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CHU SPACES AND CONDITIONAL PROBABILITY

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Abstract. Let $\tilde{\Omega} = (\Omega, P, \mathcal{A})$ and $\tilde{\Sigma} = (\Sigma, Q, \mathcal{B})$ be probability measure spaces, and let $\phi : \Omega \to \Sigma$ and $\psi : \Sigma \to \Omega$ be measurability preserving maps. The maps ϕ and ψ induce $\phi^{-1} : \mathcal{B} \to \mathcal{A}$ and $\psi^{-1} : \mathcal{A} \to \mathcal{B}$. By $(\mathcal{A}, \mathcal{A}, f)$ we denote the Chu space associated with the probability measure space $(\Omega, P, \mathcal{A})$. Our main results are:

Theorem 1. Let $P(\tilde{\Omega}) = (\mathcal{A}, \mathcal{A}, f)$ and $P(\tilde{\Sigma}) = (\mathcal{B}, \mathcal{B}, g)$ be Chu spaces associated with $\tilde{\Omega}$ and $\tilde{\Sigma}$, respectively. If $\Phi = (\psi^{-1}, \phi^{-1}) : P(\tilde{\Omega}) \longrightarrow P(\tilde{\Sigma})$ is a Chu morphism, then both ϕ and ψ are measure preserving.

Theorem 2. The pair (ϕ, ψ) is mutually measure preserving if and only if $\Phi = (\psi^{-1}, \phi^{-1}) : (\mathcal{A}, \mathcal{A}, f) \to (\mathcal{B}, \mathcal{B}, g)$ is a Chu morphism.

1. Introduction

The notion of Chu spaces has been introduced in the appendix of Barr's book [1] and presented completely by Papadopoulos and Syropoulos [5] at the Internationnal Workshop on Current Trends and Developments of Fuzzy Logic, Thessaloniki, Greece, October 1998. The application of Chu space to computer science is mainly investigated by Vaughan Pratt, see e.g. [6], [7]. Another application of Chu space was shown in the recent book by Barwise and Seligman [2]. In this paper we extend the concept of Chu spaces to probability theory. Namely, we consider the notation of measure preserving and mutually measure preserving defined in [3]. Our main results are given in Theorem 1 and Theorem 2 in Section 4. Theorem 1 establishs the relation between the Chu morphisms and measure preserving. Theorem 2 gives a characterization of mutually measure preserving through Chu morphism.

2. Chu spaces associated with probability measure spaces

By a *Chu space* we mean a triple $\tilde{\mathcal{C}} = (X, A, f)$ consisting of two sets:

1. X, called the *set of events*, or *players* of $\tilde{C}$; and
2. A, called the *set of states*, or *situations* of $\tilde{C}$.

The sets A and X are joined by a map $f : X \times A \longrightarrow K$, where K is an arbitrary set of values. In this paper we take the set K to be the unit interval $[0, 1]$. The map $f : X \times A \longrightarrow [0, 1]$ is then called the *probability function* of $\tilde{C}$.

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If (X, A, f) and (Y, B, g) are Chu spaces, then a *Chu morphism*

$$\Phi : (X, A, f) \longrightarrow (Y, B, g)$$

is a pair of maps $\Phi = (\phi, \psi)$, where $\phi : X \longrightarrow Y$ and $\psi : B \longrightarrow A$ such that the diagram below commutes:

$$\begin{array}{ccc} X \times B & \xrightarrow{(\phi,1_B)} & Y \times B \\ {\scriptstyle (1_X,\psi)}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ X \times A & \xrightarrow{\ \ f\ \ } & [0,1] \end{array}$$

where $1_X : X \longrightarrow X$ denotes the identity map, that is

$$f\big(x, \psi(b)\big) = g\big(\phi(x), b\big) \quad \text{for } x \in X \text{ and } b \in B.$$

If $\Phi = (\phi, \psi) : \tilde{C} : (X, A, f) \longrightarrow \tilde{D} = (Y, B, g)$ is a Chu morphism, then the Chu space $(X, B, f \times_\Phi g)$, where

$$(f \times_\Phi g)(x, b) = f\big(x, \psi(b)\big) = g\big(\phi(x), b\big) \text{ for } (x, b) \in X \times B$$

is called the *cross product of* $\tilde{C}$ and $\tilde{D}$ over Φ, denoted by $\tilde{C} \times_\Phi \tilde{D}$.

