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ON PROPER EFFICIENCIES IN LOCALLY CONVEX SPACES–A SURVEY

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Dedicated to Pham Huu Sach on the occasion of his sixtieth birthday

Abstract. In this paper, we consider the main definitions of proper efficiency in locally convex topological vector spaces and examine the relationships among them.

1. Introduction

One important problem in vector optimization theory is to identify the efficient points of a set (for an interesting survey on vector optimization, see for instance [1], [2]. Various restrictions on efficient points have been suggested in order to eliminate "improper" efficient points, and allow more satisfactory characterization of the proper efficient points. The original concept of proper efficiency was introduced by Kuhn-Tucker [3], Hurwciz [4], Geoffrion [5], and modified and formulated in a more general framework by Borwein [6], Benson [7], Borwein [8], Henig [9], Borwein and Zhuang [10], and others [11-19]. All definitions of proper efficiencies are rather close to each other. They partly coincide. However, exact comparisons are significant. This has been done by many authors (see [2-19]). In particular, Guerraggio, Molho and Zaffaroni [13] give a detailed and comprehensive comparison among the main definitions of proper efficiency in normed spaces. In [14], Makarov and Rachkovski present a unified form of some proper efficiencies based on the notion of a dilating cone. This new form enables them to obtain a comparison among these proper efficiencies in a normed space. In [15], Zheng also presented an investigation on the relationships among several kinds of proper efficiencies in locally convex spaces. In this paper, we make a survey on a number of definitions of proper efficiency in locally convex spaces and examine the relationships among these efficiencies.

2. Preliminaries and Definitions

Throughout this paper, we assume that X is a locally convex topological vector space (in brief, LCS) with the topological dual space X^*, $A \subset X$ is a nonempty

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subset of X, and $S \subset X$ is a closed convex pointed cone. The dual cone of a cone $S \subset X$ is the set

$$S^+ = \{f \in X^* : f(x) \geq 0, \forall x \in S\},$$

and the strict polar of S is the set

$$S^{+i} = \{f \in X^* : f(x) > 0, \text{ for all } x \in S \setminus \{0\}\}.$$

A convex subset Θ of the cone S is said to be a base for S if $0 \notin \mathrm{cl}\Theta$ and $S = \mathrm{cone}(\Theta) = \{t\theta | t \geq 0,\ \theta \in \Theta\}$. Denoted by $B(S)$ the set of all the bases of S.

A point $x \in A$ is said to be a point of continuity, written as $x \in \mathrm{PC}(A)$, if for every $\{x_\alpha\}_{\alpha \in I} \subset A, x_\alpha \xrightarrow{w} x$ implies $x_\alpha \to x$.

It has proved (see [15]) that if S has a bounded base, then $0 \in \mathrm{PC}(S)$, but the converse is not generally true.

Definition 2.1. A point $x \in A$ is said to be an efficient point of A with respect to S, written as $x \in \mathrm{E}(A, S)$, if $(A - x) \cap (-S) = \{0\}$. If a cone S is not pointed, then $x \in \mathrm{E}(A, S)$ means that $(A - x) \cap (-S) \subseteq S$.

Definition 2.2. A point $x \in A$ is said to be a positive proper efficient point of A with respect to S, written as $x \in \mathrm{Pos}(A, S)$, if there exists $f \in S^{+i}$ such that $f(x) = \inf\{f(a) | a \in A\}$.

Definition 2.3. [4] A point $x \in A$ is said to be a Hurwicz proper efficient point of A with respect to S, written as $x \in \mathrm{Hu}(A, S)$, if $\mathrm{clconvcone}((A - x) \cup S) \cap (-S) = \{0\}$.

Let Θ be a base of S. Then $0 \notin \mathrm{cl}\Theta$. By a standard separation theorem, there is $f_\Theta \in X^* \setminus \{0\}$ such that $\alpha = \inf\{f_\Theta(\theta) | \theta \in \Theta\} > f_\Theta(0) = 0$. Let $V_\Theta = \{x \in X || f_\Theta(x)| < \alpha/2\}$. It is clear that V_Θ is a neighborhood of 0. For each convex neighborhood V of 0 with $V \subset V_\Theta$, $\Theta + V$ is convex and $0 \notin \mathrm{cl}(\Theta + V)$. Let $S_V(\Theta) = \mathrm{cone}(\Theta + V)$.

