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DIRECTIONAL DIFFERENTIABILITY OF THE OPTIMAL VALUE FUNCTION IN INDEFINITE QUADRATIC PROGRAMMING

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Dedicated to Pham Huu Sach on the occasion of his sixtieth birthday

Abstract. We obtain an explicit formula for computing the directional derivative of the optimal value function in a general parametric quadratic programming problem with linear constraints. Our result can be used in some cases where the existing results on differential stability in nonlinear programming (applied to quadratic programming) cannot be used.

1. Introduction

The first essential existence result in quadratic programming (QP) was obtained by M. Frank and P. Wolfe [5] in 1956. In the last five decades QP problems have been addressed intensively in the literature. Many authors have tried to investigate QP problems in detail, both from the qualitative analysis and the numerical analysis points of view. Various theoretical and practical applications of quadratic programming can be found, for instance, in [3], [7], [10] and [14].

Let R^n and R^m be finite-dimensional Euclidean spaces equipped with the standard scalar product and the Euclidean norm, $R^{m\times n}$ the space of $(m\times n)-$matrices equipped with the matrix norm induced by the vector norms in R^n and R^m. Let $R_S^{n\times n}$ be the space of symmetric $(n\times n)-$matrices equipped with the matrix norm induced by the vector norm in R^n. Let

$$\Omega := R_S^{n\times n} \times R^{m\times n} \times R^n \times R^m.$$

Consider the following general quadratic programming problem with linear constraints

$$(1) \qquad \begin{cases} \min f(x,c,D) := c^T x + \dfrac{1}{2} x^T D x \\ x \in \Delta(A,b) := \{x \in R^n : Ax \geq b\} \end{cases}$$

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depending on the parameter $\omega = (D, A, c, b) \in \Omega$, where the superscript ^{T} denotes the transposition. The feasible region and the solution set of (1) will be denoted by $\Delta(A, b)$ and $Sol(D, A, c, b)$ respectively. The function

$$\varphi : \Omega \longrightarrow R \cup \{\pm\infty\}$$

defined by

$$\varphi(\omega) = \begin{cases} \inf\{f(x, c, D) : x \in \Delta(A, b)\}, & \text{if } \Delta(A, b) \neq \emptyset; \\ +\infty, & \text{if } \Delta(A, b) = \emptyset, \end{cases}$$

where $\omega = (D, A, c, b)$, is called the optimal value function of the parametric Problem (1).

If we require $v^T Dv \geq 0$ (resp., $v^T Dv \leq 0$) for all $v \in R^n$ then $f(\cdot, c, D)$ is a convex (resp., concave) function and (1) is called a convex (resp., concave) QP problem. If such conditions are not required then we say that (1) is an indefinite QP problem.

In [4], M. J. Best and N. Chakravarti considered parametric convex quadratic programming problems and obtained some results on the directional differentiability of the optimal value function. In [2], A. Auslender and P. Coutat investigated similar questions for the case of generalized linear-quadratic programs. A survey of recent results on stability of nonlinear programming problems was given by J. F. Bonnans and A. Shapiro [6].

This paper continues our previous investigations ([17–20]) on the stability of QP problems. More precisely, we consider indefinite QP problems and obtain an explicit formula for computing the directional derivative of the optimal value function $\varphi(\cdot)$ at a given point $\omega \in \Omega$ and in a given direction $\omega^0 \in \Omega$.

The following notations will be adopted. The scalar product of vectors x, y and the norm of a vector x in a finite-dimensional Euclidean space are denoted by $x^T y$ and $\|x\|$, respectively. Vectors in Euclidean spaces are interpreted as columns of real numbers. The $i-$th component of a vector x is denoted by x_i. The inequality $x \geq y$ (resp., $x > y$) means that each component of x is greater than or equal to (resp., greater than) the corresponding component of y. For $A \in R^{m\times n}$, the matrix norm of A is given by

$$\|A\| = \max\{\|Ax\| : x \in R^n,\ \|x\| \leq 1\}.$$

For $D \in R^{n\times n}_S$, we define

$$\|D\| = \max\{\|Dx\| : x \in R^n,\ \|x\| \leq 1\}.$$

For any $A \in R^{m\times n}$ and for any nonempty subset $I \subset \{1, \dots, m\}$, A_I denotes the submatrix of A consisting of the $i-$th rows of A, for all $i \in I$. The norm in the product space $X_1 \times \cdots \times X_k$ of the normed spaces $X_1, \dots,\ X_k$ is set to be

$$\|(x_1, \dots, x_k)\| = \left(\|x_1\|^2 + \cdots + \|x_k\|^2\right)^{1/2}.$$

The paper is organized as follows. In Section 2 we establish several lemmas. In Section 3 we introduce a condition (G), and describe a general situation where

(G) holds. Section 4 is devoted to proving a formula for computing the directional derivative of the optimal value function in indefinite QP problems. The obtained result is compared with the corresponding results on differential stability in nonlinear programming of A. Auslender and R. Cominetti [1], and L. I. Minchenko and P. P. Sakolchik [13].

2. Lemmas

In this section we establish some lemmas which will be used in the proofs of our main results, Theorems 3.1 and 4.1.

Let $\omega = (D, A, c, b)$ and $\omega^0 = (D^0, A^0, c^0, b^0)$ be two elements of the space $R_s^{n\times n} \times R^{m\times n} \times R^n \times R^m$. Denote

$$\omega + t\omega^0 = (D + tD^0, A + tA^0, c + tc^0, b + tb^0),$$

$$\varphi^+(\omega; \omega^0) = \limsup_{t\downarrow 0} \frac{\varphi(\omega + t\omega^0) - \varphi(\omega)}{t},$$

$$\varphi^-(\omega; \omega^0) = \liminf_{t\downarrow 0} \frac{\varphi(\omega + t\omega^0) - \varphi(\omega)}{t}.$$

If $\varphi^+(\omega; \omega^0) = \varphi^-(\omega; \omega^0)$ then we say that the optimal value function $\varphi(\cdot)$ is directionally differentiable at ω in direction ω^0. The common value is denoted by $\varphi'(\omega; \omega^0)$ and it is called the directional derivative of φ at ω in direction ω^0. We have

$$\varphi'(\omega; \omega^0) = \lim_{t\downarrow 0} \frac{\varphi(\omega + t\omega^0) - \varphi(\omega)}{t}.$$

For every $\bar{x} \in \Delta(A, b)$, we set

$$I = \alpha(\bar{x}) = \{i : \ (A\bar{x})_i = b_i\},$$

and define

$$F(\bar{x}, \omega, \omega^0) = \{v \in R^n : \ \exists\varepsilon > 0 \quad \text{such that} \quad \bar{x} + tv \in \Delta(A + tA^0, b + tb^0) \quad \text{for every } t \in [0, \varepsilon]\},$$

$$R(\bar{x}, \omega, \omega^0) = \begin{cases} R^n & \text{if } I = \emptyset, \\ \{v \in R^n : A_I v + A_I^0 \bar{x} - b_I^0 \geq 0\} & \text{otherwise.} \end{cases}$$

Recall ([11], [15]) that a linear inequality system $Ax \geq b$ is said to be *regular* if there exists $x_0 \in R^n$ such that $Ax_0 > b$ (the Slater condition).