The *composition* of two morphism $\Phi_1 = (\phi_1, \psi_1)$ and $\Phi_2 = (\phi_2, \psi_2)$ is given by $\Phi_1\Phi_2 = (\phi_1\phi_2, \psi_2\psi_1)$. Clearly, $1_{\tilde{C}} = (1_X, 1_A)$ is the identity map of $\tilde{C} = (X, A, f)$.

Proposition 1. [4] *If* Φ_1 *and* Φ_2 *are Chu morphism, then* $\Phi_1\Phi_2$ *is a Chu morphism.*

By means of Proposition 1 we can define the *Chu category*, denoted by $\mathcal{C}$, of Chu spaces with Chu morphisms.

We say that $\tilde{C}$ and $\tilde{D}$ are *Chu isomorphic*, denoted by $\tilde{C} \cong \tilde{D}$, if $\tilde{C}$ and $\tilde{D}$ are isomorphic objects in the category $\tilde{C}$ of Chu spaces. It is easy to see that a Chu morphism $\Phi = (\phi, \psi) : (X, \mathcal{A}, f) \longrightarrow (Y, \mathcal{B}, g)$ is an isomorphism if and only if both $\phi : X \longrightarrow Y$ and $\psi : B \longrightarrow A$ are one to one and onto.

Let $\mathcal{A}$ be a σ - field over a set Ω, and let $P : \mathcal{A} \longrightarrow [0, 1]$ be a probability measure on $\mathcal{A}$. Then the triple $\tilde{\Omega} = (\Omega, P, \mathcal{A})$ is called a *probability measure space*.

Let $\tilde{\Omega} = (\Omega, P, \mathcal{A})$ and $\tilde{\Sigma} = (\Sigma, Q, \mathcal{B})$ be probability measure spaces. We recall, see for instance [3], that $\phi : \Omega \longrightarrow \Sigma$ is measurable if $\phi^{-1}(B) \in \mathcal{A}$ for any $B \in \mathcal{B}$. If ϕ is one-two-one, $\phi(\Omega) = \Sigma$ and both ϕ and ϕ^{-1} are measurable, then we say that ϕ is *measurability preserving*. If ϕ is measurability preserving, then we say that ϕ is *measure preserving* if

$$P\big(\phi^{-1}(B)\big) = Q(B) \text{ for every } B \in \mathcal{B},$$

and ϕ is *isomorphic* if both ϕ and ϕ^{-1} are measure preserving.

Let $\tilde{\Omega} = (\Omega, P, \mathcal{A})$ be a probability measure space. We define

$$f : \mathcal{A} \times \mathcal{A} \longrightarrow [0,1]$$

by

$$f(A,B) = \begin{cases} P(A \mid B), & \text{if } P(B) > 0 \\ 0, & \text{if } P(B) = 0 \end{cases} \tag{1}$$

for $(A,B) \in \mathcal{A} \times \mathcal{A}$, where $P(A \mid B)$ means the probability of A given B.

Thus $(\mathcal{A}, \mathcal{A}, f)$ is a Chu space, called the *Chu space associated with the probability measure space* $(\Omega, P, \mathcal{A})$ and denoted by $P(\tilde{\Omega}) = (\mathcal{A}, \mathcal{A}, f)$.

3. Mutually measure preserving maps

To define Chu morphisms for Chu space associated with probability measure spaces we need the following definitions.

