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SOME EQUILIBRIUM PROBLEMS IN GENERALIZED CONVEX SPACES

SEHIE PARK

Dedicated to Pham Huu Sach on the occasion of his sixtieth birthday

ABSTRACT. We show that how the fundamental theorems on equilibrium problems can be extended to generalized convex spaces. Precisely, most of important results in the KKM theory hold without assuming the linearity in topological vector spaces. Such examples are the KKM theorem, the minimax theorem and the intersection lemma of von Neumann, the Nash equilibrium theorem, various fixed point theorems, Ky Fan's minimax inequality, variational inequalities, best approximation theorems, existence theorems for solutions of generalized quasi-equilibrium problems, and others.

1. INTRODUCTION

By an *equilibrium problem*, Blum and Oettli [2] understand the problem of finding:

(EP) $$\widehat{x} \in X \text{ such that } f(\widehat{x}, y) \le 0 \text{ for all } y \in X,$$

where X is a given set and $f : X \times X \to \overline{\mathbb{R}}$ is a given function.

We can consider more general problems as follows:

A *quasi-equilibrium problem* is to find

(QEP) $$\widehat{x} \in X \text{ such that } \widehat{x} \in S(\widehat{x}) \text{ and } f(\widehat{x}, z) \le 0 \text{ for all } z \in S(\widehat{x}),$$

where X and f are as above and $S : X \multimap X$ is a given multimap.

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A *generalized quasi-equilibrium problem* is to find

$$\text{(GQEP)} \qquad \widehat{x} \in X \text{ and } \widehat{y} \in T(\widehat{x}) \text{ such that } \widehat{x} \in S(\widehat{x}) \\ \text{and } f(\widehat{x}, \widehat{y}, z) \leq 0 \text{ for all } z \in S(\widehat{x}),$$

where X and S are the same as above, Y is another given set, $T : X \multimap Y$ is another multimap, and $f : X \times Y \times X \to \overline{\mathbb{R}}$ is a given function.

These problems contain as special cases, for instance, optimization problems, problems of the Nash type equilibrium, complementarity problems, fixed point problems, and variational inequalities, as well as many others. There are many generalizations of these problems; see [2, 19, 24, 26].

In this paper, we study some equilibrium problems, quasi-equilibrium problems, and generalized quasi-equilibrium problems in generalized convex spaces. We show that how the fundamental theorems on equilibrium problems can be extended to generalized convex spaces. Precisely, most of important results in the KKM theory hold without assuming the linearity in topological vector spaces. Such examples are the KKM theorem, the minimax theorem and the intersection lemma of von Neumann, the Nash equilibrium theorem, various fixed point theorems, Ky Fan's minimax inequality, variational inequalities, best approximation theorems, existence theorems for solutions of generalized quasi-equilibrium problems, and others. For the proofs of those results, see [28] and references therein.

2. Generalized convex spaces

For topological spaces X and Y, a *multimap* or a *map* $T : X \multimap Y$ is a function from X into the power set of Y. $T(x)$ is called the *value* of T at $x \in X$ and $T^-(y) := \{x \in X : y \in T(x)\}$ the *fiber* of T at $y \in Y$. Let $T(A) := \bigcup\{T(x) : x \in A\}$ for $A \subset X$. A map $T : X \multimap Y$ is *upper semicontinuous* (*u.s.c.*) if for each open subset G of Y, the set $\{x \in X : T(x) \subset G\}$ is open in X; *lower semicontinuous* (*l.s.c.*) if for each closed subset F of Y, the set $\{x \in X : T(x) \subset F\}$ is closed in X; *continuous* if it is u.s.c. and l.s.c.; and *compact* if the range $T(X)$ is contained in a compact subset of Y.

A *generalized convex space* or a *G-convex space* $(X, D; \Gamma)$ consists of a topological space X and a nonempty set D such that for each $A = \{a_0, a_1, \cdots, a_n\} \in \langle D \rangle$, there exist a subset $\Gamma(A) = \Gamma_A$ of X and a continuous function $\phi_A : \Delta_n \to \Gamma(A)$ such that $J \subset \{0, 1, \cdots, n\}$ implies $\phi_A(\Delta_J) \subset \Gamma(\{a_j : j \in J\})$, where Δ_n is an n-simplex with vertices $v_0, v_1, \cdots, v_n$, $\Delta_J = \mathrm{co}\{v_j : j \in J\}$ the face of Δ_n corresponding to J, and $\langle D \rangle$ denotes the set of all nonempty finite subsets of D.

In case to emphasize $X \supset D$, $(X, D; \Gamma)$ will be denoted by $(X \supset D; \Gamma)$; and if $X = D$, then $(X; \Gamma) := (X \supset X; \Gamma)$. For details on G-convex spaces, see [21-23, 28, 31, 33] and references therein, where basic theory was extensively developed.

