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INCREASING POSITIVELY HOMOGENEOUS FUNCTIONS DEFINED ON $\mathbb{R}^n$

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Dedicated to Pham Huu Sach on the occasion of his sixtieth birthday

Abstract. The theory of IPH (increasing positively homogeneous of degree one) functions defined on the cone $\mathbb{R}^n_+$ of all vectors with nonnegative coordinates is well developed. In this paper we present a suitable extension of this theory for IPH functions defined on the entire space $\mathbb{R}^n$.

1. Introduction

In this paper we study IPH (increasing positively homogeneous of degree one) functions defined on the n-dimensional space $\mathbb{R}^n$. The theory of IPH functions defined on the cone $\mathbb{R}^n_+$ of all vectors with nonnegative coordinates is well developed [6]. Two main results form the core of this theory. First, each IPH function p defined on $\mathbb{R}^n_+$ can be represented as the Minkowski gauge of a certain normal closed subset U of $\mathbb{R}^n_+$, namely $U = \{x \in \mathbb{R}^n_+ : p(x) \leq 1\}$. (A set U is called normal if $(x \in U, y \in \mathbb{R}^n_+, y \leq x) \implies y \in U)$. Vice versa, the Minkowski gauge of a normal closed set is an IPH function. The second result is based on ideas of abstract convexity: each IPH function defined on $\mathbb{R}^n_+$ can be represented as the upper envelope of a set of so-called min-type functions. This result can be considered as a certain form of a dual representation of IPH functions.

IPH functions can be useful for the description of radiant and co-radiant downward sets and, through them, can have applications to the study of some NTU games arising in Mathematical Economics [1, 8, 5] and to the analysis of topical functions, which are used in the analysis of discrete event systems [2, 3, 4]. IPH functions defined on $\mathbb{R}^n_+$ have many interesting applications (see, for example, [7], where some applications to nonlinear penalty functions have been examined). In order to extend these applications to a more general class of problems we need to extend, in a suitable way, the above mentioned results to IPH functions defined on the entire space $\mathbb{R}^n$.

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Thus we need to give two kinds of presentations of IPH functions defined on $\mathbb{R}^n$: one of them, which can be called *primal*, is an analogue of a representation through the Minkowski gauge of a normal set for IPH functions defined on $\mathbb{R}^n_+$. The other, which can be called *dual*, is based on the representation of an IPH function as the upper envelope of a certain set of simple (elementary) functions. In the current paper we examine both primal and dual representations.

As it turns out, the theory of IPH functions defined on $\mathbb{R}^n$ is much more complicated than that for IPH functions defined on $\mathbb{R}^n_+$. Namely, a primal representation for IPH functions that attend both positive and negative values can be given only through a certain pair of sets. We cannot use min-type functions for a dual representation. This presentation can be given by functions that coincide with a min-type function on a certain cone, which depends on the function, and is equal to $-\infty$ outside of this cone.

The paper has the following structure. In Section 2 we provide some brief preliminary definitions and results related to IPH functions. The primal representation of IPH functions is examined in Section 3. Abstract convexity of IPH functions is studied in Section 4. Finally in Section 5 abstract convexity and abstract concavity of nonnegative IPH and DPH functions are examined.

2. The basic properties of IPH functions

Let $I = 1, \dots, n$ be a finite set of indices. Consider the space $\mathbb{R}^n$ of all vectors $(x_i)_{i\in I}$. We shall use the following notations:

- if $x \in \mathbb{R}^n$, then x_i is the i-th coordinate of x;
- if $x, y \in \mathbb{R}^n$ then $x \geq y \iff x_i \geq y_i$ for all $i \in I$;
- if $x, y \in \mathbb{R}^n$ then $x \gg y \iff x_i > y_i$ for all $i \in I$;
- $\mathbb{R}^n_+ = \{x = (x_i)_{i\in I} \in \mathbb{R}^I : x_i \geq 0 \text{ for all } i \in I\}$;
- $\mathbb{R}^n_- = \{x = (x_i)_{i\in I} \in \mathbb{R}^I : x_i \leq 0 \text{ for all } i \in I\}$;
- $\mathbb{R} := \mathbb{R}^1$
- $\bar{\mathbb{R}} = \mathbb{R} \cup \{-\infty\} \cup \{+\infty\} \equiv [-\infty, +\infty]$.
- $\mathbb{R}_{+\infty} = \mathbb{R} \cup \{+\infty\} \equiv (-\infty, +\infty]$.
- $\mathbb{R}_{++} = \{x \in \mathbb{R} : x > 0\}$

In this paper we accept the following conventions:

$$0 \times (+\infty) = 0, \quad 0 \times (-\infty) = 0, \quad \frac{0}{0} = 0.$$

We need to define the supremum and the infimum over the empty set. This definition depends on the universal set. If $A \subseteq \mathbb{R}$ is the universal set of numbers then the supremum of the empty set is equal to $\inf A$ and the infimum of the empty set is equal to $\sup A$. Thus to define the supremum of the empty set we need to indicate the universal set A. In the sequel we consider either $A = \mathbb{R}$ or $A = \mathbb{R}_{++} := \{x \in \mathbb{R} : x > 0\}$. In the former case $\sup \emptyset = -\infty$, in the latter case $\sup \emptyset = 0$, in both cases $\inf \emptyset = +\infty$.

Let K be either the space $\mathbb{R}^n$, the cone $\mathbb{R}^n_+$ or the cone $\mathbb{R}^n_-$. A function $p : K \to \bar{\mathbb{R}}$ is called increasing if $x \geq y \implies p(x) \geq p(y)$. A function $p : K \to \bar{\mathbb{R}}$

is called positively homogeneous of degree one if $p(\lambda x) = \lambda p(x)$ for all $x \in \mathbb{R}^n$ and $\lambda \geq 0$. In this paper we shall study IPH (increasing positively homogeneous of degree one) functions $p : \mathbb{R}^n \to \bar{\mathbb{R}}$. We shall consider only IPH functions p such that $0 \in \operatorname{dom} p := \{x \in \mathbb{R}^n : -\infty < p(x) < +\infty\}$.

We now describe some simple properties of IPH functions p defined on $\mathbb{R}^n$.

Proposition 2.1. *Let $p : \mathbb{R}^n \to \bar{\mathbb{R}}$ be an IPH function. Then*

1) $p(0) = 0$;
2) $p(x) \geq 0$ *for* $x \geq 0$ *and* $p(x) \leq 0$ *for* $x \leq 0$;
3) *If* $p(x) = 0$ *for a strictly positive vector* x *then* $p(y) \leq 0$ *for all* $y \in \mathbb{R}^n$; *if* $p(x) = 0$ *for a strictly negative vector* x *then* $p(y) \geq 0$ *for all* $y \in \mathbb{R}^n$.
4) *If there exists a point* $x \in \mathbb{R}^n$ *such that* $p(x) = +\infty$, *then* $p(y) = +\infty$ *for all* $y \gg 0$; *if there exists a point* $x \in \mathbb{R}^n$ *such that* $p(x) = -\infty$ *then* $p(y) = -\infty$ *for all* $y \ll 0$.
5) p *is continuous on* $\operatorname{int} \mathbb{R}^n_+$ *and* $\operatorname{int} \mathbb{R}^n_-$.

Proof. 1) It follows from positive homogeneity of p.

2) It follows from monotonicity of p.

3) Let $p(x) = 0$ for a point $x \in \operatorname{int} \mathbb{R}^n_+$. Since $x \gg 0$, it follows that for each $y \in \mathbb{R}^n$ there exists $\lambda > 0$ such that $y \leq \lambda x$. Hence $p(y) \leq \lambda p(x) = 0$. The similar argument shows that $p(y) \geq 0$ for each $y \in \mathbb{R}^n$ if there exists $x \ll 0$ such that $p(x) = 0$.

