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LINEAR OPERATORS SATISFYING THE ASSUMPTIONS OF SOME GENERALIZED LAX-MILGRAM THEOREMS

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Dedicated to Pham Huu Sach on the occasion of his sixtieth birthday

Abstract. This paper analyzes the structure of two classes of linear operators satisfying the assumptions of two generalized Lax-Milgram theorems of J. Saint Raymond. For the first class, complete characterizations are proposed. For the second class, some preliminary results are shown. In particular, we prove that the second class is strictly larger than the first one which, in its turn, is strictly larger than the class of elliptic operators.

1. Introduction

The Lax-Milgram theorem (see [1, p.66], [2, p.84]) is an important result of functional analysis which has various applications. For example, in [1] (Chapter 13) this theorem has been used for proving the existence and uniqueness of solutions to a boundary value problem for partial differential equations of elliptic type.

Several authors have tried to generalize the above-mentioned theorem. The Stampacchia theorem is a generalized version of the Lax-Milgram theorem which is useful for studying the existence of solutions to variational inequalities. B. Ricceri [3] expressed some ideas which would allow one to get generalized Lax-Milgram theorems even in the case where the linear operator does not satisfy the elliptic condition. The problems asked by B. Ricceri were solved completely by J. Saint Raymond [4] who obtained four new generalized Lax-Milgram theorems. There is a hope that the results of J. Saint Raymond can be a tool for proving the existence and uniqueness of solutions to a boundary value problem for a large class of PDEs which includes many non-elliptic equations.

The aim of this paper is to analyze the structure of two classes of linear operators satisfying the assumptions of two generalized Lax-Milgram theorems of J. Saint Raymond. For the first class, complete characterizations are proposed. For the second class, some preliminary results are shown. In particular, we prove

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that the second class is strictly larger than the first one which, in its turn, is strictly larger than the class of elliptic operators.

The paper is organized as follows. In Section 2 we summarize without proofs the relevant material on the Lax-Milgram theorem. In Section 3 we obtain complete characterizations for the class $\mathcal{A}_0$ of the linear operators satisfying the assumptions of Theorem 2.4 (J. Saint Raymond's first generalized Lax-Milgram theorem), and give several examples on the operators of the class. In Section 4 it is shown that the class $\mathcal{A}_K$ of the linear operators satisfying the assumptions of Theorem 2.6 (J. Saint Raymond's third generalized Lax-Milgram theorem), in general, is strictly larger than $\mathcal{A}_0$. Necessary conditions for a linear operator to belong to $\mathcal{A}_K$ are also obtained in Section 4.

Throughout this paper, H is a Hilbert space over the reals. The norm and the scalar product in H are denoted by $\|\cdot\|$ and $\langle\cdot,\cdot\rangle$, respectively. If $A : H \to H$ is a continuous linear operator then $\|A\|$ denotes the norm of A. For simplicity of notation, we let S_H stand for the unit sphere of H.

2. Lax-Milgram theorem and the results of J. Saint Raymond

Definition 2.1. ([1, p.66]) A linear operator $A : H \to H$ is called an elliptic operator if there exists a constant $\alpha > 0$ such that

$$\langle Ax, x\rangle \geq \alpha\|x\|^2 \quad \forall x \in H. \tag{2.1}$$

Theorem 2.1. (Lax-Milgram; see [1, p.66], [2, p.84]) *If $A : H \to H$ is an elliptic continuous linear operator then A is invertible. That is, for every $y \in H$, the equation*

$$Ax = y \tag{2.2}$$

has a unique solution $x = A^{-1}y$. Besides, it holds $\|A^{-1}\| \leq \dfrac{1}{\alpha}$.

The above theorem is a strong tool for proving the existence and uniqueness of solutions to a boundary value problem for elliptic partial differential equations (see [1, Chapter 13]).

The following result is a type of generalized Lax-Milgram theorems.