Definition 2.4. [15] and [17] A point $x \in A$ is said to be a Henig proper efficient point of A with respect to $\Theta \in B(S)$, written as $x \in \mathrm{HE}(A, \Theta)$, if there is a convex neighborhood V of 0 with $V \subset V_\Theta$ such that $\mathrm{clcone}(A - x) \cap (-S_V(\Theta)) = \{0\}$.

A point $x \in A$ is said to be a Henig proper efficient point [resp. generalized Henig proper efficient point] of A with respect to S, written as $x \in \mathrm{HE}(A, S)$ [resp. $x \in \mathrm{GHE}(A, S)$], if $x \in \cap_{\Theta \in B(S)} \mathrm{HE}(A, \Theta)$ [resp. $x \in \cup_{\Theta \in B(S)} \mathrm{HE}(A, \Theta)$].

It is clear that $x \in \mathrm{HE}(A, \Theta)$ iff there exists a convex neighborhood V of 0 with $V \subset V_\Theta$ such that $x \in \mathrm{E}(A, S_V(\Theta))$, and $\mathrm{HE}(A, S) \subset \mathrm{HE}(A, \Theta) \subset \mathrm{GHE}(A, S)$. It has been proved (see [15]) that if Θ is bounded, then $\mathrm{HE}(A, S) = \mathrm{H}(A, \Theta)$.

Definition 2.5. [9] A point $x \in A$ is said to be a globally Henig proper efficient point of A with respect to S, written as $x \in \mathrm{GHe}(A, S)$, if there is a convex cone $S' \neq X$ with $S \setminus \{0\} \subset \mathrm{int}S'$ such that $x \in \mathrm{E}(A, S')$.

Definition 2.6. [7] A point $x \in A$ is said to be a Benson proper efficient point of A with respect to S, written as $x \in \mathrm{Be}(A, S)$ if

$$\mathrm{clcone}(A + S - x) \cap (-S) = \{0\}.$$

Definition 2.7. [8] A point $x \in A$ is said to be a global Borwein proper efficient point of A with respect to S, written as $x \in \mathrm{GBo}(A, S)$, if

$$\mathrm{clcone}(A - x) \cap (-S) = \{0\}.$$

Definition 2.8. [10] A point $x \in A$ is said to be a superefficient point of A with respect to S, written as $x \in \mathrm{SE}(A, S)$, if for each neighborhood V of 0, there is a neighborhood U of 0 such that

$$\mathrm{clcone}(A - x) \cap (U - S) \subset V.$$

Definition 2.9. [19] A point $x \in A$ is said to be a strict efficient point of A with respect to S, written as $x \in \mathrm{Str}(A, S)$, if for each neighborhood V of 0, there is a neighborhood U of 0 such that

$$(A - x) \cap (U - S) \subset V.$$

It is clear that $\mathrm{SE}(A, S) \subset \mathrm{Str}(A, S)$.

This notion was also introduced by Zheng (see [15]) (called strong efficient point).

Definition 2.10. The Bouligand tangent (in short, B-tangent) cone to A at a point $x \in \mathrm{cl}A$ is the set

$$T(A, x) = \{y : y = \lim_{n\to\infty} \lambda_n(x_n - x), \{\lambda_n\} \subseteq R^+, \{x_n\} \subseteq A, \lim_{n\to\infty} x_n = x\}.$$

$T(A, S)$ is closed when X is normed space, but not necessarily closed in a general locally convex space.

