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MEAN VALUE THEOREMS FOR CORRESPONDENCES

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Dedicated to Pham Huu Sach on the occasion of his sixtieth birthday

Abstract. We prove mean value estimates yielding Lipschitz rate for multimappings using coderivatives and subdifferential calculus.

1. Introduction

Mean value theorems have proved to be very useful in smooth and nonsmooth analysis. In particular, they allows to get estimates. Leaving apart the case of differentiable or convex functions ([14]-[16], [45]), the first appearance of a mean value theorem is due to Lebourg in the case of Lipschitz functions, by making use of the Clarke subdifferential ([29]). In [32] the author proved such a result for lower semicontinuous (l.s.c.) functions using contingent (or Hadamard) directional derivatives; the new features involved the replacement of the point ensuring a mean property by a sequence of points and the key fact that these points may be outside of the segment. Shortly after that extension, Zagrodny ([47]-[48]; see also [44]), adopting these new features and using the Ekeland variational principle instead of the equivalent Drop theorem returned to a more strinking and usable dual point of view with the Clarke-Rockafellar subdifferential. The applicability of such a result has been extended by Loewen [30] to Fréchet subdifferentials and by the author to a general class of subdifferentials [36].

The notion of pseudo-Lipschitz behavior (or Aubin property) is an important generalization to multimappings of the notion of Lipschitz mapping. For several applications, it is a realistic notion, whereas a genuine Lipschitz behavior does not hold in general. The same can be said for the notion of sub-Lipschitzian behavior introduced by Rockafellar in [41] and for the notion of boundedly Lipschitzian behavior considered here. Recall that a multimapping (or correspondence) $F : X \rightrightarrows Y$ between two metric spaces is said to be *pseudo-Lipschitzian* around $(x_0, y_0) \in F$ is there exist $c > 0$ and neighborhoods U, V of x_0, y_0 in X and Y respectively such that for any x, $x' \in U$ and any $y \in F(x) \cap V$ one has

$$d(y, F(x')) \leq cd(x, x').$$

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It is the purpose of this note to present criteria for such Lipschitzian properties in terms of coderivatives and coderivative sprouts, a notion we introduce in the next section. It is designed for quantitative purposes. We use it for getting estimates of Lipschitz rates. Let us note that the main applications of mean value inequalities are of quantitative nature as well as of qualitative character (see [8]-[13], [39], [40], [43], [44], [46] for instance). Our main results are given in section 3; the methods are close to the ones used in [24], [26], [27], [37] and elsewhere. This is not surprising because it is known that Lipschitz properties and metric regularity are intimately linked (however, the results we give may have a global character which is not shared by known results dealing with openness and metric regularity). In particular, these methods are quite different from the methods used by Pham Huu Sach in his study of calmness and regularity ([42]). We end the paper with some comparisons and open problems.

2. Normals and normal sprouts

The appearance of various concepts in nonsmooth analysis has created a need to work in a unified framework depending on a few basic properties adopted as axioms. Up to now, only the analytical side of nonsmooth analysis has received such a general treatment (see [12], [17], [19]-[23], [28], [36], [50] for example) although a number of properties deal with set-theoretical concept such as normal cones and tangent cones (see [4], [3], [6] for characterizations of generalized convexity properties by means of normal cones to sublevel sets). Here we propose a similar framework for the geometric side which can also provide a general approach. In fact, many constructions of subdifferentials for lower semicontinuous functions use a passage through normal cones to epigraphs. The following conditions are met by usual normal cones:

(N1) if S, S' are two subsets of a Banach space X such that $S \cap V = S' \cap V$ for some neighborhood V of x, then $N(S,x) = N(S',x)$;

(N2) if $S \subset X$ is convex and $x \in \mathrm{cl}\, S$, then

$$N(S,x) = \{x^* \in X^* : \forall u \in S \ \langle x^*, u - x\rangle \leq 0\};$$

(N3) if $S \subset h^{-1}(]-\infty, c])$ with $h \in X^*$, $c \in I\!R$ and $h(x) = c$ then $h \in N(S,x)$.

