ACTA MATHEMATICA VIETNAMICA
Volume 26, Number 3, 2001, pp. 257-268

SECOND-ORDER OPTIMALITY CONDITIONS FOR C[1] MULTIOBJECTIVE PROGRAMMING PROBLEMS

A. GUERRAGGIO, D. T. LUC AND N. B. MINH

Dedicated to Pham Huu Sach on the occasion of his sixtieth birthday

Abstract. In this paper we use approximate Hessian matrices of continuously differentiable vector functions to establish second order optimality conditions for constrained multiobjective programming problems with data of class C^1.

1. Introduction

One of the most important topics of multiobjective programming is to find optimality conditions for efficient solutions of problems with data in a possibly larger class of functions. Today there exists a huge number of papers (see [5, 9, 19, 24]) that deal with first order optimality conditions, starting with the pioneering work [17] by Kuhn-Tucker, who studied continuously differentiable problems with inequality constraints. Quite few publications exist on second order conditions, among which we cite the papers [1, 2, 3, 6, 7, 18, 27] for problems with C^2 and $C^{1,1}$ data, and [25, 26] for problems involving set valued data.

The purpose of the present note is to establish second order optimality conditions for constrained multiobjective programming problems with continuously differentiable data or C^1 data for short. The main tool we are going to exploit is approximate Hessian of continuously differentiable functions and its recession matrices. The notions of approximate Jacobian and approximate Hessian have been introduced and studied by Jeyakumar and Luc in [11]. Further developments and applications of these concepts are found in [8, 11, 12, 13, 14, 15, 21, 28]. It is important to notice that several known second order generalized derivatives of continuously differentiable functions are examples of approximate Hessian, including the Clarke generalized Hessian. Moreover, a $C^{1,1}$ function, i. e. function with locally Lipschitz gradient map may have an approximate Hessian whose closed convex hull is strictly smaller than the Clarke generalized Hessian. Therefore the optimality conditions we are going to establish by means of approximate

Received Marh 7, 2001.
1991 *Mathematics Subject Classification.* 90C29, 90C30.
Key words and phrases. Efficient solution, second order optimality condition, approximate Hessian.

Hessian are not only valid for $C^{1,1}$ problems when the Clarke generalized Hessian is used, but sometimes also yield a sharper result.

The paper is organized as follows. In section 2 we recall the definition of approximate Jacobian and approximate Hessian, and some elementary calculus rules that will be needed in the sequel. Section 3 and Section 4 are devoted to second order necessary conditions and sufficient conditions respectively. In the final section an example is given to illustrate our approach.

2. Approximate Hessian matrices

Let f be a continuous function from R^n to R^m. A closed set of $(m\times n)$-matrices $\partial f(x) \subseteq L(R^n, R^m)$ is said to be an *approximate Jacobian* of f at x if for every $u \in R^n$ and $v \in R^m$, one has

$$(vf)^+(x,u) \leq \sup_{M\in\partial f(x)} \langle v, M(u)\rangle,$$

where vf is the real function $\sum_{i=1}^{n} v_i f_i$, here $v_1, \dots, v_m$ are components of v and $f_1, \dots, f_m$ are components of f, and $(vf)^+(x,u)$ is the upper Dini directional derivative of the function vf at x in the direction u, that is

$$(vf)^+(x,u) := \limsup_{t\downarrow 0} \frac{(vf)(x+tu) - (vf)(x)}{t}.$$

If for every $x \in R^n$, $\partial f(x)$ is an approximate Jacobian of f at x, then the set valued map $x \mapsto \partial f(x)$ from R^n to $L(R^n, R^m)$ is called an approximate Jacobian map of f. We refer the interested reader to [11, 12, 13, 14, 15, 28] for more about properties and applications of approximate Jacobian matrices.

Now let $f : R^n \to R^m$ be continuously differentiable. The Jacobian matrix map $\bigtriangledown f$ is a continuous vector function from R^n to the space of $m \times n$-matrices $L(R^n, R^m)$. The space $L(R^n, R^m)$ is equipped with the euclidean norm

$$\|M\| := (\|M_1\|^2 + ... + \|M_n\|^2)^{1/2},$$

where $M_1, \dots, M_n$ are columns of the matrix M. This norm is equivalent to the operator norm

$$\|M\| = \max_{u\in R^n, \|u\|\leq 1} \|M(u)\|.$$

A closed set of three dimensional $m \times n \times n$- matrices $\partial^2 f(x)$ is said to be an approximate Hessian of f at x if it is an approximate Jacobian of the function $\bigtriangledown f$ at x. Approximate Hessian shares many properties of approximate Jacobian. Let us list some of them for the reader's convenience (see [11] for the proof):