Theorem 2.2. (Stampacchia; see [1, p.67], [2, p.83], [5]) *Suppose that $A : H \to H$ is an elliptic continuous linear operator and $C \subset H$ is a nonempty closed convex subset. Then, for every $y \in H$, the variational inequality problem*

$$x \in C : \quad \langle Ax - y, x' - x\rangle \geq 0 \quad \forall x' \in C \tag{2.3}$$

has a unique solution $x = x(y)$. Besides, it holds

$$\|x(y') - x(y)\| \leq \frac{1}{\alpha}\|y' - y\| \quad \forall y, y' \in H. \tag{2.4}$$

Note that if $C = H$ then x is a solution of the problem (2.3) if and only if x is a solution of the equation (2.2). Thus Theorem 2.2 extends Theorem 2.1.

Sometimes Theorem 2.2 is stated in the following equivalent form.

Theorem 2.3. (Stampacchia; see [2, p.83], [5]) *Suppose that $C \subset H$ is a nonempty closed convex subset; $a(\cdot,\cdot) : H \times H \to R$ is a bilinear form such that there are constants $\ell > 0$ and $\alpha > 0$ satisfying*

$$|a(u,v)| \le \ell \|u\| \|v\| \quad \forall u, v \in H,$$
$$|a(u,u)| \ge \alpha \|u\|^2 \quad \forall u \in H.$$

Then, for every $y \in H$, there exists a unique vector $x \in C$ such that

$$a(x, x' - x) \ge \langle y, x' - x \rangle \quad \forall x' \in C.$$

In addition, the Lipschitz property (2.4) of the function $x(\cdot)$ holds.

The equivalence between Theorems 2.2 and 2.3 can be proved easily by using the Riesz-Fréchet representation theorem (see [2, p.83]).

From what has been said it follows that the Lax-Milgram theorem and the Stampacchia theorem can be applied only to problems involving elliptic operators.

One may ask:

Is it possible to establish theorems of the Lax-Milgram type where the linear operator A needs not to satisfy the elliptic condition (2.1) *?*

Based on his generalizations of the Ky Fan inequality, B. Ricceri [4] expressed an idea that the condition (2.1) might be replaced by the following weaker condition:

$$\inf_{x \in S_H} \left(\langle Ax, x \rangle + \|Ax\| \right) > 0. \tag{2.5}$$

Observe that (2.1) implies (2.5). Indeed, from (2.1) we have

$$\inf_{x \in S_H} \langle Ax, x \rangle \ge \alpha > 0. \tag{2.6}$$

As $\langle Ax, x \rangle + \|Ax\| \ge \langle Ax, x \rangle$, (2.6) implies (2.5).

Recently J. Saint Raymond has proved that the conclusion of the Lax-Milgram theorem remains valid for the case of the linear operators satisfying (2.5). Namely, the following result has been obtained.

Theorem 2.4. [4] *If $A : H \to H$ is a linear operator satisfying the condition* (2.5) *then A is continuous and invertible. In particular, for every $y \in H$, the equation $Ax = y$ has a unique solution.*

J. Saint Raymond has obtained also the following results.

Theorem 2.5. [4] *Suppose that $A : H \to H$ is a linear operator and y_1, y_2, $\dots$, y_m is an orthogonal system of m unit vectors in H. If there exists $\gamma > 0$ such that*

$$\inf_{x \in S_H} \left(\langle Ax, x \rangle + \|Ax\| + \gamma \left(\sum_{j=1}^{m} \langle Ax, y_j \rangle^2 \right)^{1/2} \right) > 0 \tag{2.7}$$

then A is continuous and invertible. In particular, for every $y \in H$, the equation $Ax = y$ has a unique solution.

Theorem 2.6. [4] *Suppose that $A : H \to H$ is a linear operator and $K : H \to H$ is a compact linear operator. If*

$$\inf_{x \in S_H} \left(\langle Ax, x\rangle + \|Ax\| + \|KAx\| \right) > 0 \tag{2.8}$$

then A is continuous and invertible. In particular, for every $y \in H$, the equation $Ax = y$ has a unique solution.