The following lemma originates from [1], [16].

Lemma 2.1. *If the system $Ax \geq b$ is regular then*

$$\emptyset \neq \mathit{int}R(\bar{x}, \omega, \omega^0) \subseteq F(\bar{x}, \omega, \omega^0) \subseteq R(\bar{x}, \omega, \omega^0) \tag{2}$$

for every $\bar{x} \in \Delta(A, b)$. Here $\mathit{int}R(\bar{x}, \omega, \omega^0)$ denotes the interior of the set

$$R(\bar{x}, \omega, \omega^0) \subset R^n.$$

Proof. Consider $\bar{x} \in \Delta(A, b)$. If $I = \alpha(\bar{x}) = \{i : (A\bar{x})_i = b_i\} = \emptyset$ then $A\bar{x} > b$. In this case, for every $v \in R^n$, there exists an $\varepsilon = \varepsilon(v) > 0$ such that for each $t \in [0, \varepsilon]$ we have

$$A\bar{x} + t(Av + A^0\bar{x} - b^0 + tA^0v) \geq b.$$

The above inequality is equivalent to

$$(A + tA^0)(\bar{x} + tv) \geq b + tb^0.$$

Hence $\bar{x} + tv \in \Delta(A + tA^0, b + tb^0)$ for each $t \in [0, \varepsilon]$. This implies that $F(\bar{x}, \omega, \omega^0) = R^n$. By definition, in this case we also have $R(\bar{x}, \omega, \omega^0) = R^n$. Therefore

$$F(\bar{x}, \omega, \omega^0) = R^n = R(\bar{x}, \omega, \omega^0),$$

and we have (2).

Consider the case $I \neq \emptyset$. First, we show that

$$\text{int} R(\bar{x}, \omega, \omega^0) \neq \emptyset.$$

Since $Ax \geq b$ is a regular system, there exists $x_0 \in R^n$ such that $Ax_0 > b$. Therefore, we have

$$A_I x_0 > b_I.$$

As $A_I\bar{x} = b_I$ and $A_I x_0 > b_I$, we have

$$A_I(x_0 - \bar{x}) > 0.$$

Putting $\hat{v} = x_0 - \bar{x}$, we get

$$A_I\hat{v} > 0.$$

By Lemma 2.2 of [18], the inequality system (of the unknown v)

$$A_I v \geq b_I^0 - A_I^0\bar{x}$$

is regular, hence there exists $\bar{v} \in R^n$ such that

$$A_I\bar{v} > b_I^0 - A_I^0\bar{x}.$$

This proves that $\bar{v} \in \text{int}R(\bar{x}, \omega, \omega^0)$, therefore $\text{int}R(\bar{x}, \omega, \omega^0) \neq \emptyset$.

We now prove that

$$\text{int}R(\bar{x}, \omega, \omega^0) \subseteq F(\bar{x}, \omega, \omega^0).$$

Suppose that $v \in \text{int}R(\bar{x}, \omega, \omega^0)$. We have

$$A_I v + A_I^0\bar{x} - b_I^0 > 0.$$

Hence there exists $\varepsilon_1 > 0$ such that for each $t \in [0, \varepsilon_1]$

$$A_I v + A_I^0\bar{x} - b_I^0 + tA_I^0 v > 0.$$

Therefore, for each $t \in [0, \varepsilon_1]$,

$$t(A_I v + A_I^0\bar{x} - b_I^0 + tA_I^0 v) \geq 0. \tag{3}$$

As $A_i\bar{x} > b_i$ for every $i \in \{1, \ldots, m\} \setminus I$, one can find $\varepsilon_2 > 0$ such that for each $t \in [0, \varepsilon_2]$

$$A_i\bar{x} + t(A_i v + A_i^0\bar{x} - b_i^0 + tA_i^0 v) \geq b_i, \tag{4}$$

for every $i \in \{1, \dots, m\} \setminus I$. Let $\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}$. It follows from (3) and (4) that

$$A\bar{x} + t(Av + A^0\bar{x} - b^0 + tA^0v) \geq b \tag{5}$$

for every $t \in [0, \varepsilon]$. This implies that

$$\bar{x} + tv \in \Delta(A + tA^0, b + tb^0)$$

for every $t \in [0, \varepsilon]$. Hence $v \in F(\bar{x}, \omega, \omega^0)$, and we have

$$\text{int} R(\bar{x}, \omega, \omega^0) \subseteq F(\bar{x}, \omega, \omega^0).$$

Finally, we prove that

$$F(\bar{x}, \omega, \omega^0) \subseteq R(\bar{x}, \omega, \omega^0).$$

Take any $v \in F(\bar{x}, \omega, \omega^0)$. By definition, there exists an $\varepsilon > 0$ such that for each $t \in [0, \varepsilon]$ we have

$$(A_I + tA_I^0)(\bar{x} + tv) \geq b + tb^0.$$

Consequently,

$$A_I\bar{x} + t(A_Iv + A_I^0\bar{x} - b_I^0 + tA_I^0v) \geq b_I$$

for every $t \in [0, \varepsilon]$. As $A_I\bar{x} = b_I$, we have

$$t(A_Iv + A_I^0\bar{x} - b_I^0 + tA_I^0v) \geq 0$$

for each $t \in [0, \varepsilon]$. Hence, for every $t \in (0, \varepsilon]$,

$$A_Iv + A_I^0\bar{x} - b_I^0 + tA_I^0v \geq 0.$$

Letting $t \to 0$, we obtain

$$A_Iv + A_I^0\bar{x} - b_I^0 \geq 0.$$

This shows that $v \in R(\bar{x}, \omega, \omega^0)$, hence $F(\bar{x}, \omega, \omega^0) \subseteq R(\bar{x}, \omega, \omega^0)$. We have thus shown the inclusions in (2). The proof is complete. □

It is well known that if $\bar{x} \in Sol(D, A, c, b)$ then there exists a Lagrange multiplier $\lambda \in R^m$ such that

$$\begin{aligned} &D\bar{x} - A^T\lambda + c = 0, \\ &A\bar{x} \geq b, \quad \lambda \geq 0, \\ &\lambda^T(A\bar{x} - b) = 0 \end{aligned}$$

(see [7], Theorem 2.8.2). The set of all such multipliers is called the Lagrange multiplier set corresponding to $\bar{x}$ and is denoted by $\Lambda(\bar{x}, \omega)$, where $\omega = (D, A, c, b)$.