(i) If $\partial^2 f(x)$ is an approximate Hessian of f at x, then every closed set of $m \times n \times n$-matrices which contains $\partial^2 f(x)$, is an approximate Hessian of f at x;

(ii) If f is twice Gâteaux differentiable at x, then every approximate Hessian $\partial^2 f(x)$ of f at x contains the second Gâteaux derivative of f at x in its closed convex hull $\overline{co}\partial^2 f(x)$. Moreover, f is twice Gâteaux differentiable at x if and only if it admits a singleton approximate Hessian at this point.

(iii) If $f, g : R^n \to R^m$ are continuously differentiable and if $\partial^2 f(x)$ and $\partial^2 g(x)$ are approximate Hessians of f and g at x respectively, then the closure of the set $\partial^2 f(x) + \partial^2 g(x)$ is an approximate Hessian of $f + g$ at x.

(iv) Generalized Taylor expansion: Let $f : R^n \to R^m$ be continuously differentiable and let $x, y \in R^n$. Suppose that for each $z \in [x, y], \partial^2 f(z)$ is an approximate Hessian of f at z. Then

$$f(y) - f(x) - \nabla f(x)(y - x) \in \frac{1}{2}\overline{co}(\partial^2 f[x, y](y - x, y - x)).$$

We shall need some more terminologies. Let $A \subseteq R^n$ be a nonempty set. The recession cone of A, which is denoted by A_∞, consists of all limits $\lim_{i\to\infty} t_i a_i$ where $a_i \in A$ and $\{t_i\}$ is a sequence of positive numbers converging to 0. It is useful to notice that a set is bounded if and only if its recession cone is trivial. Elements of the recession cone of approximate Hessian $\partial^2 f(x)$ will be called recession Hessian matrices.

Let $F : R^n \Longrightarrow R^m$ be a set valued map. It is said to be upper semicontinuous at x if for every $\epsilon > 0$, there is some $\delta > 0$ such that $F(x + \delta B_n) \subset F(x) + \varepsilon B_m$, where B_n and B_m denote the closed unite balls in R^n and R^m respectively.

3. Necessary conditions

Let $f : R^n \to R^m$ be continuous and let $S \subseteq R^n$ be a nonempty set. Let R^m be partially ordered by a closed convex and pointed cone C with a nonempty interior $\operatorname{int} C$ as follows: for $a, b \in R^n$, we write $a \geq_C b$ if $a - b \in C$. We also use the notation $a >_C b$ when $a \geq_C b$ and $a \neq b$, and $a >>_C b$ if $a - b \in intC$. Sometimes the lower index C in $\geq_C$ is omitted with the hope that no confusion likely occurs.

We shall study the following multiobjective problem

$$\text{(P)} \qquad \begin{aligned} &\min f(x) \\ &\text{s. t. } x \in S. \end{aligned}$$

We are interested in finding two kinds of solutions:

(i) a local efficient solution $x_0 \in S$ which means that in some neighborhood U of x_0, there is no $x \in S \cap V$ such that $f(x) < f(x_0)$,

(ii) a local weakly efficient solution $x_0 \in S$ which means that in some neighborhood U of x_0, there is no $x \in S \cap V$ such that $f(x) << f(x_0)$.

Some notations are in order. For $x_0 \in S$, the first order tangent cone and the second order tangent cone to S at x_0 are defined respectively by

$$T_1(S, x_0) := \{u \in R^n : \exists t_i > 0, x_i = x_0 + t_i u + o(t_i) \in S\},$$

$$T_2(S, x_0) := \{(u, v) \in R^n \times R^n : \exists t_i > 0, x_i = x_0 + t_i u + \frac{1}{2}t_i^2 v + o(t_i^2) \in S\}.$$

We also set

$$\Lambda := \{\xi \in C' : \|\xi\| = 1\},$$

where C' is the positive polar cone of C, i. e.

$$C' = \{\xi \in R^n : \langle \xi, c\rangle \geq 0 \text{ for all } \xi \in C\},$$

and for $\delta > 0$,

$$S_\delta(x_0) = \{t(x - x_0) : t \geq 0, x \in S \text{ and } \|x - x_0\| \leq \delta\}.$$

Theorem 3.1. *Assume that f is a continuously differentiable function, $x_0 \in S$ is a local weakly efficient solution of* (*P*) *and $\partial^2 f$ is an approximate Hessian map of f which is upper semicontinuous at x_0. Then for each $(u, v) \in T_2(S, x_0)$ one has*