Let $(\Omega, P, \mathcal{A})$ and $(\Sigma, Q, \mathcal{B})$ be probability measure spaces and let $\phi : \Omega \longrightarrow \Sigma$ and $\psi : \Sigma \longrightarrow \Omega$ be measurability preserving maps. We say that the pair (ϕ, ψ) is *mutually measure preserving* if

$$P\big(A \cap \phi^{-1}(B)\big) = Q\big(\psi^{-1}(A) \cap B\big) \text{ for every } A \in \mathcal{A} \text{ and } B \in \mathcal{B}.$$

Proposition 2. *Assume that ϕ and ψ are measurability preserving.*

1. *If (ϕ, ψ) is mutually measure preserving, then both ϕ and ψ are measure preserving.*
2. *If ϕ is isomorphic, then the pair (ϕ, ϕ^{-1}) is mutually measure preserving.*

Proof. 1. If (ϕ, ψ) is mutually measure preserving, then

$$P\big(A \cap \phi^{-1}(B)\big) = Q\big(\psi^{-1}(A) \cap B\big) \text{ for every } A \in \mathcal{A} \text{ and } B \in \mathcal{B}.$$

For $B = \Sigma$ we get

$$P(A) = Q\big(\psi^{-1}(A)\big) \text{ for every } A \in \mathcal{A}.$$

And for $A = \Omega$ we have

$$P\big(\phi^{-1}(B)\big) = Q(B) \text{ for every } B \in \mathcal{B}.$$

Consequently ϕ and ψ are measure preserving.

2. If ϕ is isomorphic, then

$$\begin{aligned} P\big(A \cap \phi^{-1}(B)\big) &= P\big(\phi^{-1}\phi(A) \cap \phi^{-1}(B)\big) \\ &= P\big(\phi^{-1}[\phi(A) \cap B]\big) \\ &= Q\big(\phi(A) \cap B\big). \end{aligned}$$

For every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Hence, the pair (ϕ, ϕ^{-1}) is mutually measure preserving.

Let $\tilde{\Omega} = (\Omega, P, \mathcal{A}), \tilde{\Sigma} = (\Sigma, Q, \mathcal{B})$ and $\tilde{\Gamma} = (\Gamma, R, \mathcal{C})$ be probability measure spaces, and let $\phi : \Omega \longrightarrow \Sigma, \psi : \Sigma \longrightarrow \Omega, \theta : \Sigma \longrightarrow \Gamma$ and $\lambda : \Gamma \longrightarrow \Sigma$ be measurability preserving maps. $\square$

Proposition 3. *If (ϕ, ψ) and (θ, λ) are mutually measure preserving, then $(\theta\phi, \psi\lambda)$ is mutually measure preserving.*

Proof. Observe that

$$\begin{aligned} P\big(A \cap (\theta\phi)^{-1}(C)\big) &= P\big(A \cap \ \phi^{-1}\theta^{-1}(C)\big) \\ &= Q\big(\psi^{-1}(A) \cap \theta^{-1}(C)\big) \\ &= R\big(\lambda^{-1}\psi^{-1}(A) \cap C\big) \\ &= R\big((\psi\lambda)^{-1}(A) \cap C\big) \end{aligned}$$

for every $A \in \mathcal{A}$ and $C \in \mathcal{C}$. Consequently $(\theta\phi, \psi\lambda)$ is mutually measure preserving. □

From Proposition 3 it follows that the family of all probability measure spaces with mutually measure preserving maps forms a category, called the *mutually measure space category*, denoted by $\mathcal{M}$.

4. The main result

Let $\tilde{\Omega} = (\Omega, P, \mathcal{A})$ and $\tilde{\Sigma} = (\Sigma, Q, \mathcal{B})$ be probability measure spaces, and let $\phi : \Omega \longrightarrow \Sigma$ and $\psi : \Sigma \longrightarrow \Omega$ be measurability preserving maps. The maps ϕ and ψ induce $\phi^{-1} : \mathcal{B} \longrightarrow \mathcal{A}$ and $\psi^{-1} : \mathcal{A} \longrightarrow \mathcal{B}$.