The following notion of normal sprout fulfils a quantitative concern. The idea of taking just a part of the normal cone instead of the whole normal cone has already been used by Ioffe [21], [24], Jourani and Jourani and Thibault [26], [27]. Let us define it formally.

Definition 2.1. A *normal sprout* is a correspondence which associates to each point x of the closure $\mathrm{cl}\, S$ of a subset S of a Banach space X a subset $\widehat{N}(S,x)$ of the unit ball $B^* := B_{X^*}$ of the dual space, in such a way that the following conditions are satisfied:

(NS1) if S, S' are two subsets such that $S \cap V = S' \cap V$ for some neighborhood V of x, then $\widehat{N}(S,x) = \widehat{N}(S',x)$;

(NS2) if S is convex and $x \in \mathrm{cl}\, S$, then

$$\widehat{N}(S,x) = \{x^* \in B^* : \forall u \in S\ \langle x^*, u - x\rangle \leq 0\}\,;$$

(NS3) if $S \subset h^{-1}(]-\infty, c])$ with $h \in B^*,\ c \in I\!R$ and $h(x) = c$ then $h \in \widehat{N}(S,x)$.

Clearly if N is a normal cone satisfying properties (N1)-(N3), then $\widehat{N}$ given by $\widehat{N}(S,x) = N(S,x) \cap B^*$ satisfies properties (NS1)-(NS3). Conversely, given a normal sprout $\widehat{N}$, the normal cone N associated with $\widehat{N}$ given by $N(S,x) = \mathbb{R}_+ \widehat{N}(S,x)$ satisfies properties (N1)-(N3). We call N the normal cone generated by $\widehat{N}$.

The following example is the main source of normal sprouts.

Example. Let ∂ be a subdifferential satisfying the usual conditions:
(S1) if f, f' are two functions which coincide on some neighborhood of x, then $\partial f(x) = \partial f'(x)$;
(S2) if f is convex, then $\partial f(x) = \{x^* \in X^* : \forall u \in X\ \langle x^*, u - x\rangle \leq f(u) - f(x)\}\,;$
(S3) if for some $x^* \in X$ the function $f - x^*$ attains its infimum at x then $x^* \in \partial f(x)$.

Then, setting $\widehat{N}(S,x) := \partial d_S(x)$, where $d_S(w) := \inf_{z \in S} \|w - z\|$, we obtain a notion of normal sprout. In fact, condition (NS2) is a consequence of the fact that for any convex set S and any $x \in S$ one has

$$\partial d_S(x) = \{x^* \in B^* : \forall u \in S\ \langle x^*, u - x\rangle \leq 0\}\,.$$

Now if $S \subset h^{-1}(]-\infty, c])$ with $h \in B^*,\ c \in I\!R$ and $h(x) = c$ as in (NS3), the function $-h$ attains its infimum over S at x, so that $-h + d_S$ attains its infimum over X at x and by (S3) $h \in \partial d_S(x)$. Condition (NS1) being obviously satisfied, one gets a normal sprout.

Conversely, given a normal sprout $\widehat{N}$, the subdifferential ∂ associated with $\widehat{N}$ given for a function f finite at x by

$$\partial f(x) = \{x^* \in X^* : (0, -1) \in \widehat{N}(E_{f-x^*}, x_{f-x^*})\},$$

where $x_{f-x^*} := (x, (f - x^*)(x))$, $E_{f-x^*} := \{(x,r) \in X \times \mathbb{R} : r \geq f(x) - x^*(x)\}$ satisfies properties (S1)-(S3). The assertion is obvious for (S1) and (S2). Now assume that, as in (S3), for some $x^* \in X$ the function $f - x^*$ attains its infimum at x. Then E_{f-x^*} is contained in $h^{-1}(]-\infty, c])$ with $h := (0, -1) \in (X \times I\!R)^*,\ c := -f(x) + x^*(x)$, so that $x^* \in \partial f(x)$ in view of (NS3) and of the definition above.