(i) *there is $\lambda \in \Lambda$ such that $\langle \lambda, \nabla f(x_0)(u)\rangle \geq 0$,*

(ii) *when $\nabla f(x_0)(u) = 0$, there is $\lambda' \in \Lambda$ such that either*

$$\langle \lambda', \nabla f(x_0)(v) + M(u, u)\rangle \geq 0 \ \text{ for some } \ M \in \overline{\text{co}}\partial^2 f(x_0)$$

or

$$\langle \lambda', M_*(u, u)\rangle \geq 0 \ \text{ for some } M_* \in (\text{co}\partial^2 f(x_0))_\infty \setminus \{0\}.$$

If in addition C is polyhedral, then (i) *holds and when $< \lambda, \nabla f(x_0)(u) >= 0$, the inequalities of* (ii) *are true for $\lambda' = \lambda$.*

Proof. Let $(u, v) \in T_2(S, x_0)$, say

(1) $$x_i = x_0 = t_i u + \frac{1}{2} t_i^2 v + o(t_i^2) \in S$$

for some sequence $\{t_i\}$ of positive numbers converging to 0. Since x_0 is a local weakly efficient solusion, there is some $i_0 \geq 1$ such that

(2) $$f(x_i) - f(x_0) \in (-\text{int}C)^c \text{ for } i \geq i_0.$$

Since f is continuously differentiable, we derive

$$f(x_i) - f(x_0) = \nabla f(x_0)(x_i - x_0) + o(x_i - x_0).$$

This and (2) imply that

$$\nabla f(x_0)(u) \in (-intC)^c$$

which actually is equivalent to (i).

Now let $\nabla f(x_0)(u) = 0$. Observe first that by the upper semicontinuity of $\partial^2 f$ at x_0, for every $\varepsilon > 0$, there is $\delta > 0$ such that

$$\partial^2 f(x) \subseteq \partial^2 f(x_0) + \varepsilon B \ \text{ for each } x \text{ with } \|x - x_0\| < \delta,$$

where B is the closed unit ball in the space of matrices in which $\partial^2 f$ takes its values. Consequently, there is $i_1 \geq i_0$ such that

$$\overline{\text{co}}\partial^2 f[x_0, x_i] \subseteq \text{co}\partial^2 f(x_0) + 2\varepsilon B \ \text{ for every } i \geq i_1.$$

We apply the Taylor expansion to find $M_i \in \text{co}\partial^2 f(x_0) + 2\varepsilon B$ such that

$$f(x_i) - f(x_0) = \nabla f(x_0)(x_i - x_0) + \frac{1}{2} M_i(x_i - x_0, x_i - x_0), i \geq i_1.$$

Substituting (1) into this equality we derive

$$f(x_i) - f(x_0) = \frac{1}{2}t_i^2(\nabla f(x_0)(v) + M_i(u,v)) + \alpha_i,$$

where $\alpha_i = \frac{1}{2}M_i\Big(\frac{1}{2}t_i^2 v + o(t_i^2), t_i u + \frac{1}{2}t_i^2 v + o(t_i^2)\Big) + \nabla f(x_0)(o(t_i^2))$. This and (2) show

(3) $$\nabla f(x_0)(v) + M_i(u,v) + \alpha_i/t_i^2 \in (-intC)^c, \quad i \geq i_1.$$

Consider the sequence $\{M_i\}$. If it is bounded, we may assume that it converges to some $M_0 \in \overline{\text{co}}\partial^2 f(x_0) + 2\varepsilon B$. Then $\alpha_i/t_i^2 \to 0$ as $i \to \infty$ and relation (3) gives

$$\nabla f(x_0)(v) + M_0(u,u) \in (-intC)^c.$$

Since ε is arbitrary, the latter inclusion yields the existence of $M \in \overline{co}\partial^2 f(x_0)$ such that

$$\nabla f(x_0)(v) + M(u,u) \in (-intC)^c,$$

which actually is equivalent to the first inequality of (ii).
If $\{M_i\}$ is unbounded, say $\lim\limits_{i\to\infty} \|M_i\| = \infty$, we may assume that

$$\lim_{i\to\infty} \frac{M_i}{\|M_i\|} = M_* \in (\text{co}\partial^2 f(x_0))_\infty \setminus \{0\}.$$