Theorem 1. *Let $P(\tilde{\Omega}) = (\mathcal{A}, \mathcal{A}, f)$ and $P(\tilde{\Sigma}) = (\mathcal{B}, \mathcal{B}, g)$ be Chu spaces associated with $\tilde{\Omega}$ and $\tilde{\Sigma}$, respectively. If $\Phi = (\psi^{-1}, \phi^{-1}) : P(\tilde{\Omega}) \longrightarrow P(\tilde{\Sigma})$ is a Chu morphism, then both ϕ and ψ are measure preserving.*

Proof. Since $\Phi = (\psi^{-1}, \phi^{-1}) : (\mathcal{A}, \mathcal{A}, f) \longrightarrow (\mathcal{B}, \mathcal{B}, g)$ is a Chu morphism,

$$f\big(A, \phi^{-1}(B)\big) = g\big(\psi^{-1}(A), B\big) \text{ for } A \in \mathcal{A} \text{ and } B \in \mathcal{B}. \tag{2}$$

We claim that

$$g\big(B, \phi(A)\big) = f\big(\psi(B), A\big) \text{ for } A \in \mathcal{A} \text{ and } B \in \mathcal{B}. \tag{3}$$

In fact, since $\phi : \Omega \longrightarrow \Sigma$ is measurability preserving, for $A \in \mathcal{A}$ there exists $B' \in \mathcal{B}$ such that $A = \phi^{-1}(B')$. Hence by virtue of (2) we have

$$\begin{aligned} f\big(\psi(B), A\big) &= f\big(\psi(B), \phi^{-1}(B')\big) \\ &= g\big(\psi^{-1}\psi(B), B'\big) \\ &= g\big(B, \phi(A)\big) \end{aligned}$$

for $A \in \mathcal{A}$ and $B \in \mathcal{B}$. Whence (3) follows.

Next, we claim that both ϕ and ψ preserve the zero-measure. In fact, let $B \in \mathcal{B}$ with $Q(B) = 0$. Then by definition, see (1), we have

$$g\big(\psi^{-1}(A), B)\big) = 0 \text{ for every } A \in \mathcal{A}.$$

Hence from (2) we get

$$f\big(A, \phi^{-1}(B)\big) = 0 \text{ for every } A \in \mathcal{A}.$$

In particular $f\big(\Omega, \phi^{-1}(B)\big) = 0$, which implies

$$P\big(\phi^{-1}(B)\big) = 0.$$

Let $A \in \mathcal{A}$ with $P(A) = 0$. Then applying (2) for $B = \Sigma$, we obtain $Q\big(\psi^{-1}(A)\big) = 0$. Consequently ϕ and ψ preseve the zero-measure.

Now assume that $B \in \mathcal{B}$ with $Q(B) > 0$. Then by (2) we have $P\big(\phi^{-1}(B)\big) > 0$. Therefore

$$P\big(A \mid \phi^{-1}(B)\big) = Q\big(\psi^{-1}(A) \mid B\big) \text{ for } A \in \mathcal{A} \text{ and } B \in \mathcal{B}.$$

This means

$$\frac{P\big(A \cap \phi^{-1}(B)\big)}{P\big(\phi^{-1}(B)\big)} = \frac{Q\big(\psi^{-1}(A) \cap B\big)}{Q(B)} \tag{4}$$

for $A \in \mathcal{A}$ and $B \in \mathcal{B}$ with $P(B) > 0$. Hence for $B = \Sigma$ we get

$$P(A) = Q\big(\psi^{-1}(A)\big) \text{ for } A \in \mathcal{A}.$$

So ψ is measure preserving.

To complete the proof of the theorem it remains to show that ϕ is measure preserving.

