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VECTOR RANDOM STABLE MEASURES AND RANDOM INTEGRALS

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ABSTRACT. In this paper, the definition and basic properties of Banach space-valued symmetric independently scattered stable measures including random Gaussian measures are presented. Random integrals of real-valued deterministic functions with respect to these random measures are also investigated.

1. INTRODUCTION

Let $(T, \mathcal{A})$ be a measurable space. A mapping $M : \mathcal{A} \longrightarrow L_0(\Omega)$ is called random measure on $(T, \mathcal{A})$ if for every sequence (A_n) of disjoint sets from $\mathcal{A}$ the r.v.'s $M(A_n)$ are independent and

$$M\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} M(A_n) \quad \text{in} \quad L_0(\Omega).$$

The study of random measures and the random integral of the form $\int f\, dM$, where f is a real-valued function defined on T, has been carried out by several authors, see, e.g., [12, 7, 8, 3, 5]. Rosinski [6] considered the case in which the integrand f takes values in a Banach space X.

Vector random measures arise naturally as a Banach space generalization of random measures. In this setting, for each $A \in \mathcal{A}, M(A)$ is no longer a real-valued random variable but a random variable with values in a Banach space X. Some aspects of vector random measures and the random integral of real-valued functions with respect to vector random measures were discussed in [9, 10].

In this paper we are concerned with the study of the vector symmetric random stable measures and random integral with respect to them. Definition and properties of vector symmetric random stable measures are introduced in Section 2, where each X-valued p-stable random measure Z_p is associated with a deterministic vector measure Q_p taking values in a certain Banach space. In Section 3 we deal with the random integral of the form $\int f dZ_p$, where f is a real-valued function defined on T. Conditions for the integrability of a function f with respect to Z_p is expressed in terms of the vector measure Q_p (Theorem 3.1 and Theorem 3.4). In particular, if X is of type p then f is Z_p-integrable if and only if $|f|^p$ is Q_p-integrable.

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2. Vector symmetric random stable measures

Let X be a separable Banach space and $(T, \mathcal{A})$ be a measurable space. A mapping $M : \mathcal{A} \longrightarrow L^0_X(\Omega)$ is called an X-valued symmetric random measure on $(T, \mathcal{A})$ if for every sequence (A_n) of disjoint sets from $\mathcal{A}$ the r.v.'s $M(A_n)$ are independent, symmetric and

$$M\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} M(A_n) \quad \text{in} \quad L^0_X(\Omega)$$

An X-valued symmetric random measure F is said to be p-stable $(0 < p \leq 2)$ if for each $A \in \mathcal{A}$, $F(A)$ is a p-stable random variable. In this paper, we always denote an X-valued symmetric p-stable random measure by Z_p. For brevity we write Z for Z_2.

Definition 2.1 Let Z be an X-valued symmetric Gaussian random measure. A function on $\mathcal{A}$ whose value on a set $A \in \mathcal{A}$ is the covariance operator of the Gaussian r.v. $Z(A)$ is called the characteristic measure of Z and denoted by Q.

In order to study properties of the characteristic measure Q it is useful to introduce on $L^2_X(\Omega)$ a concept of inner product whose values are no longer scalar but operators. For $\xi, \eta \in L^2_X(\Omega)$ the inner product of ξ and η, denoted by $[\xi, \eta]$, is an element of $L(X', X)$ defined by

$$[\xi, \eta]a = \int \xi(\omega)(\eta(\omega), a)dP, \quad \forall a \in X'.$$

Here the Bochner integral exists since ξ, η belong to $L^2_X(\Omega)$.

Let us recall the notion of nuclear operators. An operator $T \in L(X', X)$ is called nuclear if there exist two sequences $(x_n) \subset X$", $(y_n) \subset X$ such that $\sum \|x_n\| \|y_n\| < \infty$ and

$$Ta = \sum (a, x_n) y_n, \quad \forall a \in X'. \tag{1}$$

If T is a nuclear operator then the nuclear norm of T is defined by

$$\|T\|_{nuc} = \inf \left\{ \sum \|x_n\| \|y_n\| \right\},$$

where the infimum is taken over all sequences $(x_n) \subset X$", $(y_n) \subset X$ satisfying (1).