By deviding (3) by $\|M_i\|$ and passing to the limit when $i \to \infty$, we deduce

$$M_*(u,u) \in (-intC)^c,$$

which is equivalent to the second inequality of (ii).
Now, assume that C is polyhedral. It follows from (2) that there is some $\lambda \in \Lambda$ such that

$$\langle \lambda, f(x_i) - f(x_0) \rangle \geq 0,$$

for infinitely many i. By taking a subsequence instead if necessary, we may assume this for all $i = 1, 2, \ldots$. Since f is continuously differentiable, we deduce

$$\langle \lambda, \nabla f(x_0)(u) \rangle \geq 0.$$

When $\langle \lambda, \nabla f(x_0)(u) \rangle = 0$, using the argument of the first part, we can find $M_i \in \text{co}\partial^2 f(x_0) + 2\varepsilon B$ such that

$$0 \leq \langle \lambda, f(x_i) - f(x_0) \rangle = \langle \lambda, \frac{1}{2}t_i^2(\nabla f(x_0)(v) + M_i(u,u)) + \alpha_i \rangle,$$

from which the two last inequalities of the theorem follow. □

Let us now study the problem where S is explicitly given by the following system

$$g(x) \leq 0$$
$$h(x) = 0,$$

where $g : R^n \to R^k$ and $h : R^n \to R^l$ are given. We shall denote this problem by (CP). Let $\xi \in C', \beta \in R^k$ and $\gamma \in R^l$. Define the Lagrangian function L by

$$L(x, \xi, \beta, \gamma) := \langle \lambda, f(x) \rangle + \langle \beta, g(x) \rangle + \langle \gamma, h(x) \rangle$$

and set

$$S_0 := \{x \in R^n : g_i(x) = 0 \text{ if } \beta_i > 0, g_i(x) \le 0 \text{ if } \beta_i = 0 \text{ and, } h(x) = 0\}.$$

In the sequel, when (ξ, β, γ) is fixed, we shall write $L(x)$ instead of $L(x, \xi, \beta, \gamma)$ and ∇L means the gradient of $L(x, \xi, \beta, \gamma)$ with respect to the variable x.

Theorem 3.2. *Assume that f, g and h are continuously differentiable and C is a polyhedral convex cone. If $x_0 \in S$ is a local weakly efficient solution of (CP), then there is a nonzero vector $(\xi_0, \beta, \gamma) \in C' \times R^k_+ \times R^l$ such that*

$$\nabla L(x_0, \xi_0, \beta, \gamma) = 0$$

and for each $(u, v) \in T_2(S_0, x_0)$, there is some $\xi \in \Lambda$ such that either

$$\nabla L(x_0, \xi, \beta, \gamma)(u) > 0,$$

or

$$\nabla L(x_0, \xi, \beta, \gamma)(u) = 0,$$

in which case either

$$\nabla L(x_0, \xi, \beta, \gamma)(v) + M(u, u) \ge 0 \text{ for some } M \in \overline{\text{co}}\partial^2 L(x_0, \xi, \beta, \gamma)$$

or

$$M_*(u, u) \ge 0 \text{ for some } M_* \in (\text{co}\partial^2 L(x_0, \xi, \beta, \gamma))_\infty \setminus \{0\}$$

provided $\partial^2 L$ is an approximate Hessian map of L which is upper semicontinuous at x_0.

Proof. The first condition about the existence of (ξ_0, β, γ) is already known and is true for any convex closed cone C with a nonempty interior. Let now $(u, v) \in T_2(S_0, x_0)$. Let $x_i = x_0 + t_i u + \frac{1}{2} t_i^2 v + o(t_i^2) \in S_0$ for some $t_i > 0, t_i \to 0$ as $i \to \infty$. Since x_0 is a local weakly efficient solution of (CP), there is some $i_0 \ge 1$ such that

$$f(x_i) - f(x_0) \in (-intC)^c, \text{ for } i \ge i_0.$$

Moreover, as C is polyhedral, there exists $\xi \in \Lambda$ such that

(4) $$\langle \xi, f(x_i) - f(x_0) \rangle \ \ge 0$$

for infinitely many i. We may assume this for all $i \ge i_0$. Since $\partial^2 L$ is upper semicontinuous at x_0, by applying the Taylor expansion to L we can find

$$M_i \in \text{co}\partial^2 L(x_0) + 2\varepsilon B,$$

where ε is an arbitrarily fixed positive, such that

$$L(x_i) - L(x_0) = \nabla L(x_0)(x_i - x_0) + \frac{1}{2} M_i(x_i - x_0, x_i - x_0)$$