There are a lot of examples of G-convex spaces:

Examples 1. If $X = D$ is a convex subset of a vector space and each Γ_A is the convex hull of $A \in \langle X \rangle$ equipped with the Euclidean topology, then $(X; \Gamma)$

becomes a *convex space* in the sense of Lassonde [18]. Note that any convex subset of a topological vector space is a convex space, but not conversely.

Examples 2. If $X = D$ and each Γ_A is assumed to be contractible or, more generally, infinitely connected (that is, n-connected for all $n \geq 0$) and if for each $A, B \in \langle X \rangle$, $A \subset B$ implies $\Gamma_A \subset \Gamma_B$, then $(X; \Gamma)$ becomes a C-*space* (or an H-*space*) due to Horvath [11-14].

Examples 3. The other major examples of G-convex spaces are metric spaces with Michael's convex structure, Pasicki's S-contractible spaces, Horvath's pseudoconvex spaces, Komiya's convex spaces, Bielawski's simplicial convexities, Joó's pseudoconvex spaces, and so on. For the literature, see [21-23]. Recently, we gave new examples of G-convex spaces in [31] as follows: L-spaces of Ben-El-Mechaiekh *et al.*, continuous images of G-convex spaces, Verma's or Stachó's generalized H-spaces, Kulpa's simplicial structures, $P_{1,1}$-spaces of Forgo and Joó, and mc-spaces of Llinares. Moreover, Ben-El-Mechaiekh *et al.* gave examples of G-convex spaces $(X; \Gamma)$ as follows: B'-simplicial convexity, hyperconvex metric spaces due to Aronszajn and Panitchpakdi, and Takahashi's convexity in metric spaces.

Examples 4. Futhermore, any hyperbolic space X in the sense of Kirk and Reich-Shafrir is a G-convex space, since the closed convex hull of any $A \in \langle X \rangle$ is contractible. This class of metric spaces contains all normed vector spaces, all Hadamard manifolds, the Hilbert ball with the hyperbolic metric, and others. Note that an arbitrary product of hyperbolic spaces is also hyperbolic; see [33].

For a G-convex space $(X \supset D; \Gamma)$, a subset $Y \subset X$ is said to be Γ-*convex* if for each $N \in \langle D \rangle$, $N \subset Y$ implies $\Gamma_N \subset Y$; and for any subset $Y \subset X$, the *convex hull* of Y is defined as follows:

$$\Gamma\text{-co}\, Y := \bigcap \{Z \subset X : Z \text{ is a } \Gamma\text{-convex subset of } X \text{ containing } Y\}.$$

It is easily seen that $\Gamma\text{-co}\, Y = \bigcup \{\Gamma\text{-co}\, N : N \in \langle Y \rangle\}$.

For a G-convex space $(X \supset D; \Gamma)$, a real function $f : X \to \mathbb{R}$ is said to be *quasiconcave* [resp. *quasiconvex*] if $\{x \in X : f(x) > r\}$ [resp. $\{x \in X : f(x) < r\}$] is Γ-convex for each $r \in \mathbb{R}$.

Recall that a real function $f : X \to \mathbb{R}$, where X is a topological space, is *lower* [resp. *upper*] *semicontinuous* (l.s.c.) [resp. u.s.c.] if $\{x \in X : f(x) > r\}$ [resp. $\{x \in X : f(x) < r\}$] is open for each $r \in \mathbb{R}$.

3. The KKM theorems

For a G-convex space $(X, D; \Gamma)$, a multimap $F : D \multimap X$ is called a *KKM map* if $\Gamma_A \subset F(A)$ for each $A \in \langle D \rangle$.

The following result is well-known [17, 27, 28]:

The KKM Principle. *Let D be the set of vertices of an n-simplex Δ_n and*

$F : D \multimap \Delta_n$ be a KKM map (that is, $\operatorname{co} A \subset F(A)$ *for each $A \subset D$) with closed [resp. open] values. Then* $\bigcap_{z \in D} F(z) \neq \emptyset$.

The following is a KKM theorem for G-convex spaces [28]:

Theorem 1. *Let $(X, D; \Gamma)$ be a G-convex space and $F : D \multimap X$ a multimap such that*

(1.1) *F has closed [resp. open] values; and*

(1.2) *F is a KKM map.*

Then $\{F(z)\}_{z \in D}$ has the finite intersection property.

Further, if

(1.3) $\bigcap_{z \in M} \overline{F(z)}$ *is compact for some $M \in \langle D \rangle$,*

then we have $\bigcap_{z \in D} \overline{F(z)} \neq \emptyset$.