Theorem 2.7. [4] *Suppose that $A : H \to H$ is a linear operator and $K : H \to H$ is a compact linear operator. If*

$$\inf_{x \in S_H} \left(|\langle Ax, x\rangle| + \|KAx\| \right) > 0$$

then A is continuous and invertible. In particular, for every $y \in H$, the equation $Ax = y$ has a unique solution.

Recall that a linear operator $K : H \to H$ is said to be compact if K maps every bounded subset of H to a subset whose closure in the norm topology of H is compact.

Note that if $y_1,\ y_2,\ \cdots,\ y_m$ is an orthogonal system of unit vectors in H and $\gamma > 0$ is an arbitrary positive number then the formula

$$K(x) = \gamma \left(\langle x, y_1\rangle y_1 + \langle x, y_2\rangle y_2 + \cdots + \langle x, y_m\rangle y_m \right) \quad \forall x \in H,$$

defines a continuous linear operator. As $K(x)$, for every $x \in H$, belongs to the finite dimensional subspace span$\{y_1,\ y_2, \cdots,\ y_m\}$ generated by the vectors $y_1,\ y_2,\ \cdots,\ y_m$, K is a compact operator. For this K we have

$$\|KAx\| = \gamma \left(\sum_{j=1}^{m} \langle Ax, y_j\rangle^2 \right)^{1/2} \qquad \forall x \in H,$$

hence it is obvious that (2.7) is a special case of (2.8). We thus conclude that Theorem 2.5 is a special case of Theorem 2.6.

For any linear operators A and K we have

$$\langle Ax, x\rangle + \|Ax\| + \|KAx\| \geq \langle Ax, x\rangle + \|Ax\|,$$

hence (2.5) implies (2.8). Thus Theorem 2.4 is a special case of Theorem 2.6 (if one chooses $K = 0$.)

Let us denote by $\mathcal{A}_0$ (resp., $\mathcal{A}_K$, $\mathcal{A}'_K$) the class of the linear operators satisfying the assumptions of Theorem 2.4 (resp., Theorem 2.6, Theorem 2.7).

It is worth noting that Theorems 2.4–2.6 cannot be derived from Theorem 2.7. For analyzing the structure of the class $\mathcal{A}'_K$ probably one cannot proceed similarly as one does for the classes $\mathcal{A}_0$ and $\mathcal{A}_K$ in the subsequent two sections.

3. The class $\mathcal{A}_0$

We shall need the following fact whose simple proof will be omitted.

Lemma 3.1. *Let H be a Hilbert space. If $x, y \in H$ are such that $\langle y, x\rangle = \|y\|$ and $\|x\| = 1$ then there exists $\mu \geq 0$ satisfying $y = \mu x$.*

In finite-dimensional spaces, the operators from $\mathcal{A}_0$ can be characterized via their eigenvalue sets.

Theorem 3.1. *If H is a finite-dimensional Hilbert space, $A : H \to H$ is a linear operator, then $A \in \mathcal{A}_0$ if and only if A has no nonpositive eigenvalues.*

Proof. a) Necessity: To obtain a contradiction, suppose that (2.5) is satisfied and there exists an eigenvalue $\lambda \leq 0$ of A. Let $v \in S_H$ be an eigenvector corresponding to that λ. Since

$$\langle Av, v\rangle + \|Av\| = \lambda\|v\|^2 - \lambda\|v\| = 0,$$

it follows that

$$\inf_{x \in S_H} (\langle Ax, x\rangle + \|Ax\|) = 0,$$

contrary to (2.5).