Definition 2.11. [6] A point $x \in A$ is said to be a Borwein proper efficient point, written as $x \in \mathrm{Bo}(A, S)$, if $x \in \mathrm{E}(A, S)$ and

$$\mathrm{cl}T(A + S, x) \cap (-S) = \{0\}.$$

Definition 2.12. [9] A point $x \in A$ is said to be locally Henig proper efficient point, written as $x \in \mathrm{LHe}(A, S)$, if for every neighborhood V of 0, there is a convex cone $S^{'} \neq X$ with $S \setminus \{0\} \subset \mathrm{int}S^{'}$ such that

$$x \in E((A + S) \cap (x + V), S^{'}).$$

Definition 2.13. [13] A point $x \in A$ is said to be a properly efficient point, written as $x \in \mathrm{LBo}(A, S)$, if $x \in \mathrm{E}(A, S)$ and $\mathrm{cl}T(A, x) \cap (-S) = \{0\}$.

3. Main Results

In this section, we shall compare the above efficiencies in locally convex spaces setting.

Theorem 3.1. (see [4, 12]) $\mathrm{Pos}(A, S) \subseteq \mathrm{Hu}(A, S)$.

When X is a separable normed space, Hurwicz (see [4]) proved the converse inclusion, i.e. $\mathrm{Hu}(A, S) \subset \mathrm{Pos}(A, S)$. *This inclusion was also proved in [12] when X is a noremed space and S has a weakly compact base, and was proved in [13] when X is locally convex space and S has a compact base.*

We shall prove that this inclusion remains true in a locally convex space when S has a weakly compact base.

Lemma 3.1. (see [18]) *Let X be a locally convex spaces, let P and C be two cones in X, and $P \cap C = \{0\}$. Assume that one of the following conditions holds.*

(a) *P is weakly closed and C has a weakly compact base;*

(b) *P is closed and C has a compact base.*

Then there exists a pointed convex cone S such that $C \setminus \{0\} \subset \mathrm{int}S$ and $P \cap C = \{0\}$. Furthermore, if P is convex, there is an element $f \in C^{+i}$ such that $-f \in P^{+}$.

Theorem 3.2. *If S has a weakly compact base, then* $\mathrm{Hu}(A, S) \subset \mathrm{Pos}(A, S)$.

Proof. If $x \in \mathrm{Hu}(A, S)$, then

$$\mathrm{clconvcone}((A - x) \cup S) \cap (-S) = \{0\}.$$

From Lemma 3.1, there is an element $f \in S^{+i}$ such that $f \in (\mathrm{clconvcone}((A - x) \cup S))^{+}$. Hence $f(a - x) \geq 0,$ for all $a \in A$. Therefore $x \in \mathrm{Pos}(A, S)$. $\square$

Lemma 3.2. (see [22]) *Let A be a convex subset of X and $x \in \mathrm{cl}A$. Then clcone$(A - x)$ is closed convex cone.*

Theorem 3.3. *If S has a base, then*

(i) $\mathrm{Pos}(A, S) \subset \mathrm{GHE}(A, S)$.

(ii) $\mathrm{Pos}(A, S) = \mathrm{GHE}(A, S)$, *whenever cl$(A + S)$ is convex.*

Proof. (i) For the proof see [15]. (ii) For every $x \in \mathrm{GHE}(A, S)$, there is $\Theta \in B(S)$ and an open convex neighborhood V of 0 with $V \subset V_\Theta$ such that

$$\mathrm{cone}(A - x) \cap (-S_V(\Theta)) = \{0\}.$$

Since $S \setminus \{0\} \subset S_V(\Theta)$, we have

$$\mathrm{clcone}(A + S - x) \cap (-\mathrm{int}S_V(\Theta)) = \emptyset.$$

Since $\mathrm{cl}(A + S)$ is convex, we have $\mathrm{clcone}(A + S - x) = \mathrm{clcone}(\mathrm{cl}(A + S) - x)$ is convex too. By a standard separation theorem, there is $f \in X^* \setminus \{0\}$ such that

$$\inf\{f(x) | x \in \mathrm{int}S_V(\Theta)\} > \alpha \geq \sup\{f(x) | x \in -\mathrm{cone}(A + S - x)\}.$$

Since $\mathrm{cone}(A+S-x)$ is a cone, we may take $\alpha = 0$. Hence

$$f(x) \leq f(a) \text{ for all } a \in A.$$

Since $\Theta \subset \mathrm{int}S_V(\Theta)$ and $f \neq 0$, we have $f(\theta) > 0$ for all $\theta \in \Theta$. Hence $f \in S^{+i}$. It follows that $x \in \mathrm{Pos}(A,S)$. Thus, $\mathrm{GHE}(A,S) \subset \mathrm{Pos}(A,S)$. $\square$

When A is a convex subset of X, $\mathrm{Pos}(A,S) = \mathrm{GHE}(A,S)$ was proved in [15].