The KKM theory, first called by the author [20], is the study of KKM maps and their applications. Nowadays, it would be better to regard as the study of applications of various equivalent formulations of the KKM principle. At the beginning, the theory was mainly devoted to study on convex subsets of topological vector spaces. Later, it has been extended to convex spaces by Lassonde [18], and to C-spaces (or H-spaces) by Horvath [11-14] and others. Nowadays the KKM theory is extended to G-convex spaces in a sequence of papers of the author [21-23, 28, 31, 33].

A milestone of the history of the KKM theory was erected by Ky Fan [4]. He extended the KKM theorem to infinite dimensional spaces and applied it to coincidence theorems generalizing the Tychonoff fixed point theorem and a result concerning two continuous maps from a compact convex set into a uniform space. Further applications were followed by himself [5-7] and many other authors; see [8, 27].

Corollary 1.1 (Fan [4]). *Let X be an arbitrary set in a topological vector space Y. To each $x \in X$, let a closed set $F(x) \subset Y$ be given such that the following two conditions are satisfied:*

(1) *The convex hull of any finite subset $\{x_1, x_2, \cdots, x_n\}$ of X is contained in* $\bigcup_{i=1}^{n} F(x_i)$.

(2) *$F(x)$ is compact for at least one $x \in X$.*

Then $\bigcap_{x \in X} F(x) \neq \emptyset$.

This is usually known as the KKMF theorem.

Examples 5. Granas [8] gave examples of KKM maps as follows:

(i) *Variational problems.* Let C be a convex subset of a vector space E and $\phi : C \to \mathbb{R}$ is a convex function. Then $G : C \multimap C$ defined by

$$G(x) = \{y \in C : \phi(y) \le \phi(x)\} \quad \text{for } x \in C$$

is a KKM map.

(ii) *Best approximation.* Let C be a convex subset of a vector space E, p a seminorm on E, and $f : C \to E$ a function. Then $G : C \multimap C$ defined by

$$G(x) = \{y \in C : p(f(y) - y) \le p(f(y) - x)\} \quad \text{for } x \in C$$

is a KKM map.

(iii) *Variational inequalities.* Let $(H, \langle \cdot, \cdot \rangle)$ be an inner product space, C a convex subset of H, and $f : C \to H$ a function. Then $G : C \multimap C$ defined by

$$G(x) = \{y \in C : \langle f(y), y - x\rangle \le 0\} \quad \text{for } x \in C$$

is a KKM map.

4. The von Neumann type minimax theorem

Let $\{(X_i, D_i; \Gamma_i)\}_{i\in I}$ be any family of G-convex spaces. Let $X := \prod_{i\in I} X_i$ be equipped with the product topology and $D = \prod_{i\in I} D_i$. For each $i \in I$, let $\pi_i : D \to D_i$ be the projection. For each $A \in \langle D\rangle$, define

$$\Gamma(A) := \prod_{i\in I} \Gamma_i(\pi_i(A)).$$

Then $(X, D; \Gamma)$ is a G-convex space; see [35].

Note also that for the case $X_i = D_i$ for each i, the product of Γ-convex subsets is also Γ-convex in the product G-convex space; see [35].

In the framework of our KKM theory on G-convex spaces, we have the following generalization of the von Neumann–Sion minimax theorem; see [28]:

Theorem 2. *Let $(X; \Gamma)$ and $(Y; \Gamma')$ be compact G-convex spaces and $f, g : X \times Y \to \mathbb{R} \cup \{+\infty\}$ be functions such that*

(2.1) *$f(x, y) \le g(x, y)$ for each $(x, y) \in X \times Y$;*

(2.2) *for each $x \in X$, $f(x, \cdot)$ is l.s.c. and $g(x, \cdot)$ is quasiconvex on Y; and*

(2.3) *for each $y \in Y$, $f(\cdot, y)$ is quasiconcave and $g(\cdot, y)$ is u.s.c. on X.*

Then we have

$$\min_{y\in Y} \sup_{x\in X} f(x, y) \le \max_{x\in X} \inf_{y\in Y} g(x, y).$$

If $f = g$ and if X and Y are convex spaces, Theorem 2 reduces to the following Sion's generalization of the von Neumann minimax theorem:

Corollary 2.1 (Sion [34]). *Let X, Y be compact convex sets in topological vector spaces. Let f be a real function defined on $X \times Y$. If*

(1) *for each fixed $x \in X$, $f(x, y)$ is a l.s.c. quasiconvex function on Y, and*

(2) *for each fixed* $y \in Y$, $f(x,y)$ *is an u.s.c. quasiconcave function on* X, *then we have*

$$\min_{y \in Y} \max_{x \in X} f(x,y) = \max_{x \in X} \min_{y \in Y} f(x,y).$$

Kakutani [19] expressed the von Neumann theorem in 1928 as follows:

Corollary 2.2 (von Neumann [36]). *Let* $f(x,y)$ *be a continuous real function defined for* $x \in K$ *and* $y \in L$, *where* K *and* L *are arbitrary bounded closed convex sets in two Euclidean spaces* $\mathbb{R}^m$ *and* $\mathbb{R}^n$. *If for every* $x_0 \in K$ *and for every real number* α, *the set of all* $y \in L$ *such that* $f(x_0, y) \leq \alpha$ *is convex, and if for every* $y_0 \in L$ *and for every real number* β, *the set of all* $x \in K$ *such that* $f(x, y_0) \geq \beta$ *is convex, then we have*

$$\max_{x \in K} \min_{y \in L} f(x,y) = \min_{y \in L} \max_{x \in K} f(x,y).$$

5. The Ky Fan type intersection theorem

Given a Cartesian product $X = \prod_{i \in I} X_i$ of sets, let $X^i = \prod_{j \neq i} X_j$ and

$$\pi_i : X \to X_i, \quad \pi^i : X \to X^i$$

be the projections; we write $\pi_i(x) = x_i$ and $\pi^i(x) = x^i$. Given $x, y \in X$, we let

$$(y_i, x^i) := (x_1, \cdots, x_{i-1}, y_i, x_{i+1}, \cdots, x_n).$$

In our KKM theory [28], we have the following generalization of Ky Fan intersection theorem:

Theorem 3. *Let* $X = \prod_{i=1}^{n} X_i$, $(X;\Gamma)$ *be a compact* G*-convex space, and* $A_1, A_2, \cdots, A_n$ n *subsets of* X *such that*

(3.1) *for each* $x^i \in X^i$ *and* $i = 1, \cdots, n$, *the set* $A_i(x^i) = \{y_i \in X_i : (y_i, x^i) \in A_i\}$ *is* Γ*-convex and nonempty; and*

(3.2) *for each* $y_i \in X_i$ *and* $i = 1, \cdots, n$, *the set* $A_i(y_i) = \{x^i \in X^i : (y_i, x^i) \in A_i\}$ *is open.*

Then $\bigcap_{i=1}^{n} A_i \neq \emptyset$.

If each X_i is a compact G-convex space, so is X.

Corollary 3.1 (Fan [5]). *Let* $X_1, X_2, \cdots, X_n$ *be* $n\,(\geq 2)$ *compact convex sets each in a real Hausdorff topological vector space. Let* $E_1, E_2, \cdots, E_n$ *be* n *subsets of* $X = \prod_{i=1}^{n} X_i$ *having the following two properties*:

(a) *For each* i *and every point* $x_i \in X_i$, *the section* $E_i(x_i)$ *formed by all points* $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$ *of* X^i *such that* $(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n) \in E_i$ *is open in* X^i.

(b) *For each* i *and every point* $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$ *of* X^i, *the section* $E_i(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$ *formed by all points* $x_i \in X_i$ *such that* $(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n) \in E_i$ *is nonempty and convex.*

Then the intersection $\bigcap_{i=1}^{n} E_i$ *is nonempty.*

6. The Nash equilibrium theorem

From Theorem 3, we can deduce the following Nash type equilibrium theorem for G-convex spaces:

Theorem 4. *Let* $X = \prod_{i=1}^{n} X_i$, $(X; \Gamma)$ *a compact* G*-convex space, and* $f_1, \cdots, f_n : X \to \mathbb{R}$ *continuous functions such that*

(4.1) *for each* $x^i \in X^i$, *each* $i = 1, \cdots, n$, *and each* $r \in \mathbb{R}$, *the set*
$\{y_i \in X_i : f_i(y_i, x^i) > r\}$ *is* Γ*-convex.*

Then there exists a point $\widehat{x} \in X$ *such that*

$$f_i(\widehat{x}) = \max_{y_i \in X_i} f_i(y_i, \widehat{x}^i) \quad \textit{for} \quad i = 1, \cdots, n.$$

The first remarkable one of generalizations of von Neumann's minimax theorem was Nash's theorem on equilibrium points of non-cooperative games. The following is formulated by Ky Fan [5, Theorem 4]:

Corollary 4.1 (Nash). *Let* $X_1, X_2, \cdots, X_n$ *be* $n\,(\geq 2)$ *nonempty compact convex sets each in a real Hausdorff topological vector space. Let* $f_1, f_2, \cdots, f_n$ *be* n *real-valued continuous functions defined on* $X = \prod_{i=1}^{n} X_i$. *If for each* $i = 1, 2, \cdots, n$ *and for any given point* $(x_1, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n) \in \prod_{j \neq i} X_j$, $f_i(x_1, \cdots, x_{i-1}, x_i, x_{i+1}, \cdots, x_n)$ *is a quasiconcave function on* X_i, *then there exists a point* $(\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n) \in \prod_{i=1}^{n} X_i$ *such that*

$$f_i(\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n) = \max_{y_i \in X_i} f_i(\widehat{x}_1, \cdots, \widehat{x}_{i-1}, y_i, \widehat{x}_{i+1}, \cdots, \widehat{x}_n) \quad (1 \leq i \leq n).$$

The point $\widehat{x} = (\widehat{x}_1, \widehat{x}_2, \cdots, \widehat{x}_n)$ in the conclusion of Theorem 4 is called the *Nash equilibrium.* This concept is a natural extension of the local maxima (for the case $n = 1,\ f = f_1$) and of the saddle points (for the case $n = 2,\ f_1 = -f,\ f_2 = f$).