Assume $A \in \mathcal{A}$ with $P(A) > 0$. Then by (3) we have $Q\big(\phi(A) > 0$. Therefore

$$Q\big(B \mid \phi(A)\big) = P\big(\psi(B) \mid A\big) \text{ for } A \in \mathcal{A} \text{ and } B \in \mathcal{B}.$$

This means

$$\frac{Q\big(B \cap \phi(A)\big)}{Q\big(\phi(A)\big)} = \frac{P\big(\psi(B) \cap A\big)}{P(A)} \quad \text{for } A \in \mathcal{A} \text{ and } B \in \mathcal{B}.$$

For $A = \Omega$ we get

$$Q(B) = P\big(\psi(B)\big) \quad \text{for } B \in \mathcal{B}. \tag{5}$$

On the other hand for $A = \phi^{-1}(B)$ we have

$$1 = \frac{Q\big(\phi(A)\big)}{Q\big(\phi(A)\big)} = \frac{P\big(\psi(B) \cap \phi^{-1}(B)\big)}{P\big(\phi^{-1}(B)\big)} \cdot$$

Hence, by (5)

$$P\big(\phi^{-1}(B)\big) = P\big(\psi(B) \cap \phi^{-1}(B)\big) \leq P\big(\psi(B)\big) = Q(B). \tag{6}$$

Again, from the equality (4) we get for $A = \phi^{-1}(B)$

$$1 = \frac{P\big(\phi^{-1}(B)\big)}{P\big(\phi^{-1}(B)\big)} = \frac{Q\big(\psi^{-1}\big(\phi^{-1}(B)\big) \cap B\big)}{Q(B)} \cdot$$

Hence, by (5)

$$\begin{aligned} Q(B) &= Q\big(\psi^{-1}\big(\phi^{-1}(B)\big) \cap B\big) \\ &\le Q\big(\psi^{-1}\big(\phi^{-1}(B)\big)\big) \\ &= P\big(\psi\psi^{-1}(\phi^{-1}(B))\big) \\ &= P\big(\phi^{-1}(B)\big), \end{aligned}$$

for $B \in \mathcal{B}$.

From the latter and (6) it follows that

$$Q(B) = P\big(\phi^{-1}(B)\big) \text{ for } B \in \mathcal{B}.$$

Consequently ϕ is measure preserving, and the proof of Theorem 1 is finished. □

Theorem 2. *The pair (ϕ, ψ) is mutually measure preserving if and only if $\Phi = (\psi^{-1}, \phi^{-1}) : (\mathcal{A}, \mathcal{A}, f) \longrightarrow (\mathcal{B}, \mathcal{B}, g)$ is a Chu morphism.*

Proof. Assume that (ϕ, ψ) is mutually measure preserving. Then by Proposition 2, ϕ and ψ are measure preserving. It is easy to see that

$$f\big(A, \phi^{-1}(B)\big) = g\big(\psi^{-1}(A), B\big) \text{ for } A \in \mathcal{A} \text{ and } B \in \mathcal{B} \text{ with } Q(B) = 0.$$

For $A \in \mathcal{A}$ and $B \in \mathcal{B}$ with $Q(B) > 0$ we have $P\big(\phi^{-1}(B)\big) > 0$, and

$$\begin{aligned} f\big(A, \phi^{-1}(B)\big) &= P\big(A|\phi^{-1}(B)\big) \\ &= \frac{P\big(A \cap \phi^{-1}(B)\big)}{P\big(\phi^{-1}(B)\big)} \\ &= \frac{Q\big(\psi^{-1}(A) \cap B\big)}{Q(B)} \\ &= Q\big(\psi^{-1}(A)|B\big) \\ &= g\big(\psi^{-1}(A), B\big). \end{aligned}$$