b) Sufficiency: Assume that A has no nonpositive eigenvalues. If (2.5) is violated then

$$\inf_{x \in S_H} (\langle Ax, x\rangle + \|Ax\|) \leq 0.$$

Since H is a finite-dimensional Hilbert space, S_H is a compact set. Since the function $\varphi(x) := \langle Ax, x\rangle + \|Ax\|$ is continuous on S_H, there exists $v \in S_H$ satisfying

$$\varphi(v) = \inf_{x \in S_H} (\langle Ax, x\rangle + \|Ax\|) \leq 0.$$

As $\varphi(v) = \langle Av, v\rangle + \|Av\| \geq 0$ we have $\varphi(v) = 0$. Hence $\langle -Av, v\rangle = \| - Av\|$. By Lemma 3.1, there exists $\mu \geq 0$ such that $-Av = \mu v$. This shows that $\lambda := -\mu$ is a nonpositive eigenvalue of A, contrary to our assumption. The proof is complete. $\square$

In the case where H is an arbitrary Hilbert space, we shall characterize the operators $A \in \mathcal{A}_0$ through the upper bound of the geometric angle between a vector $x \in S_H$ and its image Ax.

By definition, if $u,\ v \in H$ are two arbitrary nonzero vectors then the cosine of the geometric angle formed by u and v is defined by the formula

$$\cos(u, v) = \frac{\langle u, v\rangle}{\|u\|\|v\|}.$$

Clearly, if u and v are two unit vectors then $\cos(u, v) = \langle u, v\rangle$.

Theorem 3.2. *If H is a Hilbert space over the reals and $A : H \to H$ is a continuous linear operator, then $A \in \mathcal{A}_0$ if and only if the following two conditions are satisfied*

(i) *There exists a constant $\rho > 0$ such that*

$$\|Ax\| \geq \rho \quad \forall x \in S_H, \tag{3.1}$$

(ii) *There exists a constant $\mu > -1$ such that*

$$\cos(Ax, x) \geq \mu \quad \forall x \in S_H. \tag{3.2}$$

Proof. *a) Necessity:* Suppose that (2.5) is satisfied. If there exists no $\rho > 0$ satisfying (3.1) then one can find a sequence of unit vectors $\{x_k\}$ such that $\|Ax_k\| \to 0$ as $k \to \infty$. It is evident that

$$0 \leq \langle Ax_k, x_k \rangle + \|Ax_k\| \leq 2\|Ax_k\|.$$

This clearly forces

$$\inf_{x \in S_H} (\langle Ax, x \rangle + \|Ax\|) = 0,$$

a contradiction. We have thus proved that there exists a constant $\rho > 0$ such that (3.1) holds. Since (2.5) is valid, there is $\beta > 0$ such that

$$\inf_{x \in S_H} (\langle Ax, x \rangle + \|Ax\|) = \beta > 0.$$

For every $x \in S_H$, since $Ax \neq 0$,

$$\|Ax\| \left(\left\langle \frac{Ax}{\|Ax\|}, x \right\rangle + 1 \right) = \langle Ax, x \rangle + \|Ax\| \geq \beta.$$

Therefore

$$\left\langle \frac{Ax}{\|Ax\|}, x \right\rangle + 1 \geq \frac{\beta}{\|Ax\|} \geq \frac{\beta}{\|A\|\|x\|} = \frac{\beta}{\|A\|}.$$

We thus get

$$\cos(Ax, x) = \left\langle \frac{Ax}{\|Ax\|}, x \right\rangle \geq \frac{\beta}{\|A\|} - 1.$$

Setting

$$\mu := \frac{\beta}{\|A\|} - 1 > -1$$

we obtain (3.2).

b) Sufficiency: Suppose that there exist constants $\rho > 0$ and $\mu > -1$ such that (3.1) and (3.2) are valid. By (3.2), there is $\varepsilon > 0$ such that

$$\cos(Ax, x) \geq \varepsilon - 1 \quad \forall x \in S_H.$$

This means that

$$\left\langle \frac{Ax}{\|Ax\|}, x \right\rangle \geq \epsilon - 1.$$

Combining this with (3.1) gives $\langle Ax, x\rangle + \|Ax\| \geq \epsilon\rho$ for all $x \in S_H$. Thus (2.5) is satisfied. The proof is complete. $\square$

Remark 3.1. The property (ii) in Theorem 3.2 can be reformulated in the following equivalent form: There exists a geometric angle $\omega \in [0, \pi)$ such that

$$\text{angle}\{Ax, x\} \leq \omega \quad \forall x \in S_H,$$

where angle$\{Ax, x\}$ denotes the geometric angle between the vectors Ax and x.