7. The Fan–Browder type fixed point theorems

Any binary relation R in a set X can be regarded as a multimap $T : X \multimap X$ and conversely by the following obvious way:

$$y \in T(x) \ \text{ if and only if } \ (x, y) \in R.$$

Therefore, a point $x_0 \in X$ is called a *maximal element* of a multimap $T : X \multimap X$ if $T(x_0) = \emptyset$.

From the KKM Theorem 1, we can show the following:

Theorem 5. *Let $(X, D; \Gamma)$ be a compact G-convex space and $S : D \multimap X$, $T : X \multimap X$ two maps such that*

(5.1) *$S(z)$ is open for each $z \in D$; and*

(5.2) *for each $y \in X$, $N \in \langle S^-(y) \rangle$ implies $\Gamma_N \subset T^-(y)$.*

Then either

(i) *T has a fixed point $x_0 \in X$; that is, $x_0 \in T(x_0)$; or*

(ii) *there exists an $x_1 \in X$ such that $S^-(x_1) = \emptyset$.*

Theorem 5 for the case (i) holds is *the Fan–Browder type fixed point theorem* and for the case (ii) *the maximal element theorem.*

From Theorem 5, we obtain the following Fan–Browder type theorems.

Corollary 5.1. *Let $(X, D; \Gamma)$ be a compact G-convex space and $S : X \multimap D$, $T : X \multimap X$ two maps such that*

(1) *for each $x \in X$, $N \in \langle S(x) \rangle$ implies $\Gamma_N \subset T(x)$; and*

(2) *$X = \bigcup \{\operatorname{Int} S^-(z) : z \in D\}$.*

Then T has a fixed point $x_0 \in X$.

Corollary 5.2. *Let $(X \supset D; \Gamma)$ be a compact G-convex space and $S : X \multimap D$ a map such that*

(1) *for each $x \in X$, $S(x)$ is nonempty; and*

(2) *for each $z \in D$, $S^-(z)$ is open.*

Then there exists an $\widehat{x} \in X$ such that $\widehat{x} \in \Gamma\text{-co}\, S(\widehat{x})$.

The following is a simplified form of Corollary 5.1 or 5.2:

Corollary 5.3. *Let $(X; \Gamma)$ be a compact G-convex space and $T : X \multimap X$ a map such that*

(1) *for each $x \in X, T(x)$ is Γ-convex; and*

(2) *$X = \bigcup \{\operatorname{Int} T^-(y) : y \in X\}$.*

Then T has a fixed point.

From Corollary 5.3, we can easily obtain the following Browder theorem in 1968:

Corollary 5.4 (Browder [3]). *Let K be a nonempty compact convex subset of a Hausdorff topological vector space. Let T be a map of K into 2^K, where for each $x \in K$, $T(x)$ is a nonempty convex subset of K. Suppose further that for each y in K, $T^-(y)$ is open in K. Then there exists x_0 in K such that $x_0 \in T(x_0)$.*

Note that Browder's result is a reformulation of Fan's geometric lemma [4] in the form of a fixed point theorem and its proof was based on the Brouwer fixed point theorem and the partition of unity argument. Since then it is known as the Fan–Browder fixed point theorem.

Browder [3] applied his theorem to a systematic treatment of the interconnections between multi-valued fixed point theorems, minimax theorems, variational inequalities, and monotone extension theorems. For further developments on generalizations and applications of the Fan–Browder theorem, we refer to Park [27].

The Fan–Browder type fixed point theorem is used by Borglin and Keiding [1] and Yannelis and Prabhakar [38] to the existence of maximal elements in mathematical economics. We give a generalization of their result as follows:

Corollary 5.5. *Let $(X, D; \Gamma)$ be a compact G-convex space and $S : X \multimap D$, $T : X \multimap X$ two maps such that*

(1) *$S^{-}(z)$ is open for each $z \in D$;*

(2) *for each $x \in X$, $N \in \langle S(x) \rangle$ implies $\Gamma_N \subset T(x)$; and*

(3) *for each $x \in X$, $x \notin T(x)$.*

Then there exists an $\widehat{x} \in X$ such that $S(\widehat{x}) = \emptyset$.