Example. (Fréchet normal sprout) It is defined in the following way: $x^* \in \widehat{N}(S,x)$ if $\|x^*\| \leq 1$ and $\limsup_{w \to x,\, w \in S} \langle x^*, w - x\rangle / \|w - x\| \leq 0$. Then $\widehat{N}(S,x) = \partial d_S(x)$, where $\partial d_S(x)$ is the Fréchet subdifferential of d_S at x ([22] Lemma 3, [38] Lemma 1, for example).

Example. (Hadamard or contingent or directional normal sprout) It is given by $\widehat{N}(S,x) = N(S,x) \cap B^*$ where $N(S,x) := (T(S,x))^o$ is the polar cone of the tangent (or contingent) cone $T(S,x)$ of S at x, with $T(S,x) := \limsup_{t \to 0_+} \frac{1}{t}(S -$

x). Then, if X is finite dimensional, $\widehat{N}(S,x) = \partial d_S(x)$, where $\partial d_S(x)$ is the Hadamard (or contingent) subdifferential of d_S at x.

Another track yielding normals and normal sprouts is geometrical. It can be traced back to the works of Bony and Federer.

Example. (Proximal normal sprout) When X is reflexive one can take

$$\widehat{N}(S,x) := \{x^* \in B_{X^*} : \exists v \in X\backslash\{0\},\ d_S(x+v) = \|v\|,\ \langle x^*, v\rangle = \|x^*\|\|v\|\}\,.$$

We leave to the reader the task of checking properties (NS1)-(NS3), using the Ascoli formula.

Example. (Quadratic normal sprout)

$$\begin{aligned}\widehat{N}(S,x) := \bigl\{x^* \in B_{X^*} : \exists c, r \in]0,\infty[,\\ \langle x^*, w - x\rangle \le c\|w-x\|^2\ \forall w \in S \cap B(x,r)\bigr\}.\end{aligned}$$

When X is a Hilbert space, this sprout coincides with the preceding one and is associated with the proximal subdifferential of d_S at x : one has the relation $\widehat{N}(S,x) = \partial d_S(x)$, where $\partial d_S(x)$ is the proximal subdifferential of d_S at x.

Example. The sprout given by $\widehat{N}(S,x) := N^{\uparrow}(S,x) \cap B^*$, where $N^{\uparrow}(S,x)$ is the Clarke normal cone to S at x is not associated with the Clarke subdifferential $\partial^{\uparrow} d_S(x)$ of d_S: the relation $N^{\uparrow}(S,x) \cap B^* = \partial^{\uparrow} d_S(x)$ is not always valid.

Example. (Stabilized or limiting normal sprout) Given any normal sprout $\widehat{N}$ one gets another normal sprout $\overline{\widehat{N}}$ by taking the set $\overline{\widehat{N}}(S,x)$ of weak* limit points of sequences (x_n^*) such that $x_n^* \in \widehat{N}(S,x_n)$ for some sequence (x_n) in S with limit x. One can also convexify a given normal sprout. We leave to the reader the easy task of checking conditions (NS1)-(NS3) in each of these two cases.

Additional conditions to (NS1)-(NS3) may be useful for different purposes; for instance, one may impose an invariance property with respect to translations or that the normal sprout to a product is the product of the normal sprouts to its factors. In fact, we will not impose any condition at all but the following basic minimization principle contained in the following definition (which is close to a notion introduced in [34] and is modelled on the concept of trustworthiness due to A.D. Ioffe [18]). The terminology we chose reflects the fact that we do not know whether the condition exactly corresponds to trustworthiness.

Definition 1. A normal sprout $\widehat{N}$ is said to be amiable with respect to a subset S of a Banach space Z at $z \in S$ if for any convex function g on Z which is Lipschitzian with rate one and attains its infimum on S at z and for any $\varepsilon > 0$ there exists $z', z'' \in B(z,\varepsilon) := \varepsilon B_Z + z$ with $z'' \in S$ such that

$$0 \in \partial g\left(z'\right) + \widehat{N}\left(S, z''\right) + \varepsilon B_{Z^*}.$$

If $\widehat{N}$ is amiable with S at any point z of S, $\widehat{N}$ is said to be amiable with S.