4) Let $x \in \mathbb{R}^n$ be a point such that $p(x) = +\infty$ and $y \gg 0$. Then there exists $\lambda > 0$ such that $x \leq \lambda y$. Since p is IPH, it follows that $p(y) = +\infty$. A similar argument shows that $p(y) = -\infty$ if there exists $x \in \mathbb{R}^n$ such that $p(x) = -\infty$.

5) We shall prove continuity only for $x \gg 0$. Due to 4) we only need to consider functions p that are finite on $\operatorname{int} \mathbb{R}^n_+$. Let $x_n \to x$ and $\epsilon > 0$. Then

$$(1 - \varepsilon)x \leq x_n \leq (1 + \varepsilon)x$$

for sufficiently large n, so

$$(1 - \varepsilon)p(x) = p((1 - \varepsilon)x) \leq p(x_n) \leq p((1 + \varepsilon)x) = (1 + \varepsilon)p(x).$$

Thus $p(x_n) \to p(x)$. □

If $x \notin (\operatorname{int} \mathbb{R}^n_+) \cup (\operatorname{int} \mathbb{R}^n_-)$ then an IPH function p can be discontinuous at x. We now give a corresponding example.

Example 2.1. Let $n = 2$. Consider the following function p:

$$p(x) = \begin{cases} x_1 + x_2 & x \in \mathbb{R}^2_- \cup \mathbb{R}^2_+ \\ 0 & \text{otherwise} \end{cases}$$

It is easy to check that p is an IPH function. This function is discontinuous, moreover it is neither lower semicontinuous nor upper semicontinuous.

A set $U \subset \mathbb{R}^n$ is called downward if $(x \in U,\, x' \leq x) \implies x' \in U$ (in particular, the empty set is downward).

Proposition 2.2. *If p is an IPH function then its level sets $\{x \in \mathbb{R}^n : p(x) \le c\}$ are downward for all $c \in \bar{\mathbb{R}}$.*

Proof. It follows from the monotonicity of p. □

The set of all IPH functions $p : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ is a convex cone, the set of all IPH functions $p : \mathbb{R}^n \to \bar{\mathbb{R}}$ is a complete lattice (with respect to the pointwise supremum and infimum).

Let p be an IPH function and let

$$p^+(x) = \max(p(x), 0), \qquad p^-(x) = \min(p(x), 0).$$

Then p^+ and p^- are IPH functions, p^+ is nonnegative, p^- is nonpositive and $p(x) = p^+(x) + p^-(x)$ for all $x \in \mathbb{R}^n$.

A function $q : \mathbb{R}^n \to \bar{\mathbb{R}}$ is called decreasing if $x \ge y \implies q(x) \le q(y)$. We shall also study DPH (decreasing positively homogeneous of the degree one) functions.

If p is an IPH function then the function $p_*(x) = -p(-x)$ is again IPH. The function p generates also two DPH functions: $q(x) = p(-x)$ and $q_*(x) = -p(x) \equiv -q(-x)$. Thus DPH functions can be studied with the help of IPH functions. Vice versa, each DPH function q generates two IPH functions: $p(x) = q(-x)$ and $p^*(x) = -q(x)$, hence IPH functions can be studied by means of DPH ones.

3. Primal representation of IPH functions

We need the following definitions. A set $U \subset \mathbb{R}^n$ is called radiant (star-shaped with respect to zero) if $(x \in U, \lambda \in (0,1) \implies \lambda x \in U)$. If U is a nonempty closed radiant set then $0 \in U$. We now define the Minkowski gauge of a radiant set. The set $\mathbb{R}_{++}$ of positive numbers will be used as the universal set for this definition, so we assume that the supremum of the empty set is equal to $\inf \mathbb{R}_{++} = 0$ and the infimum of the empty set is equal to $\sup \mathbb{R}_{++} = +\infty$. Let U be a radiant set. The function μ_U defined by

$$\mu_U(x) = \inf\{\lambda > 0 : x \in \lambda U\}, \quad (x \in \mathbb{R}^n)$$

is called the Minkowski gauge of a set U. Clearly μ_U is a nonnegative positively homogeneous function. The following proposition holds (see, for example, [6])

Proposition 3.1. *Let $\mathcal{P}^+$ be the set of all nonnegative lower semicontinuous positively homogeneous functions defined on $\mathbb{R}^n$ and $\mathcal{U}^+$ be the set of all closed radiant sets. Then the mapping φ defined on $\mathcal{P}^+$ by*

$$\varphi(p) = \{x \in \mathbb{R}^n : p(x) \le 1\} \tag{3.1}$$

is a one-to-one mapping of $\mathcal{P}^+$ onto $\mathcal{U}^+$. The inverse mapping $\varphi^{-1} : \mathcal{U}^+ \to \mathcal{P}^+$ has the following form: $\varphi^{-1}(U) = \mu_U$.

Remark 3.1. Let p be a lower semicontinuous nonnegative positively homogeneous function. Then the set $U = \{x \in \mathbb{R}^n : p(x) \le 1\}$ is nonempty if and only if $p(0) \ne +\infty$, which is equivalent to $0 \in \operatorname{dom} p$. Indeed, if $p(0) \ne +\infty$

then $p(0) = 0$ so U is nonempty. If U is nonempty, then due to the positive homogeneity and lower semicontinuity of p, we have $p(0) = 0$.

Due to Proposition 3.1 we can easily verify that a nonnegative lower semicontinuous IPH function can be described as the Minkowski gauge of a closed downward radiant set.

Proposition 3.2. *Let $\mathcal{P}_1^+$ be the set of all nonnegative lower semicontinuous IPH functions defined on $\mathbb{R}^n$ and $\mathcal{U}_1^+$ be the set of all closed downward radiant sets. Then the restriction to $\mathcal{P}_1^+$ of the mapping φ defined on by* (3.1) *is a one-to-one mapping of $\mathcal{P}_1^+$ onto $\mathcal{U}_1^+$ and the inverse mapping $\varphi^{-1} : \mathcal{U}_1^+ \to \mathcal{P}_1^+$ has the form: $\varphi^{-1}(U) = \mu_U$.*

Proof. Due to Proposition 3.1, we need to check only that the following assertions hold:

1) for an increasing function p the set $U_p = \{x \in \mathbb{R}^n : p(x) \le 1\}$ is downward. Indeed, it follows directly from the definition of a downward set.

2) for a closed downward radiant set U the function μ_U is increasing. In fact, if U is empty then $\mu_U = +\infty$, hence μ_U is increasing. Assume that U is nonempty. Let $x \ge y$ and $\lambda > 0$ be a number such that $x/\lambda \in U$. Since U is downward, it follows that $y/\lambda \in U$. Thus $\{\lambda > 0 : x \in \lambda U\} \subset \{\lambda > 0 : y \in \lambda U\}$, so $\mu_U(x) \ge \mu_U(y)$. ☐

A set $V \subset \mathbb{R}^n$ is called co-radiant if $(x \in V, \lambda > 1) \implies \lambda x \in V$. The Minkowski co-gauge ν_V of a co-radiant set V is defined by

$$\nu_V(x) = \sup\{\lambda > 0 : x \in \lambda V\}.$$

We again assume that the infimum of the empty set is equal to $+\infty$ and the supremum of the empty set is equal to zero. The following assertions hold ([6]):

Proposition 3.3. *A set U is radiant if and only if its complement $V = \mathbb{R}^n \setminus U$ is co-radiant. If U is radiant then $\mu_U = \nu_V$.*

Proposition 3.4. *Let $\mathcal{P}^-$ be the set of all nonnegative upper semicontinuous positively homogeneous functions defined on $\mathbb{R}^n$ and $\mathcal{U}^-$ be the set of all closed co-radiant sets. Then the mapping ψ defined on $\mathcal{P}^-$ by*

$$\psi(p) = \{x \in \mathbb{R}^n : p(x) \ge 1\} \tag{3.2}$$

is a one-to-one mapping of $\mathcal{P}^-$ onto $\mathcal{U}^-$. The inverse mapping $\psi^{-1} : \mathcal{U}^- \to \mathcal{P}^-$ has the following form: $\psi^{-1}(V) = \nu_V$.