8. The Ky Fan type minimax inequalities

From Theorem 5, we obtain the following analytic alternative which is a basis of various equilibrium problems:

Theorem 6. *Let $(X, D; \Gamma)$ be a compact G-convex space, $f : D \times X \to \mathbb{R}$ and $g : X \times X \to \mathbb{R}$ two real functions, and $\alpha, \beta \in \mathbb{R}$. Suppose that*

(6.1) *$\{y \in X : f(z, y) > \alpha\}$ is open for each $z \in D$; and*

(6.2) *for each $y \in X$, $N \in \langle \{z \in D : f(z, y) > \alpha\} \rangle$ implies*

$$\Gamma_N \subset \{x \in X : g(x, y) > \beta\}.$$

Then either

(a) *there exists a $\widehat{y} \in X$ such that $f(z, \widehat{y}) \le \alpha$ for all $z \in D$; or*

(b) *there exists an $\widehat{x} \in X$ such that $g(\widehat{x}, \widehat{x}) > \beta$.*

From Theorem 6, we immediately have the following generalized form of the Ky Fan minimax inequality:

Theorem 7. *Under the hypothesis of Theorem 6, if $\alpha = \beta = \sup\{g(x, x) : x \in X\}$, then*

(c) *there exists a $\widehat{y} \in X$ such that*

$$f(z, \widehat{y}) \le \sup_{x \in X} g(x, x) \quad \textit{for all } z \in D; \textit{ and}$$

(d) *we have the minimax inequality*

$$\inf_{y \in X} \sup_{z \in D} f(z, y) \le \sup_{x \in X} g(x, x).$$

More early, Fan established a minimax inequality from the KKMF theorem:

Corollary 7.1 (Fan [7]). *Let X be a compact convex set in a topological vector space. Let f be a real function defined on $X \times X$ such that*

(a) *for each fixed $x \in X$, $f(x, y)$ is a l.s.c. function of y on X;*

(b) *for each fixed $y \in X$, $f(x, y)$ is a quasiconcave function of x on X.*

Then the minimax inequality

$$\min_{y \in X} \sup_{x \in X} f(x, y) \leq \sup_{x \in X} f(x, x)$$

holds.

Fan gave applications of his inequality as follows:

• A variational inequality (extending Hartman–Stampacchia [9] and Browder [3]).

• A geometric formulation of the inequality (equivalent to the Fan–Browder theorem).

• Separation properties of upper demicontinuous multimaps, coincidence and fixed point theorems.

• Properties of sets with convex sections (Fan [5]).

• A fundamental existence theorem in potential theory.

9. Fixed point theorems in locally G-convex spaces

A G-convex space $(X \supset D; \Gamma)$ is called an *LG-space* (or a *locally G-convex space*) if $(X, \mathcal{U})$ is a Hausdorff uniform space such that D is dense in X and if there exists a basis $\{V_\lambda\}_{\lambda \in I}$ for the uniformity $\mathcal{U}$ such that for each $\lambda \in I$, $\{x \in X : C \cap V_\lambda[x] \neq \emptyset\}$ is Γ-convex whenever $C \subset X$ is Γ-convex, where

$$V_\lambda[x] = \{x' \in X : (x, x') \in V_\lambda\}.$$

In this section, we show that the open version of the KKM theorem is also useful to deduce very general fixed point theorems for topological vector spaces or G-convex spaces. For simplicity, we give only one example in [33]:

Theorem 8. *Let $(X \supset D; \Gamma)$ be an LG-space and $T : X \multimap X$ a compact u.s.c. multimap with nonempty closed Γ-convex values. Then T has a fixed point $x_0 \in X$; that is, $x_0 \in Tx_0$.*

Note that, if $\Gamma_N \subset D$ for each $N \in \langle D \rangle$, it is sufficient to assume that T has Γ-convex values on D, not necessarily on the whole X.

In order to give an example of Theorem 8, we introduce a notion due to Himmelberg [10]:

A nonempty subset Y of a topological vector space E is said to be *almost convex* if for any neighborhood V of the origin 0 of E and for any finite set $\{y_1, y_2, \cdots, y_n\} \subset Y$, there exists a finite set $\{z_1, z_2, \cdots, z_n\} \subset Y$ such that, for each $i = 1, 2, \cdots, n$, $z_i - y_i \in V$ and $\mathrm{co}\{z_1, z_2, \cdots, z_n\} \subset Y$.