This implies that the diagram bellow commutes:

$$\begin{array}{ccc} \mathcal{A} \times \mathcal{B} & \xrightarrow{(\psi^{-1}, 1_B)} & \mathcal{B} \times \mathcal{B} \\ {\scriptstyle (1_X, \phi^{-1})}\big\downarrow & & \big\downarrow{\scriptstyle g} \\ \mathcal{A} \times \mathcal{B} & \xrightarrow{\quad f \quad} & [0,1]. \end{array}$$

Conversly, assume that (ψ^{-1}, ϕ^{-1}) is a Chu morphism. Then we have

$$f\big(A, \phi^{-1}(B)\big) = g\big(\psi^{-1}(A), B\big) \text{ for } A \in \mathcal{A} \text{ and } B \in \mathcal{B}.$$

We will show that

$$P\big(A \cap \phi^{-1}(B)\big) = Q\big(\psi^{-1}(A) \cap B\big) \text{ for } A \in \mathcal{A} \text{ and } B \in \mathcal{B}. \tag{7}$$

Firstly if $Q(B)=0$, then by Theorem 1, $P\big(\phi^{-1}(B)\big)=0$. Therefore by definition, see (1),

$$f\big(A,\phi^{-1}(B)\big)=0=g\big(\psi^{-1}(A),B\big) \quad \text{for every } A\in\mathcal{A}.$$

Thus (7) holds for $Q(B)=0$. Assume that $Q(B)>0$.

By Theorem 1, ϕ is measure preserving. Therefore $P\big(\phi^{-1}(B)\big)>0$. Thus by definition, see (1), we have

$$f(A,\phi^{-1}(B)\big)=P\big(A\mid\phi^{-1}(B)\big)=\frac{P\big(A\cap\phi^{-1}(B)\big)}{P\big(\phi^{-1}(B)\big)}$$

and

$$g\big(\psi^{-1}(A),B\big)=Q\big(\psi^{-1}(A)\mid B\big)=\frac{Q\big(\psi^{-1}(A)\cap B\big)}{Q(B)}\cdot$$

Since

$$f\big(A,\phi^{-1}(B)\big)=g\big(\psi^{-1}(A),B\big) \text{ and } P\big(\phi^{-1}(B)\big)=Q(B),$$

we have

$$P\big(A\cap\phi^{-1}(B)\big)=Q\big(\psi^{-1}(A)\cap B\big).$$

Consequently (ϕ,ψ) is mutually measure preserving, and the theorem is proved. □

From Proposition 3 and Theorem 2 we get

Corollary 1. *If* $\phi:\Omega\longrightarrow\Sigma$ *is isomorphic, then the Chu morphism* $\Phi=(\phi,\phi^{-1}):(\mathcal{A},\mathcal{A},f)\longrightarrow(\mathcal{B},\mathcal{B},f)$ *is Chu isomorphic and* $\Phi^{-1}=(\phi^{-1},\phi)$.

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References

[1] M. Barr, *Autonomous Categories*, Lecture Notes in Mathematics 752, Springs-Verlag. Electronic Notes in Theoretical Computer Science, 1979.

[2] J. Barwise J. Seligman, *Information Flow. The Logic of Distributed Systems*, Cambridge Uni. Press. 1997.

[3] P. Halmos, *Measure Theory*. Springer-Verlag, 1974.

[4] H. T. Nguyen and E. A. Walker, *A First Course in Fuzzy Logic*, CRC Press, 1997.

[5] B. K. Papadopoulos and A. Syropoulos, *Fuzzy sets and fuzzy relational structures as Chu spaces*, Procedings of the First Intenational Workshop on current Trends and Delopments of Fuzzy Logic, Thessaloniki, Greece, Oct. 16 - 20, 1998, to be published in Electronic Notes in Theoretical Computer Science.

[6] V. R. Pratt, *Types as processes, via Chu spaces*, Electronic Notes in Theoretical Computer Science **7** (1997).

[7] V. R. Pratt, *Chu space from the representation viewpoint*, http://boole.stanford.edu/chuguide.html.

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