Remark 3.2. In the Lax-Milgram theorem, since A is an elliptic continuous linear operator, there exists $\alpha > 0$ such that $\langle Ax, x\rangle \geq \alpha\|x\|^2$ for every $x \in H$. This implies $\|Ax\| \geq \langle Ax, x\rangle \geq \alpha$ for every $x \in S_H$. Noting that $Ax \neq 0$ for every $x \in S_H$, we get

$$\begin{aligned}\cos(Ax, x) = \left\langle \frac{Ax}{\|Ax\|}, x \right\rangle &\geq \frac{\alpha}{\|Ax\|} \\ &\geq \frac{\alpha}{\|A\|} > 0.\end{aligned}$$

Hence A satisfies both the conditions (i) and (ii) of Theorem 3.2. Thus the class $\mathcal{A}_0$ contains the class of elliptic continuous linear operators, which we denote by $\mathcal{A}$.

From the following examples it is clear that $\mathcal{A}_0$ is strictly larger than $\mathcal{A}$ if $\dim H \geq 2$.

Example 3.1. Consider the operator $A : R^2 \to R^2$ defined by the formula

$$Ax = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

This is the rotation with the angle θ in the two-dimensional Euclidean space. For every $x \in R^2$, $\|x\| = 1$, we have $\|Ax\| = 1$ and

$$\begin{aligned}\cos(Ax, x) &= \langle Ax, x\rangle \\ &= x_1^2\cos\theta - x_1x_2\sin\theta + x_2^2\cos\theta + x_1x_2\sin\theta \\ &= \cos\theta.\end{aligned}$$

If $0 \leq \theta < \pi$ then $\cos\theta > -1$. From the above observations and Theorem 3.2 it follows that $A \in \mathcal{A}_0$. Note that if

$$\pi/2 \leq \theta < \pi$$

then $A \notin \mathcal{A}$. Indeed, for every $x \in R^2$ with $\|x\| = 1$ we have $\langle Ax, x\rangle = \cos\theta < 0$, hence A cannot be an elliptic operator.

Example 3.2. In the space $R^n = R^2 \oplus R^{n-2}$, $n \geq 3$, define the operator $A : R^n \to R^n$ by the formula

$$A(x + y) = A_1x + y, \quad \forall x \in R^2,\ \forall y \in R^{n-2},$$

where

$$A_1 x = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}.$$

Analysis similar to that in the previous example shows that:

1. If $0 \le \theta < \pi/2$ then $A \in \mathcal{A}$.
2. If $\pi/2 \le \theta < \pi$ then $A \in \mathcal{A}_0 \setminus \mathcal{A}$.

Example 3.3. Let H be an infinite-dimensional Hilbert space having an orthogonal basis $\{e_1,\ e_2, \cdots\}$. Let $M_i = \operatorname{span}\{e_{2i-1}, e_{2i}\},\ i = 1, 2, \cdots,$ and let $A_i : M_i \to M_i$ be the rotation of an angle $\theta_i,\ \theta_i \in [0, \pi)$, which is defined similarly as in Example 3.1. The formula

$$Ax = \sum_{i=1}^{\infty} A_i x_i \quad \text{for every } x = \sum_{i=1}^{\infty} x_i,$$

where $x_i \in M_i$ for all i, defines a continuous linear operator $A : H \to H$. It can be shown that if $\omega := \sup\{\theta_i : i = 1, 2, \cdots\} < \pi$ then $A \in \mathcal{A}_0$. Especially, if $\omega < \pi/2$ then $A \in \mathcal{A}$.