Theorem 3.4. *If S has a base, then* $\mathrm{GHE}(A,S) \subset \mathrm{GHe}(A,S)$.

Proof. For each $x \in \mathrm{GHE}(A,S)$, there is $\Theta \in B(S)$ with $x \in \mathrm{HE}(A,\Theta)$. Hence there is $V \subset V_\Theta$ such that

$$\mathrm{clcone}(A-x) \cap (-S_V(\Theta)) = \{0\}.$$

This implies that

$$(A-x) \cap (-S_V(\Theta)) = \{0\}.$$

i.e. $x \in \mathrm{GHe}(A,S)$. $\square$

Theorem 3.5. *If S has a base, then*

(i) $\mathrm{Pos}(A,S) \subset \mathrm{GHe}(A,S)$.

(ii) $\mathrm{Pos}(A,S) = \mathrm{GHe}(A,S)$, *whenever* $\mathrm{cl}(A+S)$ *is convex.*

Proof. (i) From Theorem 3.3 and Theorem 3.4,

$$\mathrm{Pos}(A,S) \subset \mathrm{GHE}(A,S) \subset \mathrm{GHe}(A,S).$$

(ii) For each $x \in \mathrm{GHe}(A,S)$, there is convex cone S' with $S \setminus \{0\} \subset \mathrm{int}S'$ such that

$$(A-x) \cap (-S') \subset S'.$$

Since $S \setminus \{0\} \subset \mathrm{int}S'$, we have

$$\mathrm{cl}(A+S-x) \cap (-\mathrm{int}S') = \emptyset.$$

Since $\mathrm{cl}(A+S)$ is convex, by the separation theorem, there is $f \in (S')^+ \setminus \{0\}$ such that

$$f(x) \leq f(a), \text{ forall } a \in A.$$

Since $S \setminus \{0\} \subset \mathrm{int}S'$, we have $f \in S^{+i}$. Hence $x \in \mathrm{Pos}(A,S)$. $\square$

When X is a normed linear space, one can see a proof of (i) in [13]. From Theorem 3.3 and Theorem 3.5 we see that $\mathrm{GHE}(A,S) = \mathrm{GHe}(A,S)$ whenever S has a base and $\mathrm{cl}(A+S)$ is convex. We do not know whether this equality holds without the condition that $\mathrm{cl}(A+S)$ is convex.

Example 3.1. (see [23]) Let $X = R^2$, $S = R^2_+$, and

$$\begin{aligned} A &= \{(x,y) \mid -2 \leq x \leq -1, y \geq -\sqrt{-x^2-2x}\} \\ &\cup \{(x,y) \mid -1 < x \leq 0, y \geq -1-\sqrt{1-x^2}\} \setminus \{(0,0)\}. \end{aligned}$$

Then

$$\begin{aligned} \mathrm{E}(A,S) &= \{(x,y) \mid -2 \leq x \leq -1, y = -\sqrt{-x^2-2x}\} \\ &\cup \{(x,y) \mid -1 < x \leq 0, y = -1-\sqrt{1-x^2}\}, \end{aligned}$$

$$\mathrm{GHE}(A,S) = \mathrm{GHe}(A,S) = \mathrm{E}(A,S) \setminus \{(-1,-1),(-2,0),(0,-2)\},$$

$$\mathrm{Pos}(A,S) = \mathrm{GHe}(A,S) \setminus \{(x,y) \in \mathrm{E}(A,S) \mid -1-\sqrt{2}/2 \leq x \leq -\sqrt{2}/2\}.$$

Theorem 3.6. *(see [16]) Let A be a nonempty weakly compact subset of X and S a closed convex cone with base Θ. If $0 \in \mathrm{PC}(S)$, then $\mathrm{E}(A,S) \subseteq \mathrm{cl}(\mathrm{HE}(A,\Theta))$.*

Theorem 3.7. *(see [15]) If S has a base, then*

(i) $\mathrm{SE}(A,S) \subset \mathrm{HE}(A,S)$.