Here $\partial g(z')$ denotes the usual subdifferential of g at z' in the sense of convex analysis. If one can take $\varepsilon = 0$ in what precedes, $\widehat{N}$ is said to be exactly amiable. It is said to be amiable on X with respect to a class $\mathcal{C}$ of subsets of X if it is amiable with respect to S for each S in the class $\mathcal{C}$. In particular, if $\mathcal{C}$ is the whole family of closed subsets of X, it is said to be amiable on X. It is said to be amiable on a family of Banach spaces $\mathcal{X}$ if for any X in $\mathcal{X}$ it is amiable on X. Usually, this condition is satisfied for a whole class $\mathcal{C}$ of subsets (for instance the class of convex subsets or the class of all closed subsets of an appropriate class $\mathcal{Z}$ of spaces). When $\widehat{N}$ is the Fréchet normal sprout, one can take for $\mathcal{Z}$ the class of Asplund spaces. When $\widehat{N}$ is the stabilized (or limiting) Hadamard (or contingent) normal sprout, one can take for $\mathcal{Z}$ the class of all separable spaces. When $\widehat{N}$ is the normal sprout deduced from the approximate subdifferential of A.D. Ioffe ([21]) or in the sense of Clarke ([7]), one can take for $\mathcal{Z}$ the class of all Banach spaces. The Hadamard (or contingent) normal sprout is also amiable at z with a subset S of an arbitrary Banach space if the tangent cone (or contingent cone) $T_z S$ to S at z is convex.

Definition 2. If $F : X \rightrightarrows Y$ is a multimapping, the *coderivative sprout of* F at $z \in F$ is defined as the multimapping $\widehat{D}^* F(z) : Y^* \rightrightarrows X^*$ given by

$$\widehat{D}F(z)(y^*) := \left\{ x^* : (x^*, -y^*) \in \widehat{N}(F, z) \right\},$$

where F is identified with its graph. The *coderivative* of F is the multimapping $D^* F(z) : Y^* \rightrightarrows X^*$given by

$$D^* F(z)(y^*) := \{ x^* : (x^*, -y^*) \in N(F, z) \}.$$

This last concept is a flexible tool for the study of multimappings and a number of recent papers have been devoted to it. Let us observe that when $F : X \to Y$ is a differentiable mapping in the sense of Hadamard, i.e. when there exists a continuous linear mapping $DF(x) : X \to Y$ such that for any $u \in X$

$$\frac{1}{t} \big(F(x + tu') - F(x) \big) \to DF(x)(u)$$

as $t \to 0$, $u' \to u$, the coderivative of F at $(x, F(x))$ associated with the normal cone (i.e. the polar of the tangent cone) is the transpose of $DF(x)$. A similar assertion holds for the Fréchet derivative and the coderivative associated with the Fréchet normal cone. For the proximal coderivative and the Clarke coderivative, more stringent assumptions would be required.

3. Mean value theorems for multimappings

Given $s \in (0, 1)$ and a multimapping $H : V \rightrightarrows W$ between two normed spaces, let us set (with the usual conventions $0^{-1} = \infty$, $\inf \emptyset = +\infty$)

$$\|H\|_s := \sup \big\{ s\|v\|^{-1} : v \in V,\ w \in H(v), \|w\| \geq s \big\}.$$

When H is positively homogeneous, $\|H\|_s$ does not depend on s, so that we write $\|H\|$ which is then the least constant c such that $\|w\| \leq c\|v\|$ for each $v \in V$ and each $w \in H(v)$. Let us note that this definition differs from the one given by S. Robinson for convex processes.