for i sufficiently large. Substituting the expression $x_i - x_0 = t_i u + \frac{1}{2}t_i^2 v + o(t_i^2)$ into the above equality and taking (4) into account we derive

$$0 \le t_i \nabla L(x_0)(u) + \frac{t_i^2}{2}(\nabla L(x_0)(v) + M_i(u,u)) + \alpha_i$$

where $\alpha_i = \frac{1}{2}M_i(\frac{1}{2}t_i^2 v + o(t_i^2), t_i u + \frac{1}{2}t_i^2 v + o(t_i^2)) + \nabla L(x_0)(o(t_i^2))$. This in particular implies $\nabla L(x_0)(u) \ge 0$.
When $\nabla L(x_0)(u) = 0$, we also derive

$$0 \le \nabla L(x_0)(v) + M_i(u,u) + \alpha_i / t_i^2,$$

which by the same reason as discussed in the proof of Theorem 3.1, yields the requested inequalities. □

We notice that the second conclusion of Theorem 3.1 and the conclusion of Theorem 3.2 are no longer true if C is not polyhedral (see [7] for a counter-example when the data are smooth).

4. Sufficient conditions

In this section we provide some sufficient conditions for solutions of problems (P) and (CP).

Theorem 4.1. *Assume that f is continuously differentiable, $\partial^2 f$ is an approximate Jacobian map of f which is upper semicontinuous at $x_0 \in S$. Then each of the following conditions is sufficient for x_0 to be a locally unique efficient solution of (P):*

(i) *For each $u \in T_1(S, x_0) \setminus \{0\}$, there is some $\xi \in \Lambda$ such that*

$$\langle \xi, \nabla f(x_0)(u) \rangle > 0;$$

(ii) *There is $\delta > 0$ such that for each $v \in S_\delta(x_0)$ and $u \in T_1(S, x_0)$, one has*

$$\begin{aligned} &\langle \xi_0, \nabla f(x_0)(v) \rangle \ge 0 \ \text{ for some } \xi_0 \in \Lambda \\ \text{and} \quad &\langle \xi, M(u,u) \rangle > 0 \textit{ for every } \xi \in \Lambda \\ \text{and for every} \quad &M \in \overline{\text{co}}\partial^2 f(x_0) \cup [(\text{co}\partial^2 \text{f}(\text{x}_0))_\infty \setminus \{0\}]. \end{aligned}$$

Proof. Suppose to the contrary that x_0 is not a locally unique efficient solution of (P). There exist $x_i \in S, x_i \to x_0$ such that

$$f(x_i) - f(x_0) \in -C. \tag{5}$$

We may assume that $(x_i - x_0)/\|x_i - x_0\| \to u \in T_1(S, x_0)$ as $i \to \infty$. By deviding (5) by $\|x_i - x_0\|$ and passing to the limit we deduce

$$\nabla f(x_0)(u) \in -C.$$

This contradicts condition (i) and shows the sufficiency of this condition. For the second condition, let us apply the Taylor expansion to find $M_i \in co\partial^2 f(x_0) + 2\varepsilon B$ for an arbitrarily fixed $\varepsilon > 0$ such that

$$f(x_i) - f(x_0) = \nabla f(x_0)(x_i - x_0) + \tfrac{1}{2} M_i(x_i - x_0, x_i - x_0). \tag{6}$$

Observe that the first inequality of (ii) implies

$$\nabla f(x_0)(x_i - x_0) \in (-intC)^c$$

for i sufficiently large. For such i, there is $\xi_i \in \Lambda$ such that

$$\langle \xi_i, \nabla f(x_0)(x_i - x_0)\rangle \ \geq 0.$$

On the order hand, (5) shows that

$$\langle \xi_i, f(x_i) - f(x_0)\rangle \ \leq 0.$$

This and (6) imply

$$\langle \xi_i, M_i(x_i - x_0, x_i - x_0)\rangle \ \leq 0 \text{ for } i \text{ sufficiently large.}$$

Furthermore, since Λ is compact, we may assume $\xi_i \to \xi \in \Lambda$. By considering separately the case $\{M_i\}$ is bounded and the case $\{M_i\}$ is unbounded as in the proof of Theorem 3.1, we deduce

$$\langle \xi, M(u,u)\rangle \ \leq 0 \text{ for some } M \in \overline{\text{co}}\partial^2 f(x_0) \cup [(\text{co}\partial^2 f(x_0))_\infty \setminus \{0\}],$$

which contradicts (ii). The proof is complete. □

Theorem 4.2. *Assume that f is continuously differentiable and $\partial^2 f$ is an approximate Hessian map of f. If there is some $\delta > 0$ such that for every $v \in S_\delta(x_0)$ one has*

$$\langle \xi_0, \nabla f(x_0)(v)\rangle \geq 0 \ \textit{ for some } \ \xi_0 \in \Lambda$$

and

$$\langle \xi, M(u,v)\rangle \geq 0 \textit{ for all } \xi \in \Lambda, M \in \partial^2 f(x) \ \textit{ with } \|x - x_0\| \leq \delta,$$

then x_0 is a local weakly efficient solution of (P).