Remark 3.2. Let p be an upper semicontinuous nonnegative positively homogeneous function. Then the set $V = \{x \in IR^n : p(x) \ge 1\}$ is nonempty if and only if $p \ne 0$.

Consider now a nonpositive IPH function p with $0 \in \operatorname{dom} p$. Let $V = \{x \in \mathbb{R}^n : p(x) \le -1\}$. It is easy to check that this set is downward, co-radiant and

$0 \notin V$. Let us calculate the Minkowski co-gauge of the set V:

$$\nu_V(x) = \sup\left\{\lambda > 0 : \frac{x}{\lambda} \in V\right\} = \sup\left\{\lambda > 0 : p\left(\frac{x}{\lambda}\right) \le -1\right\}$$
$$= \sup\{\lambda > 0 : -p(x) \ge \lambda\} = -p(x).$$

Using this presentation of the Minkowski co-gauge and Proposition 3.4, we can easily prove that the following assertion holds:

Proposition 3.5. *Let $\mathcal{P}_1^-$ be the set of all nonpositive lower semicontinuous IPH functions defined on $\mathbb{R}^n$ and $\mathcal{U}_1^-$ be the set of all closed downward co-radiant sets. Then the mapping ξ defined by $\xi(p) = \{x \in \mathbb{R}^n : p(x) \le -1\}$ is a one-to-one mapping of $\mathcal{P}_1^-$ onto $\mathcal{U}_1^-$. The inverse mapping $\xi^{-1} : \mathcal{U}_1^- \to \mathcal{P}_1^-$ has the following form: $\xi^{-1}(V) = -\nu_V$.*

We now examine lower semicontinuous IPH functions $p : \mathbb{R}^n \to \mathbb{R}_{+\infty} := \mathbb{R} \cup \{+\infty\}$ that are not necessary nonpositive or nonnegative. For this purpose we introduce some definitions. Let $x \in \mathbb{R}^n$ and let $R_x = \{\lambda x : \lambda > 0\}$ be the ray emanating from the origin and passing through x. Let U be a closed downward radiant set and V be a closed downward co-radiant set. Consider the sets

$$T_U = \{x \in U : R_x \not\subset U\}, \qquad Q_V = \{x \in V : R_x \cap V \ne \emptyset\}. \tag{3.3}$$

It is well-known and easy to check that $x \in T_U$ if and only if $\mu_U(x) > 0$ and that $x \in Q_U$ if and only if $\nu_V(x) > 0$. Let $p : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be a lower semicontinuous function with $0 \in \operatorname{dom} p$. Consider the sets

$$P_+ = \{x : p(x) > 0\}, \quad P_0 = \{x : p(x) = 0\}, \quad P_- = \{x : p(x) < 0\}. \tag{3.4}$$

If p is IPH and $0 \in \operatorname{dom} p$ then P_0 is nonempty, since $0 \in P_0$. Note that

$$P_+ \cap P_- = \emptyset. \tag{3.5}$$

Denote by $\mathcal{P}_1$ the set of all lower semicontinuous IPH functions $p : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ and by $\mathcal{U}_1$ the set of all pairs (U, V) such that

1) U is a closed downward radiant set;
2) V is a closed downward co-radiant set such that $R_x \not\subset V$ for all $x \in \mathbb{R}^n$;
3) the following holds:

$$T_U \cap Q_V = \emptyset. \tag{3.6}$$

Theorem 3.1. *The mapping χ_1 defined on $\mathcal{P}_1$ by $\chi(p) = (U, V)$, where*

$$U = \{x \in \mathbb{R}^n : p(x) \le 1\} \text{ and } V = \{x \in \mathbb{R}^n : p(x) \le -1\},$$

is a one-to-one mapping of $\mathcal{P}_1$ onto $\mathcal{U}_1$. The inverse mapping $\chi_1^{-1} : \mathcal{U}_1 \to \mathcal{P}_1$ has the following form: $\chi_1^{-1}(U, V) = \mu_U - \nu_V$.

Proof. Let $p \in \mathcal{P}_1$. Consider the functions $p^+(x) = \max(p(x), 0)$ and $p^-(x) = \min(p(x), 0)$. Then

$$p = p^+ + p^-, \quad p^+ \ge 0, \quad p^- \le 0. \tag{3.7}$$

Clearly p^+ and p_- are lower semicontinuous IPH functions. We have $U = \{x : p^+(x) \le 1\}$. It follows from Proposition 3.2 that U is a closed downward radiant set and $p^+ = \mu_U$. We also have $V = \{x : p^-(x) \le -1\}$. Due to Proposition 3.5 V is a closed downward co-radiant set and $p^- = -\nu_V$. Since $p(x) > -\infty$ for all x, it follows that p^- is finite, so ν_V is finite. Applying the definition of ν_V we conclude that V does not contain any ray R_x with $x \in \mathbb{R}^n$. We have

$$p = \mu_U - \nu_V. \tag{3.8}$$

Note that $\mu_U(x) > 0$ if and only if the ray R_x is not contained in the set U and that $\nu_V(x) > 0$ if and only if $R_x \cap V \ne \emptyset$. Therefore $T_U = P_+$, and $Q_V = P_-$. It follows from (3.5) that (3.6) holds. Thus $(U,V) \in \mathcal{U}_1$, so χ maps $\mathcal{P}_1$ into $\mathcal{U}_1$.

Let $(U,V) \in \mathcal{U}_1$. Then U is a closed downward radiant set, so the function μ_U is well defined; this function is nonnegative, IPH and lower semicontinuous; V is a closed downward co-radiant set and V does not contain any ray R_x with $x \in \mathbb{R}^n$. Then the function ν_V is well-defined, nonnegative, DPH, upper semicontinuous and finite. Thus the difference $p = \mu_U - \nu_V$ is well defined. Clearly, $p \in \mathcal{P}_1$. It follows from the definition of $\mathcal{U}_1$ that (3.6) holds; hence

$$\mu_U(x) > 0 \implies \nu_V(x) = 0, \qquad \nu_V(x) > 0 \implies \mu_U(x) = 0. \tag{3.9}$$

Consider the set $U' = \{x : p(x) \le 1\}$. Let $x \in U'$. If $p(x) < 0$ then $\nu_V(x) > 0$, so $\mu_U(x) = 0$. If $p(x) = 0$ then also $\mu_U(x) = 0$. If $p(x) > 0$ then $\mu_U(x) > 0$, so $\nu_V(x) = 0$, which implies $\mu_U(x) = p(x) \le 1$. Thus for all $x \in U'$ we have $\mu_U(x) \le 1$. It follows from Proposition 3.2 that $x \in U$, therefore $U' \subset U$. Assume now that $x \in U$. Then $p(x) \le \mu_U(x) \le 1$, so $x \in U'$. We have proved that $U = \{x : p(x) \le 1\}$. Consider now a point x such that $p(x) \le -1$. We have $\nu_V(x) > 0$, so $\mu_U(x) = 0$ and $p(x) = -\nu_V(x)$. Since (see Proposition 3.5) $V = \{x : \nu_V(x) \ge 1\}$, it follows that $V = \{x : p(x) \le -1\}$. We have proved that there exists $p \in \mathcal{P}_1$ such that $(U,V) = \chi(p)$. Thus χ maps $\mathcal{P}_1$ onto $\mathcal{U}_1$. It easily follows from Propositions 3.2 and 3.5 that χ is a one-to-one one-to-one mapping. □

Remark 3.3. Let $p : \mathbb{R}^n \to \mathbb{R}_{+\infty}$ be a lower semicontinuous IPH function, $U = \{x : p(x) \le 1\}$ and $V = \{x : p(x) \le -1\}$. Assume that both U and V are nonempty. We have $\bigcup_{\lambda>0} \lambda U = \operatorname{dom} p$ and $\bigcup_{\lambda>0} \lambda V = P_-$. Since

$$\bigcap_{\lambda>0} \lambda U = \{x : p(x) \le 0\},$$

it follows that

$$P_0 = \Big(\bigcap_{\lambda>0} \lambda U\Big) \setminus P_- = \Big(\bigcap_{\lambda>0} \lambda U\Big) \setminus \Big(\bigcup_{\lambda>0} \lambda V\Big).$$

Remark 3.4. A similar scheme can be applied for a primal representation of lower semicontinuous IPH functions mapping into $\mathbb{R}_{-\infty} := \mathbb{R} \cup \{-\infty\}$. Such functions p have the form $p = \mu_U - \nu_V$, where V is a closed downward co-radiant set and U is a closed downward radiant set that enjoys the following property: for each $x \in \mathbb{R}^n$ there exists $\lambda > 0$ such that $\lambda x \in U$.