4. The class $\mathcal{A}_K$

From the definitions it follows immediately that $\mathcal{A}_0 \subseteq \mathcal{A}_K$. We now construct an example to show that the equality $\mathcal{A}_0 = \mathcal{A}_K$ is not true in general.

Let $H = \ell_2$ be the Hilbert space of the sequences of real numbers $x = (x_1, x_2, x_3, \cdots)$ satisfying the condition $\sum\limits_{i=1}^{\infty} x_i^2 < +\infty$. By definition,

$$\langle x, y\rangle = \sum_{i=1}^{\infty} x_i y_i \quad (\forall x, y \in \ell_2), \quad \text{and} \quad \|x\| = \left(\sum_{i=1}^{\infty} x_i^2\right)^{1/2}.$$

For every $x \in H$, we set

$$Ax = (-x_1, x_2, x_3, \cdots)$$

and

$$Kx = (x_1, 0, 0, \cdots).$$

Note that K is a compact linear operator. For $\bar{x} := (-1, 0, 0, \cdots)$ we have

$$\langle A\bar{x}, \bar{x}\rangle + \|A\bar{x}\| = 0.$$

Hence

$$\inf_{x \in S_H} (\langle Ax, x\rangle + \|Ax\|) = 0,$$

and we see that $A \notin \mathcal{A}_0$. For every $x \in S_H$ it holds

$$\begin{aligned}\langle Ax, x\rangle &= -x_1^2 + \sum_{i=2}^{\infty} x_i^2 \\ &= -x_1^2 + (1 - x_1^2) \\ &= 1 - 2x_1^2, \\ \|Ax\| &= \sqrt{\sum_{i=1}^{\infty} x_i^2} = 1, \\ \|KAx\| &= \|(-x_1, 0, 0, \cdots)\| = |x_1|.\end{aligned}$$

Therefore

$$\begin{aligned}\inf_{x \in S_H} (\langle Ax, x\rangle + \|Ax\| + \|KAx\|) &= \inf_{-1 \le x_1 \le 1} (2 + |x_1| - 2x_1^2) \\ &= \inf_{0 \le t \le 1} (-2t^2 + t + 2) \\ &= 1.\end{aligned}$$

This shows that $A \in \mathcal{A}_K$.

In the example above we have

$$\mathcal{A}_K \setminus \mathcal{A}_0 \neq \emptyset. \tag{4.1}$$

It is of interest to know whether this property always holds.

Theorem 4.1. *If H is an infinite-dimensional Hilbert space and $K : H \to H$ is a nonzero continuous linear operator, then* (4.1) *holds.*

Proof. Let $L := \ker K = \{u \in H : Ku = 0\}$, $M := L^\perp = \{v \in H : \langle v, u\rangle = 0 \text{ for all } u \in L\}$. Since K is a nonzero operator, we can find a unit vector $\bar{v} \in M$. Set $M' = \{t\bar{v} : t \in R\}$, $M'' = \{w \in M : \langle w, \bar{v}\rangle = 0\}$. Of course, the orthogonal decomposition $H = L \oplus M' \oplus M''$ holds. Defining a linear operator $A : H \to H$ by the formula $Ax = u - v + w$ for any $x = u + v + w$, where $u \in L,\ v \in M'$ and $w \in M''$, we claim that

$$A \in \mathcal{A}_K \setminus \mathcal{A}_0. \tag{4.2}$$

Indeed, let $x = u + v + w$, where $u \in L,\ v \in M'$ and $w \in M''$, be an arbitrary unit vector from S_H. We have $\|Ax\| = \|x\| = 1$ and

$$\begin{aligned}\langle Ax, x\rangle &= \langle u - v + w, u + v + w\rangle = \|u\|^2 - \|v\|^2 + \|w\|^2, \\ \|KAx\| &= \|K(u - v + w)\| = \| - Kv + Kw\|.\end{aligned}$$