Proof. Suppose to the contrary that x_0 is not a local weakly efficient solution of (P). There is $\overline{x} \in S$ with $\|\overline{x} - x_0\| \leq \delta$ such that

$$f(\overline{x}) - f(x_0) \in -intC. \tag{7}$$

Set $v = \overline{x} - x_0$. Then $v \in S_\delta(x_0)$. The first inequality of the hypothesis implies

$$\nabla f(x_0)(v) \in (-intC)^c$$

and the second one implies

$$M(v,v) \in C \text{ for every } M \in \partial^2 f(x), \|x - x_0\| \leq \delta.$$

Since C is convex and closed, the latter inclusion gives in particular that

$$\overline{\text{co}}\partial^2 f(x) \subseteq C.$$

Using the Taylor expansion we derive

$$\begin{aligned} f(\overline{x}) - f(x_0) &\in \nabla f(x_0)(v) + \frac{1}{2}\overline{\text{co}}\{\partial^2 f[x_0, \overline{x}](v,v)\} \\ &\subseteq (-\text{int}C)^c + C \subseteq (-\text{int}C)^c, \end{aligned}$$

which contradicts (7). The proof is complete. □

Theorem 4.3. *Assume that f, g and h are continuously differentiable and for every $u \in T_1(S, x_0) \setminus \{0\}$, there is some $(\xi, \beta, \gamma) \in \Lambda \times R^k_+ \times R^l$ such that*

$$\nabla L(x_0, \xi, \beta, \gamma) = 0, \ \beta g(x_0) = 0,$$

and

$$M(u, u) > 0 \ \ \textit{for each } M \in \overline{\mathrm{co}}\partial^2 L(x_0) \cup ((\mathrm{co}\partial^2 \mathrm{L}(\mathrm{x}_0))_\infty \setminus \{0\}),$$

where $\partial^2 L$ is an approximate Jacoban map of L which is upper semicontinuous at x_0. Then x_0 is a locally unique efficient solution of (CP).

Proof. Suppose to the contrary that x_0 is not a locally unique solution of (CP). Then there is $x_i \in S, x_i \to x_0$ such that $f(x_i) - f(x_0) \in -C$. We may assume $(x_i - x_0)/\|x_i - x_0\| \to u \in T_1(S, x_0)$. It follows that

$$L(x_i) - L(x_0) \le 0 \text{ for all } i \ge 1.$$

Applying the Taylor expansion to L and by the upper semicontinuous of $\partial^2 L$ we obtain

$$\begin{aligned} L(x_i) - L(x_0) - \nabla L(x_0)(x_i - x_0) &\in \frac{1}{2}\overline{\mathrm{co}}\{\partial^2 L[x_0, x_i](x_i - x_0, x_i - x_0)\} \\ &\subseteq \frac{1}{2}(\mathrm{co}\partial^2 L(x_0) + \|x_i - x_0\| B)(x_i - x_0, x_i - x_0), \end{aligned}$$

for i sufficiently large. These relations yield

$$M_i(x_i - x_0, x_i - x_0) \le 0$$

for some $M_i \in \mathrm{co}\partial^2 L(x_0) + \|x_i - x_0\| B$, with i sufficiently large. By the argument of the proof of Theorem 3.1, we derive the existence of some matrix $M \in \overline{co}\partial^2 L(x_0) \cup ((co\partial^2 L(x_0))_\infty \setminus \{0\})$ such that

$$M(u, u) \le 0,$$

which contradicts the hypothesis. □

Theorem 4.4. *Assume that f, g and h are continuously differentiable and there is $\delta > 0$ such that for each $v \in S_\delta(x_0)$, one can find a vector $(\xi, \beta, \gamma) \in \Lambda \times R^k_+ \times R^l$ and an approximate Hessian map $\partial^2 L(x, \xi, \beta, \gamma)$ of L such that*

$$\nabla L(x_0, \xi, \beta, \gamma) = 0, \ \ \beta g(x_0) = 0$$

and

$$M(u, u) \ge 0 \ \ \textit{for every } M \in \partial^2 L(x, \xi, \beta, \gamma) \textit{ with } \|x - x_0\| \le \delta.$$

Then x_0 is a local weakly efficient solution of (CP).