(ii) $\mathrm{SE}(A,S) = \mathrm{HE}(A,S)$, *whenever S has a bounded base.*

Example 3.2. Consider Example 3.1 again. Then

$$\mathrm{SE}(A,S) = \mathrm{HE}(A,S) = \mathrm{GHE}(A,S) = \mathrm{E}(A,S) \setminus \{(-1,-1),(-2,0),(0,-2)\}.$$

Theorem 3.8. $\mathrm{GHe}(A,S) \subset \mathrm{Be}(A,S)$.

Proof. For each $x \in \mathrm{GHe}(A,S)$. there is convex cone S' and $S \setminus \{0\} \subset \mathrm{int}S'$ such that

$$(A-x) \cap (-S') \subset S'.$$

Since $S \setminus \{0\} \subset \mathrm{int}S'$, we have

$$(A+S-x) \cap (-S') \subset S'.$$

Hence

$$\mathrm{cone}(A+S-x) \cap (-\mathrm{int}S') = \emptyset.$$

This implies

$$\mathrm{clcone}(A+S-x) \cap (-\mathrm{int}S') = \emptyset.$$

Thus

$$\mathrm{clcone}(A+S-x) \cap (-S) = \{0\},$$

i.e. $x \in \mathrm{Be}(A,S)$. Thus, $\mathrm{GHe}(A,S) \subset \mathrm{Be}(A,S)$. ☐

Theorem 3.9. (i) $\mathrm{Be}(A,S) \subset \mathrm{GBo}(A,S)$.

(ii) $\mathrm{Be}(A,S) = \mathrm{GBo}(A,S)$, *whenever S has a weakly compact base and A is closure convex.*

Proof. (i) It follows immediately from the definitions.

(ii) Let $x \in \text{GBo}(A, S)$. Then $\text{clcone}(A-x) \cap (-S) = \{0\}$, and then $\text{clcone}(A-x) \cap (-\Theta) = \emptyset$. Since $\bar{A}$ is convex, we have

$$\text{clcone}(A - x) = \text{clcone}(\bar{A} - x)$$

is convex. Hence it is also weakly closed. Noticing that Θ is weakly compact, there is an open weakly neighborhood V of 0 such that

$$(\text{clcone}(A - x) + V) \cap (V - \Theta) = \emptyset.$$

Without loss of generality, we may assume that V is balanced. Assume $x \notin \text{Be}(A, S)$, there is $s \in S \setminus \{0\}$ such that $-s \in \text{clcone}(A+S-x)$. Since $S = \text{cone}\Theta$, we have $s = \lambda\theta$ with $\lambda \geq 0$ and $\theta \in \Theta$. Without loss of generality, we may assume $s \in \Theta$(take $\lambda = 1$). This implies there is $x^{'} \in A, \lambda > 0, \eta \geq 0, \theta \in \Theta$, such that

$$\lambda(x^{'} + \eta\theta - x) \in -s + V.$$

Hence

$$\frac{\lambda}{1+\lambda\eta}(x^{'} - x) \in \{\frac{\lambda\eta}{1+\lambda\eta}(-\theta) + \frac{1}{1+\lambda\eta}(-s)\} + \frac{1}{1+\lambda\eta}V,$$

and hence

$$\frac{\lambda}{1+\lambda\eta}(x^{'} - x) \in (-\Theta) + \frac{1}{1+\lambda\eta}V \subset -\Theta + V.$$

This contradicts $\text{clcone}(A - x) \cap (V - \Theta) = \emptyset$. Therefore, $x \in \text{Be}(A, S)$. □

Dauer and Saleh (see [18]) has proved $\text{Be}(A, S) = \text{GBo}(A, S)$ when S has a compact base.