The next result is well known in nonlinear programming (see [8], [9]). For the sake of completeness, we give a proof for the case of QP problems.

Lemma 2.2. *If the system $Ax \geq b$ is regular then for every $\bar{x} \in Sol(D, A, c, b)$ the set $\Lambda(\bar{x}, \omega)$ is compact.*

Proof. Let $\omega = (D, A, c, b)$. Suppose that there exists $\bar{x} \in Sol(D, A, c, b)$ such that $\Lambda(\bar{x}, \omega)$ is noncompact. Then there exists a sequence $\{\lambda_k\}$ in R^m such that $\|\lambda_k\| \neq 0$,

$$D\bar{x} - A^T\lambda_k + c = 0, \tag{6}$$

$$\lambda_k \geq 0, \tag{7}$$

$$\lambda_k^T(A\bar{x} - b) = 0, \tag{8}$$

for every k, and $\|\lambda_k\| \to \infty$ as $k \to \infty$. Without loss of generality, we may assume that $\{\|\lambda_k\|^{-1}\lambda_k\}$ converges to $\bar{\lambda}$ with $\|\bar{\lambda}\| = 1$. Dividing each expression in (6)–(8) by $\|\lambda_k\|$ and taking the limits as $k \to \infty$, we get

$$A^T\bar{\lambda} = 0, \quad \bar{\lambda} \geq 0, \quad \bar{\lambda}^T(A\bar{x} - b) = 0. \tag{9}$$

Since $\bar{\lambda}^T A\bar{x} = \bar{x}^T(A^T\bar{\lambda}) = 0$, from (9) it follows that

$$A^T\bar{\lambda} = 0, \quad \bar{\lambda} \geq 0, \quad \bar{\lambda}^T b = 0.$$

For every $t > 0$, we set $b_t = b + t\bar{\lambda}$. Since $\bar{\lambda}^T\bar{\lambda} = \|\bar{\lambda}\|^2 = 1$,

$$\bar{\lambda}^T b_t = \bar{\lambda}^T b + t\bar{\lambda}^T\bar{\lambda} = \bar{\lambda}^T b + t = t.$$

Consequently, for every $t > 0$, $\bar{\lambda}$ is a solution of the following system

$$A^T\lambda = 0, \quad \lambda \geq 0, \quad \lambda^T b_t > 0.$$

Hence, for every $t > 0$, the system $Ax \geq b_t$ has no solutions (see [7], Theorem 2.7.8). Since $\Delta(A, b) \neq \emptyset$ and $\|b_t - b\| = t \to 0$ as $t \to 0$, the system $Ax \geq b$ is irregular (see [11], Lemma 2.1]), contradicting our assumption. The proof is complete. □

Lemma 2.3. (cf. [1], Lemma 2) *If the system $Ax \geq b$ is regular and $\bar{x} \in Sol(D, A, c, b)$ then*

$$\inf_{v \in R(\bar{x}, \omega, \omega^0)} (D\bar{x} + c)^T v = \max_{\lambda \in \Lambda(\bar{x}, \omega)} (b^0 - A^0\bar{x})^T\lambda,$$

where $\Lambda(\bar{x}, \omega)$ stands for the Lagrange multiplier set corresponding to $\bar{x}$.

Proof. Let $\bar{x} \in Sol(D, A, c, b)$. If $I = \alpha(\bar{x}) = \{i : (A\bar{x})_i = b_i\}$ is empty then, by definition, $R(\bar{x}, \omega, \omega^0) = R^n$. As $\bar{x} \in Sol(D, A, c, b)$ and $A\bar{x} > b$, the first-order necessary optimality condition ([7], Theorem 2.8.2) applied to $\bar{x}$ shows that $(D\bar{x} + c)^T v = 0$ for every $v \in R^n$. Therefore, we have

$$\inf_{v \in R(\bar{x}, \omega, \omega^0)} (D\bar{x} + c)^T v = 0.$$

Again, by the just mentioned first-order necessary optimality condition, for every $\bar{x}$ we have $\Lambda(\bar{x}, \omega) \neq \emptyset$. Since $A\bar{x} > b$, $\Lambda(\bar{x}, \omega) = \{0\}$. Therefore

$$\max_{\lambda \in \Lambda(\bar{x}, \omega)} (b^0 - A^0\bar{x})^T\lambda = 0.$$

Thus, in the case $I = \emptyset$ the assertion of the lemma is true. We now consider the case where $I = \alpha(\bar{x}) = \{i : (A\bar{x})_i = b_i\} \neq \emptyset$. We have

$$\inf_{v \in R(\bar{x},\omega,\omega^0)} (D\bar{x} + c)^T v = \inf\{(D\bar{x} + c)^T v : v \in R^n, A_I v \geq b_I^0 - A_I^0 \bar{x}\}.$$

Consider a pair of dual linear programs

$$(P) \qquad \begin{cases} (D\bar{x} + c)^T v \longrightarrow \min; \\ v \in R^n : A_I v \geq b_I^0 - A_I^0 \bar{x}, \end{cases}$$

and

$$(P^\star) \qquad \begin{cases} (b_I^0 - A_I^0 \bar{x})^T \lambda_I \longrightarrow \max; \\ \lambda_I \in R^{|I|} : A_I^T \lambda_I = D\bar{x} + c, \ \lambda_I \geq 0, \end{cases}$$

where $|I|$ denotes the number of the elements of I. By the definition of the Lagrange multiplier set $\Lambda(\bar{x}, \omega)$ we observe that if λ_I is a feasible point of $(P^\star)$ then $(\lambda_I, 0_J) \in \Lambda(\bar{x}, \omega)$, where $J = \{1, \cdots, m\} \setminus I$. Conversely, if $\lambda = (\lambda_I, \lambda_J) \in \Lambda(\bar{x}, \omega)$ then $\lambda_J = 0_J$. The regularity of the system $Ax \geq b$ and Lemma 2.2 imply that $\Lambda(\bar{x}, \omega)$ is nonempty and compact. Hence, by the above observation, the feasible domain of $(P^\star)$ is nonempty and compact. By the duality theorem in linear programming ([11]), the optimal values of (P) and $(P^\star)$ are both finite and equal to each other. Therefore

$$\begin{aligned}
&\inf_{v \in R(\bar{v},\omega,\omega^0)} (D\bar{x} + c)^T v = \\
&= \inf\{(D\bar{x} + c)^T v : \ v \in R^n, A_I v \geq b_I^0 - A_I^0 \bar{x}\} \\
&= \max\{(b_I^0 - A_I^0 \bar{x})^T \lambda_I : \ \lambda_I \in R^{|I|}, \ \lambda_I \geq 0, \ A_I^T \lambda_I = D\bar{x} + c\} \\
&= \max\{(b^0 - A^0 \bar{x})^T \lambda : \ \lambda \in R^m, \ \lambda = (\lambda_I, 0_J) \geq 0, \ A^T \lambda = D\bar{x} + c\} \\
&= \max_{\lambda \in \Lambda(\bar{x},\omega)} (b^0 - A_0 \bar{x})^T \lambda.
\end{aligned}$$