We now give a primal presentation for upper semicontinuous IPH functions $p_* : \mathbb{R}^n \to \mathbb{R}_{-\infty}$. Let p_* be such a function. Then the function p defined by $p(x) = -p_*(-x)$ is a lower semicontinuous IPH function mapping into $\mathbb{R}_{+\infty}$, hence (see Theorem 3.1) $p = \mu_U - \nu_V$, where

$$U = \{x : p(x) \leq 1\} = \{x : p_*(-x) \geq -1\},$$
$$V = \{x : p(x) \leq -1\} = \{x : p_*(-x) \geq 1\},$$

and $(U, V) \in \mathcal{U}_1$. Using this construction and Theorem 3.1 we can conclude that the following assertion holds.

Theorem 3.2. *Let $\mathcal{P}_2$ be the set of all upper semicontinuous functions $p_* : \mathbb{R}^n \to \mathbb{R}_{-\infty}$. The mapping ω defined on $\mathcal{P}_2$ by $\omega(p_*) = (U, V)$, where*

$$U = \{x \in \mathbb{R}^n : p_*(-x) \geq -1\} \quad \text{and} \quad V = \{x \in \mathbb{R}^n : p^*(-x) \geq 1\},$$

is a one-to-one mapping of $\mathcal{P}_2$ onto $\mathcal{U}_1$. The inverse mapping $\omega^{-1} : \mathcal{U}_1 \to \mathcal{P}_2$ has the following form: $\omega^{-1}(U, V) = p_*$, where $p_*(x) = -\mu_U(-x) + \nu_V(-x)$.

Let $q : \mathbb{R}^n \to \bar{\mathbb{R}}$ be a DPH function. Then the functions $p(x) = -q(x)$ and $p_*(x) = q(-x)$ are IPH. Applying the results obtained for IPH functions we can easily give a representation of lower semicontinuous and upper semicontinuous DPH functions through the Minkowski gauges and the Minkowski co-gauges of suitable sets.

4. Abstract convexity of IPH functions

We need the following definitions [6]. A set $K \subset \mathbb{R}^n$ is called conic if $\lambda K \subset K$ for all λ. Let L be the set of positively homogeneous functions $l : K \to \bar{\mathbb{R}}$ defined on a conic set K. A function $f : K \to \bar{\mathbb{R}}$ is called abstract convex with respect to L, or L-convex if $f(x) = \sup\{l(x) : l \in \operatorname{supp}(f, L)\}$, where

$$\operatorname{supp}(f, L) = \{l \in L : l(x) \leq f(x) \ (x \in K)\} \tag{4.1}$$

is the set of all L-minorants of f. Since the functions $l \in L$ are positively homogeneous, it follows that each L- convex function is positively homogeneous, too. Let f be an L-convex function. A set $\partial_L f(x) = \{l \in \operatorname{supp}(f, L) : l(x) = f(x)\}$ is called the L-subdifferential of the function f. Clearly $\partial_L f(x)$ is nonempty if and only if $f(x) = \max\{l(x) : l \in \operatorname{supp}(f, L)\}$.

Abstract concavity with respect to L can be defined in a similar manner. A function $f : K \to \bar{\mathbb{R}}$ is called abstract concave with respect to L, or L-concave, if $f(x) = \inf\{l(x) : l \in \operatorname{supp}^+(f, L)\}$, where

$$\operatorname{supp}^+(f, L) = \{l \in L : \ (\forall\, x \in K) \ l(x) \geq f(x)\}$$

is the set of all L-majorants of f.

In the study of abstract convexity of IPH functions we need a certain notation. Each vector $l \in \mathbb{R}^n$ generates the following sets of indices

(4.2)
$$I_+(l) = \{i \in I : l_i > 0\}, \qquad I_0(l) = \{i \in I : l_i = 0\}, \quad I_-(l) = \{i \in I : l_i < 0\}$$

(recall that $I = \{1, 2, \dots, n\}$). Let $l \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Denote by c/l the vector with coordinates

$$\left(\frac{c}{l}\right)_i = \begin{cases} \dfrac{c}{l_i} & i \notin I_0, \\ 0 & i \in I_0. \end{cases}$$

First we recall some results related to IPH functions defined on $\mathbb{R}^n_+$. For each $l \in \mathbb{R}^n_+$ consider the function φ_l defined on $\mathbb{R}^n_+$ by

$$\varphi_l(x) = \min_{i \notin I_0(l)} l_i x_i, \quad (l \neq 0), \qquad \varphi_l(x) = 0, \quad (l = 0). \tag{4.3}$$

We shall call φ_l the min-type function corresponding to l. Let $L = \{\varphi_l : l \in \mathbb{R}^n_+\}$. The following result holds ([6]).

Proposition 4.1. *Let $p : \mathbb{R}^n_+ \to [0, +\infty)$ be an IPH function. Then for each $x \in \mathbb{R}^n_+$ the L-subdifferential $\partial_L p(x)$ is nonempty. If $p(x) > 0$ then $p(x)/x \in \partial_L p(x)$.*

Corollary 4.1. *A lower semicontinuous function $p : \mathbb{R}^n_+ \to [0, +\infty]$ is L-convex if and only if p is IPH.*

Proof. Each L-convex function is IPH, as the upper envelope of IPH functions. On the other hand, applying Proposition 4.1 we conclude that every IPH finite-valued function $p : \mathbb{R}^n_+ \to [0, +\infty)$ is L-convex. Consider now an IPH function $p : \mathbb{R}^n_+ \to [0, +\infty]$. It is easy to check that p is the upper envelope of a family of finite-valued IPH functions (for the proof of a similar result see Proposition 4.6), hence p is L-convex as the upper envelope of a family of L-convex functions. $\square$

The question arises whether it is possible to generalize Corollary 4.1 for IPH functions defined on $\mathbb{R}^n$ and mapping into $\mathbb{R}_{+\infty}$. More precisely, for each $l \in \mathbb{R}^n_+$ consider the function $\tilde{\varphi}_l$ defined on $\mathbb{R}^n$ by

$$\tilde{\varphi}_l = \min_{i \notin I_0(l)} l_i x_i, \quad (l \neq 0), \qquad \tilde{\varphi}_l = 0, \quad (l = 0). \tag{4.4}$$

Let $\tilde{L} = \{\tilde{\varphi}_l : l \in \mathbb{R}^n_+\}$. Then each L-convex function is IPH. Does the reverse assertion hold at least for finite-valued functions? The answer is negative. Moreover, the answer is negative in the following much more general situation.

Let $\mathcal{L}^+_n$ be a set of all functions ℓ of the form

$$\ell(x) = \min_{i=1,\dots,k} [l_i, x], \quad l_i \in \mathbb{R}^n_+, \ i = 1, \dots, k, \quad k \leq n \tag{4.5}$$

Here $[u, v]$ stands for the inner product of vectors u and v. Clearly each $\mathcal{L}^+_n$-convex function is IPH and does not attain the value $-\infty$. Note that $\tilde{L} \subset \mathcal{L}^+_n$.