Example 3.3. (see [18]) Let $X = l^p, 1 \leq p \leq \infty$ S be the nonnegative orthant of X, and $A = \{a_n = (a_n^{(1)}, a_n^{(2)}, \dots) \in l^p, n \in N\}$, where $a_1 = 0$ and for $n \geq 2$ we take $a_n^{(1)} = a_n^{(n)} = -1/n$, $a_n^{(n+1)} = 1/n^2$, and otherwise $a_n^{(i)} = 0$. Then $0 \notin \text{Be}(A, S)$, $0 \in \text{GBo}(A, S)$.

Theorem 3.10. (i) $\text{Hu}(A, S) \subseteq \text{Be}(A, S)$*;*

(ii) $\text{Hu}(A, S) = \text{Be}(A, S)$*, whenever* $\text{cl}(A + S)$ *is convex.*

Proof. (i) It follows immediately from the definitions.

(ii) Since $\text{cl}(A + S)$ is convex, by Lemma 3.2

$$\text{clcone}(A + S - x) = \text{clcone}(\text{cl}(A + S) - x)$$

is convex too. Hence

$$\text{clconvcone}((A - x) \cup S) \subset \text{clcone}(A + S - x)$$

and hence $\text{Be}(A, S) \subset \text{Hu}(A, S)$. □

Khanh (see [14]) showed that $\text{Hu}(A, S) = \text{Be}(A, S)$ in normed spaces when $A + S$ is convex.

Theorem 3.11. *If S has a base, then*

(i) $\mathrm{SE}(A,S) \subseteq \mathrm{HE}(A,S) \subseteq \mathrm{GHE}(A,S) \subseteq \mathrm{GHe}(A,S) \subseteq \mathrm{Be}(A,S) \subseteq \mathrm{GBo}(A,S)$.

(ii) $\mathrm{SE}(A,S) = \mathrm{GBo}(A,S)$, *whenever S has a weakly compact base and the closure A is convex.*

Proof. (i) It is clear from Theorem 3.7, Theorem 3.4, Theorem 3.8 and Theorem 3.9.

(ii) Assume $\mathrm{GBo}(A,S) \not\subset \mathrm{SE}(A,S)$. Then there is $x \in \mathrm{GBo}(A,S)$ and $x \notin \mathrm{SE}(A,S)$; i.e. there exists a neighborhood V of 0 such that for every neighborhood U of 0,

$$\mathrm{cone}(A-x) \cap (U-S) \not\subset V.$$

Hence there is $t_u > 0, a_u \in A, x_u \in U, \lambda_u \geq 0, \theta_u \in \Theta$ such that

$$t_u(a_u - x) = x_u - \lambda_u \theta_u \notin V.$$

It is clear that $\{x_u\}_{u\in U} \to 0$. Assume $\{\lambda_u\}$ is bounded, without loss of generality, we may assume that $\lambda_u \to \lambda > 0$. So $\frac{x_u}{\lambda_u} \to 0$ (when $\{\lambda_u\}$ is unbounded, the above limits holds). Since Θ is weakly compact, without loss of generality, we may assume that $\theta_u \xrightarrow{w} \theta \in \Theta$. Hence

$$\frac{t_u(a_u - x)}{\lambda_u} = \frac{x_u}{\lambda_u} - \theta_u \xrightarrow{w} -\theta \in -S.$$

Since A is closure convex and X is a locally convex space, we have $\mathrm{clcone}(A-x) = \mathrm{clcone}(\bar{A}-x)$ is weakly closed. Hence $-\theta \in \mathrm{clcone}(A-x)$. This implies that

$$\mathrm{clcone}(A-x) \cap (-S) \neq \{0\}.$$

This contradicts $x \in \mathrm{GBo}(A,S)$. Therefore, $\mathrm{GBo}(A,S) \subset \mathrm{SE}(A,S)$. $\square$

When A is convex and S has a weakly compact base, Zheng (see [15]) has proved $\mathrm{SE}(A,S) = \mathrm{GBo}(A,S)$.