The lemma is proved. □

Lemma 2.4. *Suppose that $\omega_k = \{(D_k, A_k, c_k, b_k)\}$ is a sequence in $R_s^{n\times n} \times R^{m\times n} \times R^n \times R^m$ converging to $\omega = (D, A, c, b)$, and $\{x_k\}$ is a sequence in R^n such that $x_k \in Sol(D_k, A_k, c_k, b_k)$ for every k. If the system $Ax \geq b$ is regular and $Sol(D, A, 0, 0) = \{0\}$ then there exists a subsequence $\{x_{k_i}\}$ of $\{x_k\}$ such that $\{x_{k_i}\}$ converges to $\bar{x} \in Sol(D, A, c, b)$ as $i \to \infty$.*

Proof. Suppose that $Ax \geq b$ is a regular system and $Sol(D, A, 0, 0) = \{0\}$. As $x_k \in Sol(D_k, A_k, c_k, b_k)$, we have

$$f(x_k, c_k, D_k) = c_k^T x_k + \frac{1}{2} x_k^T D_k x_k \tag{10}$$

and

$$A_k x_k \geq b_k. \tag{11}$$

Take $x \in \Delta(A, b)$. Then there exists a sequence $\{y_k\}$ in R^n tending to x such that

$$A_k y_k \geq b_k \quad \text{for every } k \tag{12}$$

(see Lemma 2.1 in [18]). The inequality in (12) shows that $y_k \in \Delta(A_k, b_k)$. Hence

$$c_k^T x_k + \frac{1}{2} x_k^T D_k x_k \leq c_k^T y_k + \frac{1}{2} y_k^T D_k y_k. \tag{13}$$

We claim that the sequence $\{x_k\}$ is bounded. Indeed, assume the contrary that $\{x_k\}$ is unbounded. Then, without loss of generality, we may assume that $\|x_k\| \to \infty$ as $k \to \infty$ and $\|x_k\| \neq 0$ for every k. Hence, the sequence $\{\|x_k\|^{-1} x_k\}$ is bounded and has a convergent subsequence. We may assume that the sequence $\{\|x_k\|^{-1} x_k\}$ itself converges to $\hat{x} \in R^n$ with $\|\hat{x}\| = 1$. From (11) we have

$$A_k \frac{x_k}{\|x_k\|} \geq \frac{b_k}{\|x_k\|}.$$

Letting $k \to \infty$, we obtain

$$A\hat{x} \geq 0. \tag{14}$$

Dividing both sides of (13) by $\|x_k\|^2$ and taking the limit as $k \to \infty$, we obtain

$$\hat{x}^T D \hat{x} \leq 0. \tag{15}$$

Combining (14) and (15), we have $Sol(D, A, 0, 0) \neq \{0\}$, contradicting our assumptions. Thus the sequence $\{x_k\}$ is bounded and it has a convergent subsequence, say, $\{x_{k_i}\}$. Suppose that $\{x_{k_i}\}$ converges to $\bar{x}$. From (13) we have

$$c_{k_i}^T x_{k_i} + \frac{1}{2} x_{k_i}^T D_{k_i} x_{k_i} \leq c_{k_i}^T y_{k_i} + \frac{1}{2} y_{k_i}^T D_{k_i} y_{k_i}. \tag{16}$$

From (11) we have

$$A_{k_i} x_{k_i} \geq b_{k_i}. \tag{17}$$

Taking limits in (16) and (17) as $i \to \infty$, we obtain

$$c^T \bar{x} + \frac{1}{2} \bar{x}^T D \bar{x} \leq c^T x + \frac{1}{2} x^T D x, \tag{18}$$

$$A\bar{x} \geq b. \tag{19}$$

As $x \in \Delta(A, b)$ is arbitrarily chosen, (18) and (19) yield $\bar{x} \in Sol(D, A, c, b)$. The lemma is proved. □

3. Condition (G)

Let $\omega = (D, A, c, b) \in R_s^{n \times n} \times R^{m \times n} \times R^n \times R^m$ be a given parameter value and $\omega^0 = (D^0, A^0, c^0, b^0) \in R_s^{n \times n} \times R^{m \times n} \times R^n \times R^m$ be a given direction. Consider the following condition which we call condition (G):

For every sequence $\{t_k\}$, $t_k \downarrow 0$, *and for every sequence* $\{x_k\}$, $x_k \to \bar{x} \in Sol(D, A, c, b)$, *where* $x_k \in Sol(\omega + t_k\omega^0)$ *for each* k, *the following inequality is satisfied*

$$\liminf_{k\to\infty} \frac{(x_k - \bar{x})^T D(x_k - \bar{x})}{t_k} \geq 0.$$

Remark 3.1. If D is a positive semidefinite matrix then condition (G) holds. Indeed, if D is positive semidefinite then $(x_k - \bar{x})^T D(x_k - \bar{x}) \geq 0$; hence the inequality in (G) is obviously satisfied.

Remark 3.2. If the constraint system $Ax \geq b$ is regular then (G) is weaker than the condition that the $(SOSC)_u$ property, introduced by A. Auslender and R. Cominetti [1] (applied to QP problems), holds at every $\bar{x} \in Sol(D, A, c, b)$. Note that if the system $Ax \geq b$ is regular then (G) is also weaker than the condition $(H3)$ introduced by L. I. Minchenko and P. P. Sakolchik in [13] (applied to QP problems). There are many QP problems where the conditions $(SOSC)_u$ and $(H3)$ are not satisfied but condition (G) is. A detailed comparison of our results with the ones in [1] and [13] will be given in Section 4.

Now, we describe a general situation where (G) is fulfilled.

Theorem 3.1. *If* $Ax \geq b$ *is a regular system and every solution* $\bar{x} \in Sol(D, A, c, b)$ *is a locally unique solution of Problem (1), then condition* (G) *is satisfied.*

Proof. From the statement of (G) it is obvious that the condition is trivially satisfied if $Sol(D, A, c, b) = \emptyset$. Consider the case $Sol(D, A, c, b) \neq \emptyset$. For any given $\bar{x} \in Sol(D, A, c, b)$, set $I = \alpha(\bar{x}) = \{i : (A\bar{x})_i = b_i\}$ and

$$\mathcal{F}_{\bar{x}} = \{v \in R^n : (Av)_i \geq 0 \text{ for every } i \in I\}.$$

It is easy to show that, for every $\bar{x} \in Sol(D, A, c, b)$, the following two conditions are equivalent (cf. [12]):

(a) $\bar{x}$ is a locally unique solution of Problem (1),

(b) For every $v \in \mathcal{F}_{\bar{x}} \setminus \{0\}$, if $(D\bar{x} + c)^T v = 0$ then $v^T Dv > 0$.