Proposition 4.2. *There exists a continuous IPH function p defined on $\mathbb{R}^n$ and mapping into $\mathbb{R}$ such that the support set $supp(p, \mathcal{L}^+_n)$ is empty.*

Proof. Let $p : \mathbb{R}^n \to \mathbb{R}$ be a superlinear (concave positively homogeneous) function. The function p is continuous. It is well known that there exists a unique convex compact set U such that $p(x) = \inf_{u \in U}[u, x]$. The function p is increasing (hence IPH) if and only if $U \subset \mathbb{R}^n_+$. Consider now a function $\ell \in \mathcal{L}^+_n$. Then there exists a positive integer $k \le n$ such that $\ell(x) = \min_{i=1,\dots,k}[l_i, x]$, where $l_i \in \mathbb{R}^n_+$, $i = 1, \dots, k$. This function is superlinear and $\ell(x) = \min_{u \in U_\ell}[u, x]$, where U_ℓ is the convex hull of n vectors $l_1, \dots, l_n$. It is well-known (and easy to check) that the inequality $\ell(x) \le p(x)$ for all $x \in \mathbb{R}^n$ is equivalent to the inclusion $U_\ell \supseteq U$. Note that the convex hull U_ℓ of n points in the n-dimensional space has an empty interior. Hence if the interior of the set U is nonempty, then the inclusion $U_\ell \supseteq U$ is not possible. $\square$

Min-type functions are very simple and convenient. In order to apply these functions to the study of IPH functions defined on $\mathbb{R}^n$ we shall consider their restriction to certain cones. We begin with the definition of these cones.

For each $l \in \mathbb{R}^n$ consider the set

$$K_l = \{y \in \mathbb{R}^n : y_i \ge 0 \ (i \in I_0(l));\ \max_{i \in I_-(l)} l_i y_i \le \min_{i \in I+(l)} l_i y_i\}. \tag{4.6}$$

Since we consider functions mapping into the real line, we assume that the maximum over the empty set is equal to $\inf \mathbb{R} = -\infty$ and the minimum over the empty set is equal to $\sup \mathbb{R} = +\infty$.

Due to the equality $\max_{i \in I_-(l)} l_i y_i = -\min_{i \in I} |l_i| y_i$, we have

$$K_l = \{y \in \mathbb{R}^n : y_i \ge 0 \ (i \in I_0(l));\ \min_{i \in I_+(l)} |l_i| y_i + \min_{i \in I_-(l)} |l_i| y_i \ge 0\}. \tag{4.7}$$

Consider also the sets

$$K_l^+ = \{y \in K_l :\ (\forall i \in I_+(l))\ y_i \ge 0\}; \tag{4.8}$$

$$K_l^- = \{y \in K_l :\ (\exists i \in I_+(l))\ y_i \le 0\}. \tag{4.9}$$

Let us describe some properties of the sets K_l, K_l^+ and K_l^-.

1) K_l is a closed convex cone. This fact easily follows from the sublinearity of the function $x \mapsto \max\limits_{i \in I_-(l)} l_i y_i$ and the superlinearity of the function $x \mapsto \min\limits_{i \in I_+(l)} l_i y_i$.
2) K_l^+ is a closed convex cone; K_l^- is a closed cone, which is the union over the finite set I_l^+ of convex closed cones $(K_l^-)_i = \{y \in K_l : y_i \le 0\}$;
3) $K_l^+ \cup K_l^- = K^l$; $\quad K_l^+ \cap K_l^- = \{y \in K_l :\ (\forall\, i \in I_+)\, y_i \ge 0\ (\exists\, i \in I_+)\, y_i = 0\}$.
4) If $l \in \mathbb{R}^n_+$ then $K_l = \{y \in \mathbb{R}^n :\ (\forall i \in I_0(l)))\ y_i \ge 0\}$ and $K_l^+ = \mathbb{R}^n_+$. Indeed for $l \in \mathbb{R}^n_+$ we have $I_-(l) = \emptyset$, so $\max\limits_{i \in I_-(l)} l_i y_i = -\infty$. In particular if $l \gg 0$ then $K_l = \mathbb{R}^n$.
5) If $l \in \mathbb{R}^n_-$ then $K_l = \{y \in \mathbb{R}^n :\ (\forall i \in I_0)\ y_i \ge 0\}$. We also have $K_l^+ = K_l$. If $l \ll 0$ then $K_l = K_l^+ = \mathbb{R}^n$.

6) The set $\{y : y \geq 1/l\} = (1/l) + \mathbb{R}^n_+$ is contained in K^+_l. Indeed if $y \geq 1/l$ then $y_i \geq 0$ for $i \in I_0(l) \cup I_+(l)$ and $\max_{i \in I_-(l)} l_i y_i \leq 1 \leq \min_{i \in I+(l)} l_i y_i$.
7) The set $\{y : y \geq (-1)/l\} = (1/l) + \mathbb{R}^n_+$ is contained in K_l. In fact if $y \geq (-1)/l$ then $\max_{i \in I_-(l)} l_i y_i \leq -1 \leq \min_{i \in I_+(l)} l_i y_i$ and $y_i \geq 0$ for $i \in I_0(l)$.

The following property of the cone K^l_+ will be very important in the sequel.

Proposition 4.3. *For each $l \in \mathbb{R}^n$ the cone K^+_l is upward, that is, $(x \in K^+_l, x' \geq x) \implies x' \in K^+_l$.*

Proof. It follows directly from (4.7) and (4.8). □

Lemma 4.1. *Let $x = 1/l$, where $l \notin \mathbb{R}^n_-$. Then*

$$K_l = \Big\{ y \in \mathbb{R}^n : y \geq \Big(\min_{i \in I_+(x)} \frac{y_i}{x_i} \Big) x \Big\}.$$

Proof. Since $l \notin \mathbb{R}^n_-$, it follows that $I_+(l) \neq \emptyset$. We have $I_+(l) = I_+(x)$, $I_0(l) = I_0(x)$, $I_-(l) = I_-(x)$. Let us verify that for all $y \in K_l$ the inequality

$$y \geq \Big(\min_{i \in I_+(x)} \frac{y_i}{x_i} \Big) x \tag{4.10}$$

holds. Let $j \in I_+(x)$. Then $x_j > 0$ and $y_j/x_j \geq \min_{i \in I_+(x)} (y_i/x_i)$. So for each $y \in \mathbb{R}^n$:

$$y_j = x_j \frac{y_j}{x_j} \geq x_j \min_{i \in I_+(x)} \frac{y_i}{x_i}, \qquad j \in I_+(x). \tag{4.11}$$

We also have for $y \in K_l$:

$$y_j \geq 0 = x_j = x_j \min_{i \in I_+(x)} \frac{y_j}{x_j}, \qquad j \in I_0(x). \tag{4.12}$$

Assume now that $j \in I_-(x)$. Since $y \in K_l$, it follows that

$$\frac{y_j}{x_j} \leq \max_{i \in I_-(x)} \frac{y_i}{x_i} \leq \min_{i \in I_+(x)} \frac{y_i}{x_i}.$$

Since $x_i < 0$, we have

$$y_j = x_j \frac{y_j}{x_j} \geq x_j \min_{i \in I_+(x)} \frac{y_i}{x_i}. \tag{4.13}$$

The inequality (4.10) follows directly from (4.11), (4.12) and (4.13).