Proof. Similar to the proof of Theorem 4.1. □

5. Example

In this section we give an example which is adapted from [22] to show that the recession Hessian matrices in Theorem 3.2 cannot be removed when the data of the problem are of class C^1. Examples for the sufficient conditions of Section 4 can be constructed in a similar way.

Let us consider the following two-objective problem

$$\min(x, x^{4/3} - y^4)$$
$$\text{s.t.} - x^2 + y^4 \leq 0.$$

The partial order of R^2 is given by the positive orthant R^2_+. It is easy to see that $(0,0)$ is a local efficient solution of the problem. By taking $\xi_0 = (0,1)$ and $\beta = 1$, the Lagrangian function of the problem is

$$L((x,y),\xi_0,\beta)) = x^{4/3} - y^4 - x^2 + y^4 = x^{4/3} - x^2$$

and satisfies the necessary condition

$$\nabla L((0,0),\xi_0,\beta) = (0,0).$$

The set S_0 is given by

$$S_0 = \{(x,y) \in R^2 : x^2 = y^4\}.$$

Let us take $u = (0,1)$ and $v = (-2,0)$. It is evident that $(u,v) \in T_2(S_0,(0,0))$. According to Theorem 3.2, there is some $\xi = (\xi_1,\xi_2) \in R^2_+$ with $\|\xi\| = 1$ such that $\nabla L((0,0),\xi,\beta)(u) \geq 0$. Actually we have

$$\nabla L((0,0),\xi,\beta) = (\xi_1, 0).$$

Hence $\nabla L((0,0),\xi_o,\beta)(u) = 0$, and the second order conditions of that theorem must hold. Observe first that if $\xi_2 = 0$, then

$$\partial^2 L(x,y) := \left\{ \begin{pmatrix} -2 & 0 \\ 0 & 12y^2 \end{pmatrix} \right\}$$

is an approximate Hessian map of L, which is upper semicontinuous at $(0,0)$. It is not hard to verify that the second order condition of Theorem 3.2 does not hold for this ξ. Consequently, $\xi_2 > 0$. Let us define

$$\partial^2 L(x,y) := \left\{ \begin{pmatrix} \frac{4}{9}\xi_2 x^{-2/3} - 2 & 0 \\ 0 & 12(1-\xi_2)y^2 \end{pmatrix} \right\}, \text{for } x \neq 0,$$

and

$$\partial^2 L(0,y) := \left\{ \begin{pmatrix} \frac{4}{9}\xi_2\alpha - 2 & 0 \\ 0 & 12(1-\xi_2)y^2 - 1/\alpha \end{pmatrix} : \alpha \geq \frac{9}{\xi_2} \right\}.$$

A direct calculation confirms that the set valued map $(x,y) \to \partial^2 L(x,y)$ is an approximate Hessian map of L which is upper semicontinuous at $(0,0)$. Moreover, for each $M \in \overline{\text{co}}\partial^2 L(0,0)$, one has

$$\nabla L(0,0)(v) + M(u,u) = -2\xi_1 - \frac{1}{\alpha} < 0,$$

which shows that the first inequality of the second order condition of Theorem 3.2 is not true. The recession cone of $\partial^2 L(0,0)$ is given by

$$(\partial^2 L(0,0))_\infty = \left\{ \begin{pmatrix} \alpha & 0 \\ 0 & 0 \end{pmatrix} : \alpha \geq 0 \right\}.$$

By choosing

$$M_* = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in (\mathrm{co}\partial^2 L(0,0))_\infty \setminus \{0\}$$

we have

$$M_*(u,u) \geq 0$$

as requested.

References

[1] B. Aghezzaf and M. Hachimi, *Second-order optimality conditions in multiobjective optimization problems,* J. Optimization Theory Appl. **102** (1999), 37-50.

[2] S. Bolintineau and M. El Maghri, *Second-order efficiency conditions and sensitivity of efficient points,* J. Optimization Theory Appl. **98** (1998), 569-592.

[3] A. Cambini and L. Martein, *Second order necessary optimality conditions in the image space: preliminary results,* on Proceedings of the Workshop Scalar and Vector Optimization in Economic and Financial Problems, Edited by E. Castagnoli and G. Giorgi , Milan, Italy, pp. 27-38, 1995.

[4] F. H. Clarke, *Optimization and Nonsmooth Analysis*, Wiley, New York, 1983.