We shall use the above equivalence to prove our theorem. Suppose, on the contrary that (G) is not satisfie. Then there exist a sequence $\{t_k\}$, $t_k \downarrow 0$, and a sequence $\{x_k\}$, $x_k \to \bar{x} \in Sol(D, A, c, b)$, $x_k \in Sol(D + t_kD^0, A + t_kA^0, c + t_kc^0, b + t_kb^0)$ for every k, such that

$$\lim_{k\to\infty} \frac{(x_k - \bar{x})^T D(x_k - \bar{x})}{t_k} < 0. \tag{20}$$

By taking a subsequence, if necessary, we may assume that

$$(x_k - \bar{x})^T D(x_k - \bar{x}) < 0, \tag{21}$$
$$\|x_k - \bar{x}\| \neq 0 \quad \text{for every } k,$$

and

$$\lim_{k\to\infty} \frac{\|x_k - \bar{x}\|}{t_k} = +\infty. \tag{22}$$

Then the sequence $\{\|x_k - \bar{x}\|^{-1}(x_k - \bar{x})\}$ has a convergent subsequence. Without loss of generality, we may assume that $\{\|x_k - \bar{x}\|^{-1}(x_k - \bar{x})\}$ converges to $v \in R^n$ with $\|v\| = 1$. Dividing both sides of the inequality in (21) by $\|x_k - \bar{x}\|^2$ and letting $k \to \infty$, we get

$$v^T D v \le 0. \tag{23}$$

Since $x_k \in Sol(D + t_k D^0, A + t_k A^0, c + t_k c^0, b + t_k b^0)$, we have

$$(A_I + t_k A_I^0) x_k \ge b_I + t_k b_I^0,$$

where $I = \{i : (A\bar{x})_i = b_i\}$. Since $b_I = A_I \bar{x}$,

$$A_I(x_k - \bar{x}) \ge t_k(b_I^0 - A_I^0 x_k).$$

Dividing both sides of the inequality above by $\|x_k - \bar{x}\|$, letting $k \to \infty$ and taking into account (22), we obtain

$$A_I v \ge 0.$$

Thus

$$v \in \mathcal{F}_{\bar{x}} \setminus \{0\}. \tag{24}$$

Next, show that $(D\bar{x} + c)^T v = 0$. We have

$$\begin{aligned} &\varphi(\omega + t_k\omega^0) - \varphi(\omega) = \\ &= (c + t_k c^0)^T x_k + \frac{1}{2} x_k^T (D + t_k D^0) x_k - c^T \bar{x} - \frac{1}{2} \bar{x}^T D \bar{x} \\ &= (D\bar{x} + c)^T (x_k - \bar{x}) + \frac{1}{2}(x_k - \bar{x})^T D (x_k - \bar{x}) + \\ &\quad + t_k \Big(\frac{1}{2} x_k^T D^0 x_k + (c^0)^T x_k \Big). \end{aligned} \tag{25}$$

Since $Ax \ge b$ is a regular system, by Lemma 2.1 we have $F(\bar{x}, \omega, \omega^0) \ne \emptyset$. Take $\bar{v} \in F(\bar{x}, \omega, \omega^0)$. Then, for every sufficiently small positive number t_k, we have

$$\bar{x} + t_k \bar{v} \in \Delta(A + t_k A^0, b + t_k b^0).$$

Hence, for t_k small enough, we have

$$\begin{aligned} \varphi(\omega + t_k\omega^0) - \varphi(\omega) &= (c + t_k c^0)^T x_k + \frac{1}{2} x_k^T (D + t_k D^0) x_k + \\ &\quad + \Big(- c^T \bar{x} - \frac{1}{2} \bar{x}^T D \bar{x} \Big) \\ &\le (c + t_k c^0)^T (\bar{x} + t_k \bar{v}) + \\ &\quad + \frac{1}{2} (\bar{x} + t_k \bar{v})^T (D + t_k D^0)(\bar{x} + t_k \bar{v}) + \\ &\quad + \Big(- c^T \bar{x} - \frac{1}{2} \bar{x}^T D \bar{x} \Big). \end{aligned} \tag{26}$$

From (25) and (26), for k large enough, we have

$$(D\bar{x}+c)^T(x_k-\bar{x})+\frac{1}{2}(x_k-\bar{x})^TD(x_k-\bar{x})+t_k\Big(\frac{1}{2}x_k^TD^0x_k+(c^0)^Tx_k\Big)$$

(27)

$$\leq t_k(c^0)^T(\bar{x}+t_k\bar{v})+\frac{1}{2}t_k(\bar{x}+t_k\bar{v})^TD^0(\bar{x}+t_k\bar{v})+t_k\big(c^T\bar{v}+\bar{v}^TD\bar{v}+\frac{1}{2}t_k\bar{v}^TD\bar{v}\big).$$

Dividing both sides of (27) by $\|x_k-\bar{x}\|$, letting $k\to\infty$ and taking into account (22), we get

$$(D\bar{x}+c)^Tv\leq 0. \tag{28}$$

As $\bar{x}$ is a solution of (1) and (24) is valid, we have $(D\bar{x}+c)^Tv\geq 0$ (see [7], Theorem 2.8.4). Combining this with (28), we conclude that

$$(D\bar{x}+c)^Tv=0. \tag{29}$$

Properties (23), (24) and (29) show that (b) does not hold. Thus $\bar{x}$ cannot be a locally unique solution of (1), a contradiction to our assumption. The proof is complete. □

4. Directional differentiability of the function $\varphi(\cdot)$

The following theorem describes a sufficient condition for $\varphi(\cdot)$ to be directionally differentiable and gives an explicit formula for computing the directional derivative of $\varphi(\cdot)$.

Theorem 4.1. *Let $\omega=(D,A,c,b)\in R_s^{n\times n}\times R^{m\times n}\times R^n\times R^m$ be a given point and $\omega^0=(D^0,A^0,c^0,b^0)\in R_s^{n\times n}\times R^{m\times n}\times R^n\times R^m$ be a given direction. If condition (G) and the following two conditions*

(i) *The system $Ax\geq b$ is regular,*

(ii) *$Sol(D,A,0,0)=\{0\}$,*

are satisfied, then the optimal value function φ is directionally differentiable at $\omega=(D,A,c,b)$ in direction $\omega^0=(D^0,A^0,c^0,b^0)$, and

$$\varphi'(\omega;\omega^0)=\inf_{\bar{x}\in Sol(D,A,c,b)}\max_{\lambda\in\Lambda(\bar{x},\omega)}((c^0)^T\bar{x}+\frac{1}{2}\bar{x}^TD^0\bar{x}+(b^0-A^0\bar{x})^T\lambda), \tag{30}$$

where $\Lambda(\bar{x},\omega)$ is the Lagrange multiplier set corresponding to the solution $\bar{x}\in Sol(D,A,c,b)$.