Corollary 3.1. *If S has a weakly compact base,* $\mathrm{cl}A$ *is convex, then*

$$\begin{aligned} \mathrm{SE}(A,S) &= \mathrm{HE}(A,S) = \mathrm{GHE}(A,S) = \mathrm{GHe}(A,S) \\ &= \mathrm{Be}(A,S) = \mathrm{GBo}(A,S) = \mathrm{Pos}(A,S) = \mathrm{Hu}(A,S). \end{aligned}$$

Theorem 3.12. *(see [19])*

(i) $\mathrm{SE}(A,S) \subseteq \mathrm{Str}(A,S) \subseteq \mathrm{E}(A,S)$.

(ii) $\mathrm{Str}(A,S) = \mathrm{E}(A,S)$, *whenever* $0 \in \mathrm{PC}(S)$ *and A is weakly compact.*

Proof. (i) It follows immediately from the definitions.

(ii) For each $x \in \mathrm{E}(A,S)$, we suppose that $x \notin \mathrm{Str}(A,S)$. Then there exists a neighborhood V of 0 and for each a neighborhood U of 0, we have

$$(A-x) \cap (U-S) \not\subset V.$$

There is $\{x_v\}$ such that $x_v \in (A-x)\cap(U-S)$ and $x_v \notin V$. So there is $\{a_v\} \subset A$ and $\{u_v\} \subset U$ and $\{s_v\} \subset S$ such that

$$x_v = a_v - x = u_v - s_v.$$

We obtain $s_v = u_v - x_v = u_v - a_v + x$. Since A is weakly compact and $x \in \mathrm{E}(A,S)$, the net a_v tends weakly to some $a \in A$ and $a = x$. Since $0 \in \mathrm{PC}(S)$ and $s_v = u_v - a_v + x \xrightarrow{w} 0$, we have $s_v \to 0$. Hence $x_v \to 0$. This contradicts $x_v \notin V$. Therefore, $x \in \mathrm{Str}(A,S)$. □

Zheng (see [15]) proved (ii) whenever S has a bounded base.

Theorem 3.13. $\mathrm{GHe}(A,S) \subset \mathrm{LHe}(A,S)$.

Proof. For each $x \in \mathrm{GHe}(A,S)$, there exists convex cone S' such that $S \setminus \{0\} \subset \mathrm{int}S'$ and $x \in \mathrm{E}(A,S')$. Hence $x \in E(A+S,S')$. For every neighborhood V of 0, one has

$$((A+S)\cap(x+V)-x)\cap(-S') \subset S'.$$

Thus, $x \in \mathrm{LHe}(A,S)$. □

Theorem 3.14. *If* $\mathrm{cl}(A+S)$ *is convex and* S *has a weakly compact base, then* $\mathrm{Be}(A,S) \subset \mathrm{LHe}(A,S)$.

Proof. For each $x \in \mathrm{Be}(A,S)$, one has

$$\mathrm{clcone}(A+S-x)\cap(-S) = \{0\}.$$

Since $\mathrm{cl}(A+S)$ is convex and S has a weakly compact base, by Lemma 3.1, there exists a pointed convex cone S' such that $S \setminus \{0\} \subset \mathrm{int}S'$ and

$$\mathrm{clcone}(A+S-x)\cap(-S') = \{0\}.$$

It follows that $x \in \mathrm{LHe}(A,S)$. □

Theorem 3.15. $\mathrm{LHe}(A,S) \subset \mathrm{Bo}(A,S)$.

Proof. Let $x \in \mathrm{LHe}(A,S)\setminus \mathrm{Bo}(A,S)$. Since $x \in \mathrm{E}(A,S)$, we have $\mathrm{cl}T(A+S,x)\cap(-S) \neq \{0\}$, i.e., there is $s_0 \neq 0$ such that

$$\mathrm{cl}T(A+S,x)\cap(-S) = \{-s_0\}.$$

Since $x \in \mathrm{LHe}(A,S)$, for every neighborhood V of 0, there is a convex cone S' with $S\setminus\{0\} \subset \mathrm{int}S'$ such that $((A+S)\cap(x+V)-x)\cap(-S') \subset S'$. Hence