Given a subset C of the Banach space Y and $\beta > 0$, we denote by $B(C, \beta)$ the set $C + \beta B_Y$. The *excess* of C over a subset D of Y is given by

$$e(C, D) := \inf\{\varepsilon > 0 : \ C \subset B(D, \varepsilon)\} = \sup_{x \in C} d(x, D).$$

The Pompeiu-Hausdorff distance is defined by $d(C, D) := \max(e(C, D), e(D, C))$. Given $r > 0$ and a point y_0 of Y we set

$$\begin{aligned} e_{r,y_0}(C, D) &: = e(C \cap B(y_0, r), D), \\ d_{r,y_0}(C, D) &= \max(e_{r,y_0}(C, D), e_{r,y_0}(D, C)). \end{aligned}$$

These notions have been extensively used during the last few years to study the variations of families of subsets (see for instance [1], [5], [33]).

The following result is a versatile tool which yields various Lipschitzian properties, depending of different choices of the subset V one makes. In the statements which follow the assumption that $\widehat{N}$ is amiable with respect to the graph of F is satisfied if $\widehat{N}$ is adapted to the class of spaces to which X belongs, as explained above, or if F belongs to a sufficiently regular class of multimappings (for instance the class of multimappings whose graphs are tangentially convex, in the sense that the tangent cones to their graphs are convex).

Theorem 3.1. *Suppose $\widehat{N}$ is amiable with respect to F and there exist $s \in (0, 1)$, $c \geq s$, $\alpha, \beta > 0$, $x_0 \in X$ and a nonempty subset V of Y such that*

$$\|\widehat{D}^* F(x, y)\|_s \leq c \ \text{ for any } x \in B := B(x_0, \alpha), \ y \in F(x) \cap B(V, \beta).$$

Then there exists $\rho \in]0, \alpha[$ depending only on α, β and c such that the multimapping F satisfies the Lipschitz type property

$$e(F(x) \cap V, F(x')) \leq c\|x - x'\| \quad \forall x, x' \in U := B(x_0, \rho).$$

Taking $V = Y$, $\beta > 0$ arbitrary, we get the following Lipschitz estimate.

Corollary 3.1. *Suppose $\widehat{N}$ is amiable with respect to F and there exist $s \in (0, 1)$, $c \geq s$, $\alpha > 0$, $x_0 \in X$ such that*

$$\|\widehat{D}^* F(x, y)\|_s \leq c \ \text{ for any } x \in B := B(x_0, \alpha), \ y \in F(x).$$

Then there exists $\rho \in (0, \alpha)$ such that the multimapping F satisfies the following Lipschitz property with respect to the Pompeiu-Hausdorff distance

$$d(F(x), F(x')) \leq c\|x - x'\| \quad \forall x, x' \in U := B(x_0, \rho).$$

It is more realistic to look for estimates on balls.

Corollary 3.2. *Suppose $\widehat{N}$ is amiable with respect to F and there exist $r > 0$, $s \in (0, 1)$, $c \geq s$, $\alpha, \beta > 0$, $x_0 \in X$, $y_0 \in Y$ such that*

$$\|\widehat{D}^* F(x, y)\|_s \leq c \ \text{ for any } x \in B := B(x_0, \alpha), \ y \in F(x) \cap B(y_0, r + \beta).$$

Then there exists $\rho > 0$ depending on α, β, c only such that F satisfies the Lipschitz type property

$$d_{r,y_0}(F(x), F(x')) \leq c\|x - x'\| \quad \forall x, x' \in U := B(x_0, \rho).$$

When one considers the coderivative of F instead of a coderivative sprout, one can suppress any reference to s. In such a case, the proof below becomes simpler.