Proof. 1) Suppose that the conditions (i) and (ii) are satisfied. According to Lemma 2.4 of [18], $Sol(D,A,c,b)$ is a nonempty compact set. Take an arbitrary $\bar{x}\in Sol(D,A,c,b)$. By (i) and Lemma 2.1, $F(\bar{x},\omega,\omega^0)\neq\emptyset$. Take $v\in F(\bar{x},\omega,\omega^0)$. For $t>0$ small enough, we have

$$\bar{x}+tv\in\Delta(A+tA^0,b+tb^0),$$

hence

$$\begin{aligned}
&\varphi(\omega + t\omega^0) - \varphi(\omega) \le \\
&\le (c + tc^0)^T(\bar{x} + tv) + \frac{1}{2}(\bar{x} + tv)^T(D + tD^0)(\bar{x} + tv) - (c^T\bar{x} + \frac{1}{2}\bar{x}^T D\bar{x}) \\
&= t(D\bar{x} + c)^T v + t\big((c^0)^T\bar{x} + \frac{1}{2}\bar{x}^T D^0\bar{x}\big) + \frac{1}{2}t^2 v^T Dv + t^2 v^T D\bar{x} + \frac{1}{2}t^3 v^T D^0 v.
\end{aligned}$$

Multiplying the above inequality by t^{-1} and taking $\limsup$ as $t \to 0^+$, we obtain

$$\varphi^+(\omega;\omega^0) \le (D\bar{x} + c)^T v + \frac{1}{2}\bar{x}^T D^0\bar{x} + (c^0)^T\bar{x}.$$

This inequality holds for any $v \in F(\bar{x},\omega,\omega^0)$ and any $\bar{x} \in Sol(D,A,c,b)$. Consequently,

$$\varphi^+(\omega;\omega^0) \le \inf_{\bar{x}\in Sol(D,A,c,b)} \inf_{v\in F(\bar{x},\omega,\omega^0)} \left[(D\bar{x} + c)^T v + \frac{1}{2}\bar{x}D^0\bar{x} + (c^0)^T\bar{x}\right].$$

By Lemmas 2.2 and 2.3,

$$\begin{aligned}
\inf_{v\in F(\bar{x},\omega,\omega^0)} (D\bar{x} + c)^T v &= \inf_{v\in R(\bar{x},\omega,\omega^0)} (D\bar{x} + c)^T v \\
&= \max_{\lambda\in\Lambda(\bar{x},\omega)} (b^0 - A^0\bar{x})^T\lambda.
\end{aligned}$$

Hence

$$\varphi^+(\omega;\omega^0) \le \inf_{\bar{x}\in Sol(D,A,c,b)} \max_{\lambda\in\Lambda(\bar{x},\omega)} [(b^0 - A^0\bar{x})^T\lambda + \frac{1}{2}\bar{x}^T D^0\bar{x} + (c^0)^T\bar{x}]. \tag{31}$$

2) Let $\{t_k\}$ be a sequence of real numbers such that $t_k \downarrow 0$ and

$$\varphi^-(\omega;\omega^0) = \lim_{k\to\infty} \frac{\varphi(\omega + t_k\omega^0) - \varphi(\omega)}{t_k}\cdot$$

Due to the assumptions (i) and (ii), and according to Lemmas 2.1, 2.3 and 2.4 of [18] we may assume that

$$Sol(\omega + t_k\omega^0) \ne \emptyset \quad \text{for every } k.$$

Let $\{x_k\}$ be an arbitrary sequence in R^n such that $x_k \in Sol(\omega + t_k\omega^0)$ for every k. By Lemma 2.4, we may assume without loss of generality that $x_k \to \hat{x} \in Sol(D,A,c,b)$ as $k \to \infty$. We have

(32)

$$\varphi(\omega + t_k\omega^0) - \varphi(\omega) = (c + t_kc^0)^T x_k + \frac{1}{2}x_k^T(D + t_kD^0)x_k + \big(-c^T\hat{x} - \frac{1}{2}\hat{x}^T D\hat{x}\big).$$

Take $\lambda \in \Lambda(\hat{x},\omega)$. Since

$$\lambda^T(A\hat{x} - b) = 0, \quad \lambda \ge 0,$$

and

$$(A + t_kA^0)x_k \ge b + t_kb^0,$$

we get from (32)

$$\begin{aligned}\varphi(\omega+t_k\omega^0)-\varphi(\omega) &\geq (c+t_kc^0)^Tx_k+\frac{1}{2}x_k^T(D+t_kD^0)x_k-c^T\hat{x}-\frac{1}{2}\hat{x}^TD\hat{x}+\\ &\quad+\lambda^T(A\hat{x}-b)-[(A+t_kA_0)x_k-b-t_kb^0]^T\lambda\\ &=(D\hat{x}-A^T\lambda+c)^T(x_k-\hat{x})+\frac{1}{2}(x_k-\hat{x})^TD(x_k-\hat{x})+\\ &\quad+t_k\left[(c^0)^Tx_k+\frac{1}{2}x_k^TD^0x_k+(b^0-A^0x_k)^T\lambda\right].\end{aligned}$$

Since $\lambda\in\Lambda(\hat{x},\omega),\ D\hat{x}-A^T\lambda+c=0$. Hence, we have

$$\begin{aligned}\varphi(\omega+t_k\omega^0)-\varphi(\omega) &\geq \frac{1}{2}(x_k-\hat{x})^TD(x_k-\hat{x})+\\ &\quad+t_k\left[(c^0)^Tx_k+\frac{1}{2}x_k^TD^0x_k+(b^0-A^0x_k)^T\lambda\right].\end{aligned}$$

Multiplying both sides of this inequality by $(t_k)^{-1}$, taking $\liminf$ as $k\to\infty$ and using condition (G), we obtain

$$\varphi^-(\omega;\omega^0)\geq(c^0)^T\hat{x}+\frac{1}{2}\hat{x}^TD^0\hat{x}+(b^0-A^0\hat{x})^T\lambda.$$

As $\lambda\in\Lambda(\hat{x},\omega)$ can be chosen arbitrarily, we conclude that

$$\begin{aligned}\varphi^-(\omega;\omega^0) &\geq \max_{\lambda\in\Lambda(\hat{x},\omega)}\left[(c^0)^T\hat{x}+\frac{1}{2}\hat{x}^TD^0\hat{x}+(b^0-A^0\hat{x})^T\lambda\right]\\ &\geq \inf_{\bar{x}\in Sol(D,A,c,b)}\max_{\lambda\in\Lambda(\bar{x},\omega)}[(c^0)^T\bar{x}+\frac{1}{2}\bar{x}^TD^0\bar{x}+(b^0-A^0\bar{x})^T\lambda].\end{aligned}$$

Combining this with (31), we have

$$\varphi^-(\omega;\omega^0)=\varphi^+(\omega;\omega^0),$$

and therefore,

$$\varphi'(\omega;\omega^0)=\inf_{\bar{x}\in Sol(\omega)}\max_{\lambda\in\Lambda(\bar{x},\omega)}[(c^0)^T\bar{x}+\frac{1}{2}\bar{x}^TD^0\bar{x}+(b^0-A^0\bar{x})^T\lambda].$$

The proof is complete. □

Let us apply Theorem 4.1 to a concrete example.

