0$. Using the results just obtained and Lemma 2.1, we infer that

$$f(t) = \frac{1}{at^2+1}\ (a > 0) \quad \forall t \in R.$$

This is the characteristic function of two-sided exponential distribution.

We now turn to the proof of the sufficiency. It follows readily from the fact that F is a two-sided exponential distribution that the characteristic function of X_j has the following form

$$f(t) = \frac{1}{at^2+1} \quad \text{(for some constants } a > 0).$$

Then, by relation (1),

$$g_{\sqrt{p}}(t) = \frac{1}{at^2+1}\cdot$$

That means $G_{\sqrt{p}} = F$.

The proof of the theorem is complete. □

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