Corollary 8.1. *Let X be a convex subset of a locally convex Hausdorff topological vector space E and Y an almost convex dense subset of X. Let $T : X \multimap X$ be a compact u.s.c. multimap with nonempty closed values such that Ty is convex for all $y \in Y$. Then T has a fixed point.*

Corollary 8.2 (Himmelberg [10]). *Let X be a convex subset of a locally convex Hausdorff topological vector space E and $T : X \multimap X$ a compact u.s.c. multimap with nonempty closed convex values. Then T has a fixed point $x_0 \in T(x_0)$.*

Recall that the Himmelberg theorem unifies and generalizes historically well-known fixed point theorems due to Brouwer, Schauder, Tychonoff, Kakutani, Bohnenblust and Karlin, Fan, Glicksberg, Hukuhara, Rhee, and others. For the literature, see [27].

In order to give simple proofs of von Neumann's Lemma and the minimax theorem, Kakutani obtained the following generalization of the Brouwer theorem to multimap:

Corollary 8.3 (Kakutani [15]). *If $x \mapsto \Phi(x)$ is an upper semicontinuous point-to-set mapping of an r-dimensional closed simplex S into the family of closed convex subsets of S, then there exists an $x_0 \in S$ such that $x_0 \in \Phi(x_0)$.*

Equivalently,

Corollary 8.4 (Kakutani [15]). *Corollary 8.3 is also valid even if S is an arbitrary bounded closed convex set in an Euclidean space.*

As Kakutani noted, Corollary 8.4 readily implies von Neumann's Lemma [37], and later it is known that those two results are equivalent.

This was the beginning of the fixed point theory of multimaps having a vital connection with the minimax theory in game theory and the equilibrium theory in economics.

The following is von Neumann's Lemma:

Corollary 8.5 (von Neumann [37]). *Let K and L be two compact convex sets in the Euclidean spaces $\mathbb{R}^m$ and $\mathbb{R}^n$, and let us consider their Cartesian product $K \times L$ in $\mathbb{R}^{m+n}$. Let U and V be two closed subsets of $K \times L$ such that for any $x_0 \in K$ the set U_{x_0}, of $y \in L$ such that $(x_0, y) \in U$, is nonempty, closed and convex and such that for any $y_0 \in L$ the set V_{y_0}, of all $x \in K$ such that $(x, y_0) \in V$, is nonempty, closed and convex. Under these assumptions, U and V have a common point.*

Von Neumann proved this by using a notion of integral in Euclidean spaces and applied this to the problems of mathematical economics. We adopt the above formulation from Kakutani [15].

According to Debreu (A commentary on the Kakutani fixed point theorem, in *Collected Works of Kakutani*), "Ironically that Lemma, which, through Kakutani's Corollary, had a major influence in particular on economic theory and on the theory of games, was not required to obtain either one of the results that von Neumann wanted to establish. The Minimax theorem, as well as his theorem on optimal balanced growth paths, can be proved by elementary means."

10. Variational inequalities

From Theorem 7, we obtain the following type of variational inequalities:

Theorem 9. *Let $(X;\Gamma)$ be a compact G-convex space and $p, q : X \times X \to \mathbb{R}$ and $h : X \to \mathbb{R}$ functions satisfying*

(9.1) *$p(x,y) \le q(x,y)$ for each $(x,y) \in X \times X$, and $q(x,x) \le 0$ for all $x \in X$;*

(9.2) *for each $x \in X$, $q(x,\cdot) + h(\cdot)$ is quasiconcave on X; and*

(9.3) *for each $y \in X$, $p(\cdot,y) - h(\cdot)$ is l.s.c. on X.*

Then there exists a $y_0 \in X$ such that

$$p(x,y_0) + h(y_0) \le h(x) \quad \text{for all } x \in X.$$

In 1966, Hartman and Stampacchia introduced the following variational inequality:

Corollary 9.1 (Hartman–Stampacchia [9]). *Let K be a compact convex subset in $\mathbb{R}^n$ and $f : K \to \mathbb{R}^n$ a continuous map. Then there exists $u_0 \in K$ such that*

$$(f(u_0), v - u_0) \ge 0 \quad \text{for all } v \in K,$$

where $(\cdot,\cdot)$ denotes the scalar product in $\mathbb{R}^n$.

Using this result, its authors obtained existence and uniqueness theorems for (weak) uniformly Lipschitz continuous solutions of Dirichlet boundary value problems associated with certain nonlinear elliptic differential functional equations.

Later the preceding corollary is known to be equivalent to the Brouwer fixed point theorem. Corollary 9.1 was extended as follows:

Corollary 9.2 (Browder [3]). *Let E be a locally convex Hausdorff topological vector space, K a compact convex subset of E, and T a continuous mapping of K into E^*. Then there exists an element u_0 of K such that*

$$(T(u_0), u - u_0) \ge 0$$

for all $u \in K$.

Here, E^* is the topological dual of E equipped with an adequate topology and $(\ ,\ )$ denotes the pairing between elements of E^* and elements of E. This theorem is later extended and improved by many authors by pointing out that the local convexity is superfluous.