Example 4.1. Let $n=2,\ m=3$,

$$D=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\quad A^T=\begin{bmatrix}1&-1&0\\-1&0&1\end{bmatrix},\quad b^T=(0,\ -1,\ 0),\quad c=\begin{pmatrix}0\\0\end{pmatrix},$$

$$D^0=\begin{bmatrix}1&0\\0&-1\end{bmatrix},\ (A^0)^T=\begin{bmatrix}0&0&0\\0&0&0\end{bmatrix},\ (b^0)^T=(0,\ -1,\ 0),\ c^0=\begin{pmatrix}0\\0\end{pmatrix},$$

$$\omega=(D,A,c,b),\quad\omega^0=(D^0,A^0,c^0,b^0).$$

It is easy to verify that $Ax \geq b$ is a regular system, $Sol(D, A, 0, 0) = \{0\}$ and

$$Sol(D, A, c, b) = Sol(\omega) = \{(x_1, x_2)^T \in R^2 : x_1 = x_2,\ 0 \leq x_1 \leq 1\}$$
$$Sol(\omega + t\omega^0) = \{(x_1, x_2)^T \in R^2 : x_1 = x_2, \quad 0 \leq x_1 \leq 1 + t\}$$

for every $t \geq 0$. For $\bar{x} = (\bar{x}_1, \bar{x}_2) \in Sol(\omega)$, we have

$$\Lambda(\bar{x}, \omega) = \{(\lambda_1, \lambda_2, \lambda_3)^T \in R^3 : \lambda_1 = \bar{x}_1, \lambda_2 = \lambda_3 = 0\}.$$

Suppose that the sequence $\{x^{(k)}\}$, $x^{(k)} = (x_1^{(k)}, x_2^{(k)}) \in Sol(\omega + t_k\omega^0)$, converges to $\bar{x} = (\bar{x}_1, \bar{x}_2) \in Sol(\omega)$. We have $x_1^{(k)} = x_2^{(k)}$ and $\bar{x}_1 = \bar{x}_2$. Hence

$$\frac{(x^{(k)} - \bar{x})^T D(x^{(k)} - \bar{x})}{t_k} = \frac{(x_1^{(k)} - \bar{x}_1)^2 - (x_2^{(k)} - \bar{x}_2)^2}{t_k} = 0,$$

and condition (G) is satisfied. By Theorem 4.1,

$$\begin{aligned}\varphi'(\omega; \omega^0) &= \inf_{\bar{x} \in Sol(\omega)} \max_{\lambda \in \Lambda(\bar{x}, \omega)} \Big((b^0)^T\lambda + \frac{1}{2}\bar{x}^T D^0 \bar{x} + (c^0)^T \bar{x}\Big) \\ &= \inf_{\bar{x} \in Sol(\omega)} 0 = 0.\end{aligned}$$

Observe that, in Example 4.1, $x^T Dx$ is an indefinite quadratic form (the sign of the expression $x^T Dx$ depends on the choice of x) and the solutions of the QP problem are not locally unique, thus the assumptions of Theorem 3.1 are not satisfied.

Consider Problem (1) and assume that $\bar{x} \in Sol(D, A, c, b)$ is one of its solutions. Let $u = \omega^0 = (D^0, A^0, c^0, b^0) \in R_s^{n \times n} \times R^{m \times n} \times R^n \times R^n$ be a given direction. Condition $(SOSC)_u$ in [1] applied to the solution $\bar{x}$ of Problem (1), is stated as follows:

$$(SOSC)_u \qquad \begin{cases} \textit{For every vector } v \in \mathcal{F}_{\bar{x}} \setminus \{0\},\ \textit{if } (D\bar{x} + c)^T v = 0 \\ \textit{then} \quad v^T D v > 0, \end{cases}$$

where $\mathcal{F}_{\bar{x}}$ is the cone of the feasible directions of $\Delta(A, b)$ at $\bar{x}$. That is

$$\mathcal{F}_{\bar{x}} = \{v \in R^n : (Av)_i \geq 0 \textit{ for every } i \textit{ satisfying } (A\bar{x})_i = b_i\}.$$

Notice that, in the case of QP problems, condition $(SOSC)_u$ is equivalent to the requirement that $\bar{x}$ is a locally unique solution of (1) (see [12]). This remark allows us to deduce from Theorem 1 of [1] the following result.

Proposition 4.1. *Let $\omega = (D, A, c, b) \in R_s^{n\times n} \times R^{m \times n} \times R^n \times R^m$ be a given point and $u = \omega^0 = (D^0, A^0, c^0, b^0) \in R_s^{n \times n} \times R^{m \times n} \times R^n \times R^m$ be a given direction. If all the solutions of Problems* (1) *are locally unique and the two conditions*

(i) *The system $Ax \geq b$ is regular,*

(ii) $Sol(D, A, 0, 0) = \{0\}$

are satisfied, then the optimal value function φ is directionally differentiable at $\omega = (D, A, c, b)$ in direction $u = \omega^0 = (D^0, A^0, c^0, b^0)$, and formula (30) *is valid.*

Proof. By a result of N. H. Nhan (see [17], Theorem 2.1), from the assumptions (i) and (ii) it follows that the map $Sol(\cdot)$ is upper semicontinuous at (D, A, c, b). Besides, by Lemma 3.3 in [17], $Sol(D, A, c, b)$ is a nonempty compact set. Hence there exist a compact set $B \subset R^n$ and a constant $\varepsilon > 0$ such that

$$\emptyset \neq Sol(\omega + t\omega^0) \subset B \quad \text{for every } t \in [0, \varepsilon].$$

We see that, under the conditions of Proposition 4.1, all the assumptions of Theorem 1 of [1] are fulfilled. So the desired conclusion follows from applying this theorem. □

Observe that Proposition 4.1 is a direct corollary of our Theorems 3.1 and 4.1. It is worth noting that the result stated in Proposition 4.1 cannot be applied to the problem described in Example 4.1 (because condition $(SOSC)_u$, where $u := \omega^0$, does not hold at any solution $\bar{x} \in Sol(\omega)$). Neither can this result be applied to convex QP problems whose solution sets have more than one element. This is because, for such a problem, the solution set is a convex set consisting of more than one element. Using Remark 3.1 we can conclude that Theorem 4.1 is applicable to convex QP problems.