Consider now a vector $y \in \mathbb{R}^n$ such that (4.10) holds. Clearly $y_i \geq 0$ for $i \in I_0(x)$. Let $I_-(x)$ be nonempty and $i \in I_-(x)$. Then $l_i y_i = y_i/x_i \leq \min_{i \in I_+(x)} l_i y_i$, so

$$\max_{i \in I_-(x)} l_i y_i \leq \min_{i \in I_+(l)} l_i y_i.$$

We have proved that $y \in K_l$. □

Corollary 4.2. *Let $l = 1/x$. Then*

$$K_l^+ = \{y \in \mathbb{R}^n : y \geq \Big(\min_{i \in I_+(x)} \frac{y_i}{x_i}\Big)x, \ \min_{i \in I_+(x)} \frac{y_i}{x_i} \geq 0\}.$$

Lemma 4.2. *Let $l = (-1)/x$ and $I_+(l) \neq \emptyset$. Then*

$$K_l^- = \Big\{y \in \mathbb{R}^n : y \geq \Big(\max_{i \in I_-(x)} \frac{y_i}{x_i}\Big)x, \ \max_{i \in I_-(x)} \frac{y_i}{x_i} \geq 0\Big\}.$$

Proof. We have $I_+(l) = I_-(x)$, $I_0(l) = I_0(x)$ and $I_-(l) = I_+(x)$. Let $y \in K_l^-$. Then

$$\max_{i \in I_-(x)} \frac{y_i}{x_i} = \max_{i \in I_+(l)} (-l_i)y_i = -\min_{i \in I_+(l)} l_i y_i \geq 0.$$

Let us check that

$$y \geq \Big(\max_{i \in I_-(x)} \frac{y_i}{x_i}\Big)x. \tag{4.14}$$

If $j \in I_-(x)$ then $y_j = x_j(y_j/x_j) \geq x_j \max\limits_{i \in I_-(x)} (y_i/x_i)$. The same inequality trivially holds for $j \in I_0(x)$. Let $j \in I_+(x)$. Since $y \in K_l^-$, it follows that

$$\frac{y_j}{x_j} \geq \min_{i \in I+(x)} \frac{y_i}{x_i} \geq \min_{i \in I_-(x)} \frac{y_i}{x_i}.$$

Thus

$$y_j = x_j \frac{y_j}{x_j} \geq x_j \min_{i \in I_-(x)} \frac{y_i}{x_i}.$$

We have checked (4.14).

Consider now a vector y such that $c \equiv \max\limits_{i \in I_-(x)} (y_i/x_i) \geq 0$ and $y \geq cx$. Clearly $y_i \geq 0$ for $i \in I_0(x)$. Let $j \in I_+(x)$. Then $y_j/x_j \geq c = \max\limits_{i \in I_-(x)} (y_i/x_i)$, so

$$\max_{i \in I_-(l)} l_i y_i = -\min_{j \in I_+(x)} \frac{y_j}{x_j} \leq -\max_{i \in I_-(x)} \frac{y_i}{x_i} = \min_{i \in I_+(l)} l_i y_i.$$

Thus $y \in K_l$. Since $\min\limits_{i \in I_+(l)} l_i y_i = -c \leq 0$, it follows that $y \in K_l^-$. $\square$

Each vector $l \in \mathbb{R}^n$ with nonempty $I_+(l)$ generates the following functions:

$$g_l^+(x) = \begin{cases} \min\limits_{i \in I_+(l)} l_i x_i & x \in K_l^+ \\ -\infty & x \not\in K_l^+ \end{cases} \tag{4.15}$$

$$g_l^-(x) = \begin{cases} \min\limits_{i \in I_+(l)} l_i x_i & x \in K_l^- \\ -\infty & x \not\in K_l^- \end{cases} \tag{4.16}$$

Remark 4.1. The function g_l^+ is IPH for each $l \in \mathbb{R}^n$ with nonempty $I_+(l)$. This follows easily from Proposition 4.3. The functions g_l^- are not IPH.

The following simple statement will be useful in the sequel.

Proposition 4.4. *Let $p : \mathbb{R}^n \to \bar{\mathbb{R}}$ be an IPH function and $x \in \operatorname{dom} p$ be a point such that $p(x) \neq 0$. Then $l = (p(x)/x) \notin \mathbb{R}^n_-$.*

Proof. If $x \geq 0$ then $p(x) \geq 0$, so $l \geq 0$. If $x \leq 0$ then $p(x) \leq 0$, so again $l \geq 0$. If there exist indices i_1 and i_2 such that $x_{i_1} < 0$ and $x_{i_2} > 0$ then $I_+(l)$ is again nonempty. $\square$

Let G be the set of all functions of the form (4.15) or (4.16) with $l \in (\mathbb{R}^n \setminus \mathbb{R}^n_-)$. We also add the functions g_0 and $g_{-\infty}$ to the set G. By definition $g_0(x) = 0$ and $g_{-\infty}(x) = -\infty$ for all $x \in \mathbb{R}^n$.

Theorem 4.1. *Let p be an IPH function and $x \in \operatorname{dom} p$ be a point such that $p(x) \neq 0$. Let $l = p(x)/x$. Then the subdifferential $\partial_G p(x)$ is nonempty and the following statements hold:*

1) *If $p(x) > 0$ then $g_l^+ \in \partial p_G(x)$.*
2) *If $p(x) < 0$ then $g_l^- \in \partial p_G(x)$.*

Proof. It follows from Proposition 4.4 that $I_+(l) \neq \emptyset$.

1) Let $p(x) > 0$. In this case $I_+(l) = I_+(x), I_0(l) = I_0(x)$ and $I_-(l) = I_-(x)$. The cone K_l^+ can be represented in the following form:

$$K_l^+ = \{y \in \mathbb{R}^n : \ (\forall\, (i \in I_+(x) \cup I_0(x)) y_i \geq 0; \quad \max_{i \in I_-(x)} \frac{y_i}{x_i} \leq \min_{i \in I_+(x)} \frac{y_i}{x_i}\}.$$

Let $y \in K_l^+$. Applying the monotonicity of p and Lemma 4.1 (with x replaced by $\dfrac{x}{p(x)}$), we can conclude that $p(y) \geq p(cx)$, where $c = \min\limits_{i \in I_+(x)} (y_i/x_i)$. It follows from the definition of K_l^+ that $y_i \geq 0$ for $i \in I_+(x)$, so $c \geq 0$. The positive homogeneity of p implies

$$p(y) \geq p(cx) = cp(x) = p(x) \min_{i \in I_+(x)} \frac{y_i}{x_i} = \min_{i \in I_+(x)} \frac{p(x)}{x_i} y_i = \min_{i \in I_+(l)} l_i y_i.$$

It follows from this inequality and (4.15) that $g_l^+(y) \leq p(y)$ for all $y \in \mathbb{R}^n$. Since $x \in K_l^+$, we have $g_l^+(x) = \min\limits_{i \in I_+(x)} l_i x_i = p(x)$. Thus $g_l^+ \in \partial_G p(x)$.

2) Let $p(x) < 0$. If $p(x) = -\infty$ then $g_{-\infty} \in \partial_G(x)$. Assume that $p(x) > -\infty$. In this case we have $I_+(l) = I_-(x), I_0(l) = I_0(x), I_-(l) = I_+(x)$. We also have

$$K_l^- = \{y : \ (\forall i \in I_0(x)) \ y_i \geq 0;$$
$$\max_{i \in I_+(x)} \frac{p(x)}{x_i} y_i \leq \min_{i \in I_-(x)} \frac{p(x)}{x_i} y_i, \quad \min_{i \in I_-(x)} \frac{p(x)}{x_i} y_i \leq 0\}.$$

Since $p(x) < 0$, it follows that the following inequalities hold for $y \in K_l^-$:

$$\min_{i \in I_+(x)} \frac{y_i}{x_i} \geq \max_{i \in I_-(x)} \frac{y_i}{x_i}; \qquad \max_{i \in I_-(x)} \frac{y_i}{x_i} \geq 0. \tag{4.17}$$

Let $y \in K_l^-$. Since p is increasing it follows from Lemma 4.2 (with x replaced by $\frac{x}{|p(x)|}$) that $p(y) \geq p(c(y)x)$, with $c(y) = \max_{i \in I_-(x)} y_i/x_i \geq 0$. Hence

$$p(y) \geq c(y)p(x) = p(x) \max_{i \in I_-(x)} \frac{y_i}{x_i} = \min_{i \in I_+(l)} l_i y_i. \tag{4.18}$$

Thus $p(y) \geq g_l^-(y)$ for all $y \in \mathbb{R}^n$. Let $y = x$. We have

$$\min_{i \in I_+(l)} l_i x_i = p(x) < 0.$$

So $x \in K_l^-$. Thus $g_l^-(x) = \min_{i \in I_+(l)} l_i x_i = p(x)$ □

Let p be an IPH function. The following example demonstrates that the subdifferential $\partial_G p(x)$ may be empty if $p(x) = 0$.