11. Best approximations

A simple consequence of Theorem 9 is the following well-known existence result on best approximations originated from Ky Fan [6]:

Theorem 10. *Let X be a compact convex subset of a topological vector space E and $f : X \to E$ a continuous function. Then for any continuous seminorm p on E, there exists a point $y_0 \in X$ such that*

$$p(y_0 - f(y_0)) \leq p(x - f(y_0)) \quad \text{for all } x \in X.$$

Theorem 10 immediately implies the following generalization of the Schauder fixed point theorem:

Corollary 10.1 (Fan [6]). *Let X be a nonempty compact convex set in a normed vector space E. For any continuous map $f : X \to E$, there exists a point $y_0 \in X$ such that*

$$\|y_0 - f(y_0)\| = \min_{x \in X} \|x - f(y_0)\|.$$

(*In particular, if $f(X) \subset X$, then y_0 is a fixed point of f.*)

Fan also obtained a generalization of this theorem to locally convex Hausdorff topological vector spaces. Those are known as best approximation theorems and applied to obtain generalizations of the Brouwer fixed point theorem and some nonseparation theorems on upper demicontinuous (u.d.c.) multimaps in Fan [6].

12. Generalized quasi-equilibrium problems

In this section, we deal with existence of solutions of certain quasi-equilibrium problems in G-convex spaces without any linear structure.

We obtained the following in [26]:

Theorem 11. *Let $(X; \Gamma)$ be a compact G-convex space, and let $S : X \multimap X$ be a map with nonempty Γ-convex values and open fibers such that $\overline{S} : X \multimap X$ is u.s.c. Suppose that $\psi : X \times X \to \mathbb{R}$ is a continuous function such that $\psi(x, \cdot)$ is quasiconvex and*

$$\psi(x, x) \geq 0 \qquad \text{for all } x \in X.$$

Then there exists an $\widehat{x} \in X$ such that

$$\widehat{x} \in S(\widehat{x}) \quad \text{and} \quad \psi(\widehat{x}, x) \geq 0 \qquad \text{for all } x \in S(\widehat{x}).$$

A nonempty subset X of a Hausdorff topological vector space (t.v.s.) E is said to be *admissible* (in the sense of Klee [16]) provided that, for every compact subset K of X and every neighborhood V of the origin 0 of E, there exists a continuous map $h : K \to X$ such that $x - h(x) \in V$ for all $x \in K$ and $h(K)$ is contained in a finite dimensional subspace L of E.

Note that every nonempty convex subset of a locally convex t.v.s. is admissible. Other examples of admissible t.v.s. are l^p and $L^p(0,1)$ for $0 < p < 1$, the space $S(0,1)$ of equivalence classes of measurable functions on $[0,1]$, the Hardy space

H^p for $0 < p < 1$, certain Orlicz spaces, ultrabarrelled t.v.s. admitting Schauder basis, and others. Note also that any locally convex subset of an F-normable t.v.s. or any compact convex locally convex subset of a t.v.s. is admissible.

The present author obtained the following fixed point theorem [25, 29] which generalizes the Himmelberg theorem [10]:

Theorem 12. *Let E be a t.v.s. and X an admissible convex subset of E. Then any compact acyclic map $F : X \multimap X$ has a fixed point $x \in X$; that is, $x \in F(x)$.*

From Theorem 12, we deduced the following existence theorem [30] for a generalized quasi-equilibrium problem:

Theorem 13. *Let X and Y be admissible convex subsets of t.v.s. E and F, respectively, $S : X \multimap X$ a compact closed map, $T : X \multimap Y$ a compact acyclic map, and $\phi : X \times Y \times X \to \mathbb{R}$ an u.s.c. function. Suppose that*

(13.1) *the function $m : X \times Y \to \mathbb{R}$ defined by*

$$m(x,y) = \max_{s \in S(x)} \phi(x,y,s) \quad \textit{for } (x,y) \in X \times Y$$

is l.s.c.; and

(13.2) *for each $(x,y) \in X \times Y$, the set*

$$M(x,y) = \{u \in S(x) : \phi(x,y,u) = m(x,y)\}$$

is acyclic.

Then there exists an $(\widehat{x}, \widehat{y}) \in X \times Y$ such that

$$\widehat{x} \in S(\widehat{x}),\ \widehat{y} \in T(\widehat{x}),\ \textit{and } \phi(\widehat{x},\widehat{y},\widehat{x}) \geq \phi(\widehat{x},\widehat{y},s) \quad \textit{for all } s \in S(\widehat{x}).$$

Moreover, in [32], Theorem 12 is applied to deduce collectively fixed point theorems, intersection theorems for sets with convex sections, and quasi-equilibrium theorems. For related results, see [20-33] and references therein.

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Department of Mathematics
Seoul National University
Seoul 151-742, Korea