PROOF OF THE THEOREM. Let $\rho > 0$ be such that $\rho < \alpha/3$, $\rho < \beta/3c$. Let us show that for any $c' \in]c, s^{-1}c[$ with $c' < 3c/2$ and any $u, u' \in U := B(x_0, \rho)$, $v \in F(u) \cap V$ one has

$$d(v, F(u')) \leq c'\|u - u'\|.$$

Since c' can be arbitrarily close to c, this inequality will prove the result. Let us set $b := 1/c' < 1/c$ and let us pick $m \in]1, s^{-1}[$ with $m > c^{-1}$, $m > s^{-1}bc$, $m > 2(s+1)^{-1}$(this is possible since $s^{-1} > c^{-1}$, $bc < 1$, $s^{-1} > 2(s+1)^{-1}$). Let us take $a \in]0, m-1[$ with $a < 1 - ms$ (this is possible since $m < s^{-1}$). Suppose on the contrary there exist $c' \in]c, s^{-1}c[$, $u, u' \in B(x_0, \rho)$, $v \in F(u) \cap V$ such that

$$d(v, F(u')) > c'\|u - u'\|. \tag{3.1}$$

Then we have $u \neq u'$. Applying Ekeland's theorem to $f : (x, y) \mapsto \|x - u'\|$ on F with $X \times Y$ endowed with the norm given by $\|(x, y)\| = a\|x\| + b\|y\|$, we get some $z_1 := (u_1, v_1) \in F$ such that for each $(x, y) \in F$

$$\left\|u_1 - u'\right\| \leq \left\|x - u'\right\| + a\left\|x - u_1\right\| + b\left\|y - v_1\right\| \tag{3.2}$$

$$\left\|u_1 - u'\right\| \leq \left\|u - u'\right\| - a\left\|u - u_1\right\| - b\left\|v - v_1\right\|. \tag{3.3}$$

If we had $u_1 = u'$, this last relation would yield

$$d(v, F(u')) \leq \|v - v_1\| \leq b^{-1}(1-a)\left\|u - u'\right\|,$$

a contradiction with $u \neq u'$, (3.1) and the choice $c' = b^{-1} > b^{-1}(1-a)$. Moreover, $\|u_1 - u'\| \leq \|u - u'\| \leq 2\rho$ and $\|u_1 - x_0\| \leq 3\rho < \alpha$, hence $u_1 \in B(x_0, \alpha)$ and $\|v - v_1\| \leq b^{-1}\|u - u'\| \leq 3c\rho < \beta$.

Since $m \geq 1 + a$, $m \geq c^{-1} \geq b$, the function g given by

$$g(x, y) = m^{-1}\left(\left\|x - u'\right\| + a\left\|x - u_1\right\| + b\|y - v_1\|\right),$$

is Lipschitzian with rate 1 on $X \times Y$ endowed with the sum norm. Let $\varepsilon > 0$ be such that

$$\varepsilon < \left\|u_1 - u'\right\|, \quad 3\rho + \varepsilon < \alpha, \quad 3c\rho + \varepsilon < \beta,$$
$$m^{-1}b + \varepsilon < c^{-1}s, \quad m^{-1}(1-a) - \varepsilon > s$$

(note that this is possible since $bc < ms < 1 - a$). We note that (3.2) means that g attains its infimum over F at $z_1 := (u_1, v_1)$. Using the amiability of $\widehat{N}$ with respect to F, we can find some $z_2 := (u_2, v_2)$, $z_3 := (u_3, v_3)$ in $B(z_1, \varepsilon)$ and some z_2^*, $z_3^* \in Z^*$ with

$$z_2^* \in \partial g(z_2), \quad z_3^* \in \widehat{N}(F, z_3), \quad \|z_2^* + z_3^*\| \leq \varepsilon.$$

Then we have $u_2,\ u_3 \in B(x_0, \alpha)$ and $v_3 \in B(v_1, \varepsilon) \subset B(v, 3c\rho + \varepsilon) \subset B(v, \beta) \subset B(V, \beta)$.