Consider Problem (1) and denote $\omega = (D, A, c, b)$. Suppose that

$$\omega^0 = (D^0, A^0, c^0, b^0) \in R_s^{n\times n} \times R^{m\times n} \times R^n \times R^m$$

is a given direction. In this case, condition $(H3)$ in [13] can be stated as follows:

- $(H3)$ *For every sequence $\{t_k\}$, $t_k \downarrow 0$, and every sequence $\{x_k\}$, $x_k \in Sol(\omega + t_k\omega^0)$, $x_k \to \bar{x} \in Sol(D, A, c, b)$, the following inequality is satisfied*

$$\limsup_{k\to\infty} \frac{\|x_k - \bar{x}\|^2}{t_k} < +\infty.$$

Applying Theorem 4.1 of [13] to Problem (1) we get the following result.

Proposition 4.2. *Let $\omega = (D, A, c, b)$ and $\omega^0 = (D^0, A^0, c^0, b^0)$ be given as in Proposition 4.1. If $(H3)$ and the two conditions*

(i) *The system $Ax \geq b$ is regular,*

(ii) *There exist a compact set $B \subset R^n$ and a neighborhood U of $(A, b) \in R^{m\times n} \times R^m$ such that $\Delta(A', b') \subset B$ for every $(A', b') \in U$, are satisfied, then the optimal value function φ is directionally differentiable at $\omega = (D, A, c, b)$ in direction $u = \omega^0 = (D^0, A^0, c^0, b^0)$, and formula (30) is valid.*

Consider again the problem described in Example 4.1. Choose $\bar{x} = (0, 0) \in Sol(\omega)$, $t_k = k^{-1}$,

$$x_k = (k^{-\frac{1}{4}}, k^{-\frac{1}{4}}) \in Sol(\omega + t_k\omega^0).$$

We have $x_k \to \bar{x}$ as $k \to \infty$ and

$$\limsup_{k\to\infty} \frac{\|x_k - \bar{x}\|^2}{t_k} = \limsup_{k\to\infty} \frac{k^{-\frac{1}{2}} + k^{-\frac{1}{2}}}{k^{-1}} = +\infty,$$

so $(H3)$ does not hold and Proposition 4.2 cannot be applied to this QP problem.

We have shown that Theorem 4.1 can be applied even to some kinds of QP problems where the existing results on differential stability in nonlinear programming cannot be used. Now, we want to show that, for Problem (1), if the system $Ax \geq b$ is regular then $(H3)$ implies (G).

Proposition 4.3. *Let* $\omega = (D, A, c, b)$ *and* $\omega^0 = (D^0, A^0, c^0, b^0)$ *be given as in Proposition 4.1. If the system* $Ax \geq b$ *is regular, then condition* $(H3)$ *implies condition* (G).

Proof. Suppose that $(H3)$ holds. Consider sequences $\{t_k\}$, $t_k \downarrow 0$, and $\{x_k\}$, where $x_k \in Sol(\omega + t_k\omega^0)$ for each k. If

$$x_k \to \bar{x} \in Sol(D, A, c, b)$$

then, by $(H3)$, we have

$$\limsup_{k\to\infty} \frac{\|x_k - \bar{x}\|^2}{t_k} < +\infty. \tag{33}$$

We have to verify condition (G). Let $\{t_{k'}^{-1}(x_{k'} - \bar{x})^T D(x_{k'} - \bar{x})\}$ be a subsequence of $\{t_k^{-1}(x_k - \bar{x})^T D(x_k - \bar{x})\}$ satisfying

$$\liminf_{k\to\infty} t_k^{-1}(x_k - \bar{x})^T D(x_k - \bar{x}) = \lim_{k'\to\infty} t_{k'}^{-1}(x_{k'} - \bar{x})^T D(x_{k'} - \bar{x}). \tag{34}$$

From (33) it follows that the sequence $\{t_k^{-1}\|x_k - \bar{x}\|^2\}$ is bounded. Then the sequence $\{t_k^{-1/2}\|x_k - \bar{x}\|\}$ is bounded. Without loss of generality, we may assume that

$$t_k^{-1/2}\|x_k - \bar{x}\| \to v \in R^n. \tag{35}$$

As $x_k \in Sol(D + t_kD^0, A + t_kA^0, c + t_kc^0, b + t_kb^0)$, we have

$$(A_I + t_kA_I^0)x_k \geq b_I + t_kb_I^0,$$

where $I = \{i : (A\bar{x})_i = b_i\}$. Since $b_I = A_I\bar{x}$,

$$A_I(x_k - \bar{x}) \geq t_k(b_I^0 - A_I^0x_k).$$

Multiplying both sides of this inequality by $t_k^{-1/2}$ and letting $k \to \infty$, according to (35) we conclude that $A_Iv \geq 0$. Hence $v \in \mathcal{F}_{\bar{x}}$, where $\mathcal{F}_{\bar{x}}$ is defined as in the formulation of condition $(SOSC)_u$. Furthermore, note that the expression (25) holds. As $Ax \geq b$ is a regular system, by Lemma 2.1 we have $F(\bar{x}, \omega, \omega^0) \neq \emptyset$. Take an arbitraryy $\bar{v} \in F(\bar{x}, \omega, \omega^0)$. Then, for k large enough,

$$\bar{x} + t_k\bar{v} \in \Delta(A + t_kA^0, b + t_kb^0).$$

Therefore, for k large enough, we have (26). From (25) and (26) we obtain (27). Multiplying both sides of (27) by $t_k^{-1/2}$, letting $k \to \infty$ and taking into account (35), we get (28). As $\bar{x}$ is a solution of Problem (1) and $v \in \mathcal{F}_{\bar{x}}$, the case

$(D\bar{x}+c)^T v < 0$ cannot happen. Hence $(D\bar{x}+c)^T v = 0$. Since $\bar{x} \in Sol(\omega)$, we should have $v^T D v \geq 0$ (see [7], Theorem 2.8.4). By (34) and (35),

$$\begin{aligned}
&\liminf_{k\to\infty} t_k^{-1}(x_k-\bar{x})^T D(x_k-\bar{x}) \\
&= \lim_{k'\to\infty} \left(t_{k'}^{-1/2}(x_{k'}-\bar{x})\right)^T D \left(t_{k'}^{-1/2}(x_{k'}-\bar{x})\right) \\
&= v^T D v \geq 0.
\end{aligned}$$

Thus (G) is satisfied. □

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