Therefore

$$\begin{aligned}f(x) &:= \langle Ax, x\rangle + \|Ax\| + \|KAx\| \\ &= \|u\|^2 - \|v\|^2 + \|w\|^2 + 1 + \| - Kv + Kw\| \\ &= 2 - 2\|v\|^2 + \| - Kv + Kw\|.\end{aligned}$$

Fix a value $\bar\varepsilon \in (0,1)$ as small as

$$(1-\bar\varepsilon)\|K\bar v\| - \|K\|\left(2\bar\varepsilon - \bar\varepsilon^2\right)^{1/2} > 0. \tag{4.3}$$

We have

$$\inf_{x\in S_H} f(x) = \min\left\{ \inf_{x\in S_H,\, 0\le\|v\|\le 1-\bar\varepsilon} f(x), \quad \inf_{x\in S_H,\, 1-\bar\varepsilon\le\|v\|\le 1} f(x)\right\}. \tag{4.4}$$

It is obvious that

$$\begin{aligned}\inf_{x\in S_H,\, 0\le\|v\|\le 1-\bar\varepsilon} f(x) &\ge \inf_{0\le\|v\|\le 1-\bar\varepsilon}(2-2\|v\|^2)\\ &= 2-2(1-\bar\varepsilon)^2 = 2\bar\varepsilon(2-\bar\varepsilon).\end{aligned} \tag{4.5}$$

Besides,

$$\begin{aligned}\inf_{x\in S_H,\, 1-\bar\varepsilon\le\|v\|\le 1} f(x) &\ge \inf_{x\in S_H,\, 1-\bar\varepsilon\le\|v\|\le 1} \|-Kv+Kw\|\\ &\ge \inf_{x\in S_H,\, 1-\bar\varepsilon\le\|v\|\le 1} (\|Kv\| - \|K\|\|w\|)\\ &\ge (1-\bar\varepsilon)\|K\bar v\| - \|K\|\left(1-(1-\bar\varepsilon)^2\right)^{1/2}\\ &= (1-\bar\varepsilon)\|K\bar v\| - \|K\|\left(2\bar\varepsilon-\bar\varepsilon^2\right)^{1/2}.\end{aligned}$$

Combining this with (4.3)–(4.5) yields

$$\inf_{x\in S_H}(\langle Ax,x\rangle + \|Ax\| + \|KAx\|) > 0,$$

hence $A \in \mathcal{A}_K$. Since

$$\langle A\bar v, \bar v\rangle + \|A\bar v\| = 0,$$

(2.5) is violated. So $A \notin \mathcal{A}_0$. We have thus obtained (4.2), and the proof is complete. □

The next statement gives necessary conditions for the inclusion $A \in \mathcal{A}_K$ to hold.

Theorem 4.2. *Suppose that H is a Hilbert space over the reals, $K : H \to H$ is a nonzero continuous linear operator, and $A : H \to H$ is a continuous linear operator. Then $A \in \mathcal{A}_K$ only if the following two conditions are satisfied*

(i) *There exists a constant $\rho > 0$ such that*

$$\|Ax\| \ge \rho \quad \forall x \in S_H, \tag{4.6}$$

(ii) *There exists a constant $\mu > -1$ such that*

$$\cos(Ax,x) \ge \mu \quad \forall x \in S_H \cap A^-(\mathit{ker}\, K), \tag{4.7}$$

where $A^-(\mathit{ker}\, K) = \{x \in H : Ax \in \mathit{ker}\, K\}$.

Proof. The proof runs similarly as the first part of the proof of Theorem 3.2. □

One may conjecture that the conditions (4.6) and (4.7) are not sufficient for having the inclusion $A \in \mathcal{A}_K$. It would be desirable to obtain a complete characterization for the operators of $\mathcal{A}_K$ but we have not been able to do this.

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