The familiar subdifferential calculus rules for convex continuous functions provides $u_2^* \in m^{-1}aB_{X^*}$, $v_2^* \in m^{-1}bB_{Y^*}$ and $w_2^* \in m^{-1}S_{X^*}$ such that $z_2^* = (w_2^*, 0) + (u_2^*, 0) + (0, v_2^*)$ (note that $u_2 - u' \neq 0$). Then, if $z_3^* := (u_3^*, v_3^*)$, one has $u_3^* \in \widehat{D}F\,(z_3)\,(-v_3^*)$ and $\|v_3^*\| \leq \|v_2^*\| + \varepsilon \leq m^{-1}b + \varepsilon < c^{-1}s$, $\|u_3^*\| \geq \|u_2^* + w_2^*\| - \varepsilon \geq m^{-1}(1-a) - \varepsilon > s$, a contradiction with our assumption $\|\widehat{D}F(u_3, v_3)\|_s \leq c$. □

The estimate of Corollary 3.1 can be applied when for some continuously differentiable mapping $g : B \to Y$ one has

$$\|D^*F(x, y) - Dg(x)^T\| \leq c \ \text{ for any } x \in B := B(x_0, \alpha),\ y \in F(x)$$

D^* being the coderivative associated with a subdifferential amiable on $X \times Y$. Then, for some $\rho > 0$, for any $c' > c$ and any $x, x' \in B(x_0, \rho)$ one has

$$F(x) \subset F(x') + g(x) - g(x') + B(0, c'\|x - x'\|).$$

Similar consequences are available for the other results we gave.

4. Related results and open problems

A. The ordinary mean value theorem for a nonsmooth function f provides estimates about the variation $f(x) - f(x')$ using some information on the subdifferential ∂f of f around the segment $[x, x']$. Therefore it is not directly comparable to the results above which require some information on an open set and do not involve any element of ∂f. However, both results share a common ground. Let us first observe that to any function $f : X \to \mathbb{R} \cup \{+\infty\}$ we can associate its epigraph multimapping $F : X \rightrightarrows \mathbb{R}$ given by $F(x) := [f(x), +\infty[$ whose graph is the epigraph of f. Then, if $\widehat{N}$ is the normal sprout associated with a subdifferential ∂ satisfying (S1)-(S3) and if f is continuous on an open convex subset U of X, for $x \in U$, $y = f(x)$ we have

$$x^* \in \widehat{D}^*F(x, y)(y^*) \Leftrightarrow x^* \in \partial f(x),\ y^* = 1$$

and $\widehat{D}^*F(x, y)$ is empty if $y \neq f(x)$. It follows that for $x \in U$, $y = f(x)$, $s \in]0, 1[$, one has

$$\left\|\widehat{D}^*F(x, y)\right\|_s = \sup\{\|x^*\| :\ x^* \in \partial f(x)\}$$

if the set $\{x^* \in \partial f(x) : \|x^*\| \geq s\}$ is not empty, 0 otherwise. Therefore, can deduce from the preceding section a result known under slightly different assumptions (compare with [8], [11], [13]).

Corollary 4.1. *Suppose the normal sprout associated with a subdifferential ∂ is amiable. Let f be a lower semicontinuous function on some open convex subset U of X such that* $\sup\{\|u^*\| : u \in U,\ u^* \in \partial f(u)\} \leq c$ *for some $c > 0$. Then f is Lipschitzian with rate c.*

B. The multidirectional mean value theorems which appeared during the last few years represent an important step forward in nonsmooth analysis ([10]-[11], [31][1], [49]). It is known that they are equivalent to several fuzzy calculus rules ([23], [28], [50]). One way wonder whether the mean value estimates of the preceding section also imply such rules. Moreover, putting some light on the relationships between these results and multidirectional inequalities would be of interest.

C. Mean value properties have been used in connection with order properties (see [10]-[11] for instance). The following proposition gives the flavour of what can be expected in this direction. Let us raise the question of a possible extension of such a property to multimappings. Let us suppose the Banach spaces X, Y are pre-ordered by closed convex cones X_+ and Y_+ respectively. We denote by Q^o the polar cone of a cone Q. We say that a set-valued mapping $F : X \rightrightarrows Y$ is *homotone* (resp. *antitone*) if for any $x, x' \in X$, with $x \leq x'$, $y \in F(x)$, $y' \in F(x')$, we have $y \leq y'$ (resp. $y \geq y'$). Here $x \leq x'$ (resp. $y \leq y'$) means that $x' - x \in X_+$ (resp. $y' - y \in Y_+$) and again we identify F with its graph. The conclusion of the following easy result can be established under various assumptions.