$$\mathrm{cl}T(A+S,x)\cap(-\mathrm{int}S') \neq \emptyset$$

and hence

$$T(A+S,x)\cap(-\mathrm{int}S') \neq \emptyset.$$

Thus there exist $\{\lambda_n\} \subseteq R^+$, $\{x_n\} \subseteq A+S$ with $\lim\limits_{n\to\infty} x_n = x$ such that $\lim\limits_{n\to\infty} \lambda_n(x_n - x) \in -\mathrm{int}S'$. There exists $N > 0$ large enough such that

$$\lambda_N(x_N - x) \in -\mathrm{int}S' \tag{3.1}$$

and

$$(x_N - x) \in -\mathrm{int}S'. \tag{3.2}$$

Thus

$$(x_N - x) \in (-\mathrm{int}S') \cap ((A+S) \cap (x+V) - x) \not\subset S'.$$

This contradicts $x \in \mathrm{LHe(A,S)}$. □

Lemma 3.3. *(see [21]) Let $x \in A$. Then $T(A,x) \subset \mathrm{clcone}(A-x)$. If A is starshape at x, then $\mathrm{cone}(A-x) \subset T(A,x)$. Consequently, $\mathrm{cl}T(A,x) = \mathrm{clcone}(A-x)$.*

Theorem 3.16. *(see [19])*

(i) $\mathrm{Be}(A,S) \subset \mathrm{Bo}(A,S)$.

(ii) $\mathrm{Be}(A,S) = \mathrm{Bo}(A,S)$, *whenever* $\mathrm{cl}(A+S)$ *is convex.*

As a consequence of Theorem 3.9 and Theorem 3.16, we obtain

Corollary 3.2. *If* $\mathrm{cl}(A+S)$ *is convex, then* $\mathrm{Bo}(A,S) \subset \mathrm{GBo}(A,S)$.

As a consequence of Theorem 3.14 and Theorem 3.16, we obtain:

Corollary 3.3. *If* $\mathrm{cl}(A+S)$ *is convex and S has a weakly compact base, then* $\mathrm{Bo}(A,S) \subset \mathrm{LHe}(A,S)$.

Guerraggio, Molho and Zaffaroni (see [13]) proved $\mathrm{Bo}(A,S) \subset \mathrm{LHe}(A,S)$ in normed space whenever S has a compact base.

Theorem 3.17. (i) $\mathrm{GBo}(A,S) \subset \mathrm{LBo}(A,S)$.

(ii) $\mathrm{GBo}(A,S) = \mathrm{LBo}(A,S)$, *whenever* $\mathrm{cl}A$ *is convex.*

Proof. (i) By Lemma 3.3

$$\mathrm{cl}T(A,x) \subset \mathrm{clcone}(A-x).$$

Hence $\mathrm{GBo}(A,S) \subset \mathrm{LBo}(A,S)$.

(ii) Since $\mathrm{cl}A$ is convex, by Lemma 3.3,

$$\mathrm{cl}T(A,x) = \mathrm{clcone}(A-x).$$

If $x \in \mathrm{LBo}(A,S)$, then

$$\mathrm{cl}T(A,x) \cap (-S) = \{0\}.$$

Hence

$$\mathrm{clcone}(A-x) \cap (-S) = \{0\}.$$

i.e. $x \in \mathrm{GBo}(A,S)$. □

From Corollary 3.1 and Theorem 3.17, we have

Corollary 3.4. *If* $\mathrm{cl}A$ *is convex and* S *has a weakly compact base, then* $\mathrm{LBo}(A,S) = \mathrm{Hu}(A,S) = \mathrm{Pos}(A,S)$.

When X *is a normed space and* S *has a compact base,* $\mathrm{LBo}(A,S) \subset \mathrm{Pos}(A,S)$ *was proved in [13].*

In Fig. 1, we give a inclusion structure, where the relations among various proper efficiencies will be made clear. The symbol "$\longrightarrow$" denotes the usual inclusion relation between sets.

H_1: S has a base;

H_2: S has a bounded base;

H_3: S has a weakly compact base;

H_4: cl(A+S) is convex;

H_5: clA is convex.

Fig. 1 The relationships among the proper efficiencies

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