[5] B. D. Craven, *Nonsmooth multiobjective programming,* Numer. Funct. Anal. Optimization **10** (1989), 49-64.

[6] A. Guerraggio and D. T. Luc, *Optimality conditions for $C^{1,1}$ vector optimization problems*, J. Optimization Theory Appl. **109** (2001), 615-529.

[7] A. Guerraggio and D. T. Luc, *Optimality conditions for $C^{1,1}$ constrained multiobjective problems*, J. Optimization Theory Appl. (to appear)

[8] A. Fischer, V. Jeyakumar and D. T. Luc, *Solution point characterizations and convergence analysis of a descent algorithm for nonsmooth continuous complementarity problems*, J. Optimization Theory Appl. **110** (2001), 493-513.

[9] J. Jahn, *Mathematical Theory of Vector Optimization in Partially Ordered Spaces*, Peter Lang, Germany, 1985.

[10] A. D. Ioffe, *Nonsmooth analysis: Differential calculus of non-differentiable mappings,* Trans. Am. Math. Soc. **266** (1981), 1-56.

[11] V. Jeyakumar and D. T. Luc, *Approximate Jacobian matrices for nonsmooth continuous maps and C^1-optimization,* SIAM J. Control Optimization **36** (1998), 1815-1832.

[12] V. Jeyakumar, D. T. Luc, and S. Schaible, *Characterizations of generalized monotone nonsmooth continuous maps using approximate Jacobians,* J. Convex Anal. **5** (1998), 119-132.

[13] V. Jeyakumar and D. T. Luc, *Nonsmooth calculus, minimality and monotonicity of convexificators,* J. Optimization Theory Appl. **101** (1999), 599-621.

[14] V. Jeyakumar and D. T. Luc, *Open mapping theorem using unbounded generalized Jacobians,* Nonlinear Analysis (to appear).

[15] V. Jeyakumar, D. T. Luc and Y. Wang, *Lagrange multipliers for equality constraints without Lipschitz continuity,* Applied Mathematics Report AMR00/1, University of New South Wales, Australia (submitted).

[16] J-B. Hiriart-Urruty, J-J. Strodiot and V. H. Nguyen, *Generalized Hessian matrix and second order optimality conditions for problems with $C^{1,1}$ data,* Appl. Math. Optimization **11** (1984), 43-56.

[17] H. W. Kuhn and F. H. Tucker, *Nonlinear programming*, Proceedings of the second Berkeley Symposium on Mathematical Statistics and Probability, California, 1951, pp. 481-492.

[18] L. P. Liu, *The second-order conditions of nondominated solutions for* $C^{1,1}$ *generalized multiobjective mathematical programming,* Systems Sciences and Mathematical Sciences **4** (1991), 128-138.

[19] D. T. Luc, *Theory of Vector Optimization*, LNEMS 319, Springer-Verlag,Germany, 1989.

[20] D. T. Luc, *Contingent derivative of set-valued maps and applications to vector optimization,* Math. Program. **50** (1991), 99-111.

[21] D. T. Luc, *A multiplier rule in multiobjective programming problems with continuous data,* SIAM J. Optim. (to appear).

[22] D. T. Luc, *Second order optimality conditions for problems with continuously differentiable data,* Optimization (to appear).

[23] D. T. Luc, *Variational inequalities: local uniqueness of solutions for problems with continuous data,* J. Math. Anal. Appl. (to appear).

[24] D. T. Luc and C. Malivert, *Invex optimization problems,* Bull. Aust. Math. Soc. **46** (1992), 47-66.

[25] P. H. Sach, *Second order necessary optimality conditions for optimization problems involving set-valued maps*, Appl. Math. Optimization **22** (1990), 189-209.

[26] P. H. Sach and W. Oettli, *Prederivatives and second order conditions for infinite optimization problems*, in "Recent Advances in Nonsmooth Optimization", Eds. D. Z. Du, L. Qi and R. S. Womersley, World Scientific Publishers, 1995, pp. 243-259.

[27] S. Wang, *Second-order necessary and sufficient conditions in multiobjective programming,* Numer. Funct. Anal. Optimization **12** (1991), 237-252.

[28] Y. Wang and V. Jeyakumar, *A sharp Lagrange multiplier rule for nonsmooth mathematical programming problems involving equality constraints*, SIAM J. Optim. **10** (2000), 1136-1148.

University of Bocconi, Milan, Italy

University of Avignon, Avignon, France

Institute of Mathematics, Hanoi, Vietnam