Lemma 4.1. *Suppose that for some open subset W of $X \times Y$, for each $z = (x, y) \in F \cap W$ and for each $u \in X_+$ there exist some $v \in Y$, $(v_n) \to v$, $(t_n) \to 0_+$ such that $(u, v) \in N(F, z)^o$ and $y + t_n v_n \in F(x + t_n u)$ for each n. If F is homotone, then, for each $z = (x, y) \in F \cap W$ one has $D^*F(x, y)(Y_+^o) \subset X_+^o$.*

When Y is finite dimensional and N is the contingent normal cone, the assumption is satisfied whenever for each $z = (x, y) \in F \cap W$ and for each $u \in X_+$ one has $\limsup_{t \searrow 0} t^{-1} d(y, F(x + tu)) < \infty$.

Proof. It suffices to check that for any $(x^*, -y^*) \in N(F, z)$ with $y^* \in Y_+^o$ one has $x^* \in X_+^o$. Given $u \in X_+$, let $v \in Y$, $(v_n) \to v$, $(t_n) \to 0_+$ be such that $(u, v) \in N(F, z)^o$ and $y + t_n v_n \in F(x + t_n u)$ for each n. Since F is homotone, we have $t_n v_n \in Y_+$; moreover $(u, v) \in N(F, z)^o$, hence

$$\langle x^*, u \rangle \leq \langle y^*, v \rangle = \lim_n \langle y^*, v_n \rangle \leq 0.$$

Thus $x^* \in X_+^o$. $\square$

Let us turn to the converse which is more interesting. Here we suppose that N is derived from a reliable subdifferential ∂ in the sense of [36].

Proposition 4.1. *Suppose F is a continuous mapping from an open convex subset U of X into Y such that $D^*F(x, y)(Y_+^o) \subset X_+^o$ for each $(x, y) = (x, F(x)) \in U \times Y$. If the following scalarization formula*

$$\partial(y^* \circ F)(x) \subset D^*F(x, y)(y^*)$$

[1]The idea of multidirectional mean value theorems has also been independently developped by D.T. Luc who presented his results at a seminar in the University of Pau during his stay in 1992 and at the international conference "Applied Analysis and its Applications" in Hanoi in September 1993.

holds for any $(x,y)=(x,F(x))\in U\times Y$ *and for any* $y^*\in Y_+^o$ *(as it is the case with most subdifferentials), then* F *is homotone on* U.

Proof. Suppose, on the contrary, that there exist $a,b\in U$ such that $b-a\in X_+$ and $F(b)-F(a)\notin Y_+$. The Hahn-Banach separation theorem yields some $y^*\in Y_+^o$ such that $\langle y^*,F(b)-F(a)\rangle>0$. The mean value theorem of [36] ensures the existence of $c\in[a,b]$, $(c_n)\to c$, $c_n^*\in\partial(y^*\circ F)(c_n)$ such that

$$\liminf_n\langle c_n^*,b-a\rangle\geq(y^*\circ F)(b)-(y^*\circ F)(a).$$

Thus, for n large enough one has $\langle c_n^*,b-a\rangle>0$, a contradiction with our scalarization assumption which yields

$$c_n^*\in\partial(y^*\circ F)(c_n)\subset D^*F(c_n,F(c_n))(y^*)\subset X_+^o.$$

□

D. In [42] Pham Huu Sach studied differentiability properties of multimappings and established calmness (and regularity) criteria. It would be interesting to compare the results of the present paper with the ones in [42]. One can observe that here we do not assume that the values of the multifunction F are closed convex. Therefore the use of the support functions of these values does not reflects accurately their behavior. However, some links between the two approaches may exist.

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