Example 4.1. Let $n = 2$ and

$$p(x) = \begin{cases} -\sqrt{x_1 x_2} & x \in \mathbb{R}^2_-, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\partial p_G p(-1, 0) = \emptyset$.

Indeed, assume on the contrary that there exists $g \in \partial_G p(-1, 0)$. First assume that $g = g_l^-$ for a certain $l \notin \mathbb{R}^2_-$. If $l = (l_1, l_2)$, with $l_1 > 0$ and $l_2 > 0$ then, due to Lemma 4.2, we have $(-1, 0) \in K_l^-$. So

$$0 = p(-1, 0) = g_l^-(-1, 0) = \min(-l_1, 0),$$

and hence $l_1 \leq 0$, which is a contradiction. Thus at least one of the numbers l_1, l_2 is nonnegative. Assume now that $l_1 > 0$ and $l_2 = 0$. Then $K_l^- = \{y : y_2 \geq 0, y_1 \leq 0\}$. Since $(-1, 0) \in K_l^-$, it follows that $g_l^-(-1, 0) = -l_1 < 0$, and hence $g_l^-(-1, 0) \neq p(-1, 0)$. If $l_1 = 0$ and $l_2 > 0$ then $K_l^- = \{y : y_1 \geq 0, y_2 \leq 0\}$. Since $(-1, 0) \notin K_l^-$, it follows that $g_l^- = -\infty \neq p(-1, 0)$. Thus l has at least one negative coordinate. If $l_1 > 0$ and $l_2 <$, then $K_l^- = \{y : l_2 y_2 \leq l_1 y_1, y_1 \leq 0\}$. If $l_1 < 0$, $l_2 > 0$ then $K_l^- = \{y : l_1 y_1 \leq l_2 y_2, y_2 \leq 0\}$. Since $(-1, 0) \notin K_l^-$ in both cases, it follows that $g_l^- \notin \partial_G p(-1, 0)$ for all $l \notin \mathbb{R}^2_-$.

We now demonstrate that the subdifferential $\partial_G p(-1, 0)$ does not contain functions of the form g_l^+ with $l \notin \mathbb{R}^2_-$. We have $g_l^+(x) > 0 = p(x)$ for all $x \gg 0$, so $g_l^+ \notin \text{supp}(p, G)$ for all such l.

We shall now discuss abstract convexity of IPH functions with respect to the set G. Note that the functions in G are not lower semicontinuous, so the upper envelope of a subset of G is not necessary a lower semicontinuous function.

Proposition 4.5. *Let p be an IPH function and $x \in dom\, p$, $p(x) \neq 0$. Then $p(x) = \max\{g(x) : g \in G\}$.*

Proof. It follows directly from Theorem 4.1. □

Proposition 4.6. *Let $p : \mathbb{R}^n \to \bar{\mathbb{R}}$ be an IPH function and $y \notin dom\, p$. Then $p(y) = \sup\{g(y) : g \in supp(p, G)\}$.*

Proof. Since $y \notin \operatorname{dom} p$, it follows that either $p(y) = -\infty$ or $p(y) = +\infty$. If $p(y) = -\infty$ then $p(y) = g_{-\infty}(y) = \sup\{g(y) : g \in \operatorname{supp}(p, G)\}$. Assume $p(y) = +\infty$. Let $\bar{p}$ be a finite IPH function such that $\bar{p}(y) > 0$ (such a function exists). Let $p_n(x) = \inf(p(x), n\bar{p}(x))$. It follows from Proposition 4.5 that $p_n(y) = \max\{g(y) : g \in G, g \le p_n\}$. We also have $p(y) = \sup_n p_n(y)$. Hence, $p(y) = \sup\{g(y) : g \in G, g \le p\}$. □

Thus the equality $p(x) = \sup\{g(x) : g \in \operatorname{supp}(p, G)\}$ holds at all points x such that $p(x) \neq 0$. We now give a simple sufficient condition, which guarantees that this equality holds also at points x such that $p(x) = 0$.

Let $y \in \mathbb{R}^n, I_+(y) \neq \emptyset$. For each $\delta > 0$ consider the element $y_\delta = y - \delta\mathbf{1}$, where $\mathbf{1} = (1, \ldots, 1)$.

Proposition 4.7. *Let $p : \mathbb{R}^n \to \bar{\mathbb{R}}$ be an IPH function and $y \notin \mathbb{R}^n_-$ be a point such that $p(y) = 0$ and p is lower semicontinuous at y. Assume that $p(y_\delta) < 0$ for all $\delta > 0$. Then $p(y) = \sup\{g(y) : g \in supp(p, G)\}$.*

Proof. Since p is lower semicontinuous at the point y we have $\liminf_{\delta \to +0} p(y_\delta) \ge 0$. Due to the inequality $p(y_\delta) < 0$, we can conclude that $\lim_{\delta \to +0} p(y_\delta) = 0$. Let $l_\delta = p(y_\delta)/y_\delta$. Since $p(y_\delta) < 0$, it follows that $g^-_{l_\delta} \in \partial_G p(y_\delta)$, that is, $g^-_{l_\delta}(x) \le p(x)$ for all $x \in \mathbb{R}^n$, and $g^-_{l_\delta}(y_\delta) = p(y_\delta)$. Since $p(y_\delta) \to 0 = p(y)$, the result follows. □

Remark 4.2. Using the functions $p_*(x) = -p(-x)$, we can give a representation of upper semicontinuous IPH functions as lower envelopes of a certain set of functions that are of max-type on the cones K^+_l or K^-_l and identically $+\infty$ on their respective complements. Using the functions $q(x) = p(-x)$ and $q_*(x) = -q(x)$, we can give a representation of lower semicontinuous (upper semicontinuous, respectively) DPH functions.

5. Abstract convexity and concavity of nonnegative IPH and DPH functions

For nonnegative IPH and DPH functions representations by means of restrictions of min-type functions or max-type functions are much simpler. It follows from Theorem 4.1 that we do not need functions of the form (4.16) in the study of nonnegative functions. Also we can consider the functions

$$h_l(x) = \begin{cases} \min_{i \in I_+(l)} l_i x_i & x \in K^+_l \\ 0 & x \notin K^+_l \end{cases} \tag{5.1}$$

instead of the functions g^+_l defined by (4.15), for l with nonempty $I_+(l)$. Let H be the set of all functions of the form (5.1) with $l \in \mathbb{R}^n \setminus \mathbb{R}^n_-$. We also add the function $g_0 \equiv 0$ to the set H. Since K^+_l is an upward set (see Proposition 4.3), it follows that H consists of nonnegative IPH functions, hence each H-convex function is IPH.

Proposition 5.1. *Let p be a nonnegative IPH function. Then $\partial_H p(y)$ is nonempty for all $y \in \mathbb{R}^n$.*

Proof. If $p(y) > 0$, we can use the same argument as in the proof of the first part of Theorem 4.1. If $p(y) = 0$ then $g_0 \in \partial p(y)$, so $\partial p(y) \neq \emptyset$. □

Let

$$h'_l(x) = \begin{cases} -\max_{i \in I_+(l)} l_i x_i & x \in K_l^+ \\ 0 & x \notin K_l^+ \end{cases} \tag{5.2}$$

and $H' = \{h'_l : l \in \mathbb{R}^n \setminus \mathbb{R}^n_-\} \cup \{g_0\}$.

Considering the functions $q(x) = p(-x)$, we can conclude that the following proposition holds:

Proposition 5.2. *Let q be a nonnegative DPH function. Then $\partial_{H'} q(y)$ is nonempty for all $y \in \mathbb{R}^n$.*

Corollary 5.1. *A nonnegative function p is H-convex if and only if p is IPH. A nonnegative function q is H'-convex if and only if q is DPH.*

Proof. It follows directly from Proposition 5.1 and from Proposition 5.2, respectively. □

We now give an infimal representation of DPH nonnegative functions and a supremal representation of IPH nonpositive functions. Let us begin with decreasing functions. For each $l \in \mathbb{R}^n \setminus \mathbb{R}^n_+$ consider the function s_l defined on $\mathbb{R}^n$ by

$$s_l(x) = \begin{cases} (\max_{i \in I_-(l)} l_i x_i)^+ & x \in K_l^+ \\ +\infty & \text{otherwise}, \end{cases} \tag{5.3}$$

with the convention that the maximum over the empty set is equal to $-\infty$. Here $a^+ = \max(a, 0)$ is the positive part of a number a. Let $S = \{s_l : l \in \mathbb{R}^n \setminus \mathbb{R}^n_+\} \cup \{g_{+\infty}\}$, where $g_{+\infty}(x) = +\infty$ for all $x \in \mathbb{R}^n$.

Theorem 5.1. *A function $q : \mathbb{R}^n \to \mathbb{R}$ is nonnegative and DPH if and only if it is abstract concave with respect to S, that is, there exists a nonempty set $U \subset \mathbb{R}^n \setminus \mathbb{R}^n_+$ such that*

$$q(x) = \inf_{l \in U} s_l(x) \qquad x \in \mathbb{R}^n. \tag{5.4}$$

If $q \neq 0$ then the infimum in (5.4) is attained if either $q(x) > 0$ or $x \in \mathbb{R}^n_+$.

Proof. **If** For each $l \in \mathbb{R}^n \setminus \mathbb{R}^n_+$ the function s_l is nonnegative. Since the cone K_l^+ is upward (see Proposition 4.3), it follows that s_l is DPH, so each S-concave function is nonnegative and DPH.

Only if Let q be a nonnegative DPH function and let $U = \text{supp}^+(q, S)$ be the upper support set of q with respect to S, that is, $U = \{s \in S : (\forall x \in \mathbb{R}^n)\ s(x) \geq q(x)\}$. Consider separately three cases.

1) First assume that $q \neq 0$ and consider a point $x \in \mathbb{R}^n$ such that $q(x) > 0$. Since q is decreasing and $q(0) = 0$, it follows that $x \notin \mathbb{R}^n_+$, so $I_-(x) \neq \emptyset$. Let $l = q(x)/x$, then also $l \notin \mathbb{R}^n_+$ and $I_-(l) = I_-(x)$.

Let us check that $s_l \in U$, that is $s_l(y) \geq q(y)$ for all $y \in K_l^+$. It is enough to consider points $y \in K_l^+$ such that $q(y) > 0$. Let $I_-^y = \{i \in I_-(l) : y_i < 0\}$. If $I_-^y = \emptyset$ then $y_i \geq 0$ for all $i \in I_-(l)$. Since $y \in K_l^+$, it follows that also $y_i \geq 0$ for all $i \in I_0(l) \cup I_+(l)$. We have $y \geq 0$, which is impossible due to the inequality $q(y) > 0$. Thus I_-^y is nonempty. We have

$$\max_{i \in I_-(l)} l_i y_i = \max_{i \in I_-^y} l_i y_i = \max_{i \in I_-^y} \frac{q(x)}{x_i} y_i = q(x) \max_{i \in I_-^y} \frac{y_i}{x_i} = q\Big(\Big(\max_{i \in I_-^y} \frac{y_i}{x_i}\Big) x\Big). \tag{5.5}$$

We now check that

$$\left(\max_{i \in I_-^y} \frac{y_i}{x_i}\right) x \leq y, \tag{5.6}$$

which is equivalent to

$$\Big(\max_{i \in I_-^y} \frac{y_i}{x_i}\Big) x_j \leq y_j, \qquad j = 1, \dots, n. \tag{5.7}$$

Indeed, if $j \in I_-^y$, this inequality is obviously satisfied. If $j \in I_-(l) \setminus I_-^y$, it also holds, as the left hand side is nonpositive and the right hand side is nonnegative. The same situation occurs if $j \in I_0(l)$. Assume now that $j \in I_+(l)$, that is $x_j > 0$. Since $y \in K_l^+$ and $l = q(x)/x$, we have

$$\max_{i \in I_-(x)} \frac{y_i}{x_i} \leq \min_{i \in I_+(x)} \frac{y_i}{x_i},$$

so

$$\Big(\max_{i \in I_-^y} \frac{y_i}{x_i}\Big) x_j \leq \Big(\max_{i \in I_(l)} \frac{y_i}{x_i}\Big) x_j \leq \frac{y_j}{x_j} x_j = y_j.$$

Thus (5.6) has been verified. Applying (5.6) and the monotonicity of q, we conclude that

$$q\Big(\Big(\max_{i \in I_-^y} \frac{y_i}{x_i}\Big) x\Big) \geq q(y). \tag{5.8}$$

It follows from (5.5) and (5.8) that

$$s_l(y) = (\max_{i \in I_-(l)} l_i y_i)^+ = \max_{i \in I_-(l)} l_i y_i \geq q(y).$$

We have proved that $s_l \in U$. Since $x \in K_l^+$, we have

$$s_l(x) = \max_{i \in I_-(l)} l_i x_i = p(x). \tag{5.9}$$

It follows from (5.9) that

$$q(x) = \min_{s \in U} s(x).$$

2) Consider now a nonnegative decreasing function $q \neq 0$ and let $x \in \mathbb{R}^n_+$. Then $q(x) = 0$. It follows from 1) than the set $U = \operatorname{supp}^+(q, S)$ is nonempty. Since $s_l(x) = 0$ for all $l \in \mathbb{R}^n \setminus \mathbb{R}^n_+$, it follows that $q(x) = \min_{s\in U} s(x)$.

3) Consider now an arbitrary nonnegative decreasing function q and let $x \in \mathbb{R}^n \setminus \mathbb{R}^n_+$ be a point such that $q(x) = 0$. Observe that $q = \inf\limits_{\varepsilon>0} q_\varepsilon$, with $q_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ defined by

$$q_\varepsilon(y) = q(y) - \varepsilon(\min_{i=1,\dots,n} y_i)^-. \qquad (y \in \mathbb{R}^n).$$

Here $a^- = \min(a, 0)$ is the negative part of the number a. Since each q_ε is nonnegative, DPH and strictly positive on $\mathbb{R}^n \setminus \mathbb{R}^n_+$, from the first part of the proof it follows that there exists a nonempty set $U_\varepsilon \subset \mathbb{R}^n \setminus \mathbb{R}^n_+$ such that $q_\varepsilon(x) = \min_{s\in U_\varepsilon} s(x)$. Hence $q(x) = \inf\limits_{s\in U} s(x)$, where $U = \bigcup\limits_{\varepsilon>0} U_\varepsilon$. □

Let $l \in \mathbb{R}^n \setminus \mathbb{R}^n_+$. Consider the function s_l defined by (5.3). Then

$$-s_l(-x) = \begin{cases} (\min_{i\in I_-(l)} l_i x_i)^- & -x \in K_l^+ \\ -\infty & \text{otherwise.} \end{cases} \tag{5.10}$$

Let $s'_l(x) = -s_l(-x)$. Denote by S' the set of all the functions s'_l, i.e. the set of functions defined by (5.10), with $l \in \mathbb{R}^n \setminus \mathbb{R}^n_+$, together with $g_{-\infty}$, where $g_{-\infty}(x) = -\infty$ for all $x \in \mathbb{R}^n$.

Remark 5.1. Let p be a nonpositive IPH function. Then the function $q(x) = -p(-x)$ is nonnegative IPH, therefore by Theorem 5.1 there exists a set $U \subset \mathbb{R}^n \setminus \mathbb{R}^n_+$ such that (5.4) holds. We have

$$p(x) = -q(-x) = -\inf_{l\in U} s_l(-x) = \sup_{l\in U}(-s_l(-x)) = \sup_{l\in U} s'_l(x).$$

Thus each nonpositive IPH function is S'-convex. Clearly the inverse assertion also holds: each S'-convex function is nonpositive IPH. Applying Theorem 5.1 we can assert that the S'-subdifferential $\partial_{S'} p(x)$ is nonempty if $p(x) < 0$ or $x \in \mathbb{R}^n_+$.

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