The nuclear operators from X' into X form a Banach space under the nuclear norm and denoted by $N(X', X)$. The intersection of $N(X', X)$ and $L^+(X', X)$ is denoted by $N^+(X', X)$.

Theorem 2.1.

1. $[\xi, \eta]$ *is a nuclear operator and*

$$\|[\xi, \eta\|_{nuc.} \leq \|\xi\|_{L_2} \|\eta\|_{L_2}. \tag{2}$$

2. $[\xi, \eta] = [\eta, \xi]^*$
   $[\xi, \eta_1 + \eta_2] = [\xi, \eta_1] + [\xi, \eta_2]$
   $[\eta_1 + \eta_2, \xi] = [\eta_1, \xi] + [\eta_2, \xi]$
   $[t\xi, \eta] = [\xi, t\eta] = t[\xi, \eta] \qquad \forall t \in R.$
3. $[\xi, \xi] \in L^+(X', X)$ *and* $\|[\xi, \xi\|_{nuc.} \leq \|\xi\|^2_{L_2}$.
4. *If* X *is of type 2 then there exists a constant* $C > 0$ *such that*
$$\|\xi\|^2_{L_2} \leq C\|[\xi, \xi\|_{nuc}$$
   for all X*-valued symmetric Gaussian r.v.'s* ξ.
5. *If* $\lim \xi_n = \xi, \lim \eta_n = \eta$ *in* $L^2_X(\Omega)$ *then*
$$\lim[\xi_n, \eta_n] = [\xi, \eta] \quad \textit{in} \quad N(X', X).$$

Proof. 1) Let ξ, η be simple r.v.'s
$$\xi = \sum x_i I_{A_i}, \quad \eta = \sum y_i I_{A_i}$$
where I_A denotes the characteristic function of the set A. We have
$$[\xi, \eta]a = \sum_{i=1}^{n} x_i(y_i, a)P(A_i).$$
Hence
$$\|[\xi, \eta]\|_{nuc} \leq \sum_{i=1}^{n} \|x_i\|.\|y_i\|P(A_i) = \int \|\xi\|.\|\eta\|dP \leq \|\xi\|_{L_2}\|\eta\|_{L_2}.$$
Now let ξ, η be arbitrary elements in $L^2_X(\Omega)$. There exist simple r.v.'s $(\xi_n), (\eta_n)$ such that $\xi_n \to \xi$ and $\eta_n \to \eta$ in $L^2_X(\Omega)$. We have
$$\|[\xi_n, \eta_n]\|_{nuc} \leq \|\xi_n\|_{L_2}\|\eta_n\|_{L_2} \tag{3}$$
and
$$\|[\xi_n, \eta_n] - [\xi_m, \eta_m]\|_{nuc} \leq \|\xi_n\|_{L_2}\|\eta_n - \eta_m\|_{L_2} + \|\eta_m\|_{L_2}\|\xi_n - \xi_m\|_{L_2}$$
which converges to 0 as $m, n \to \infty$. Since $([\xi_n, \eta_n]a, b) \to ([\xi, \eta]a, b)$ for all $a, b \in X'$ it follows that $[\xi, \eta]$ is a nuclear operator. Letting $n \to \infty$ in (3) we get (2).

The assertions 2), 3) and 5) are easy to prove. The assertion 4) follows immediately from a result of Chevet (see [1]). □

We call $[\xi, \xi]$ the covariance operator of the X-valued r.v. $\xi \in L^2_X(\Omega)$.

Let $G(X)$ denote the set of covariance operators of X-valued Gaussian symmetric r.v.'s. In view of Theorem 2.1 we have the inclusion $G(X) \subset N^+(X', X)$. Moreover, it was shown in [1] that the equality $G(X) = N^+(X', X)$ holds if and only if X is type 2.

Theorem 2.2. *The characteristic measure* Q *is a function from* $\mathcal{A}$ *into* $G(X)$ *having the following properties*

1. $[Z(A), Z(B)] = Q(A, B)$ *for all* $A, B \in \mathcal{A}$.
2. Q *is* σ*-additive in the nuclear norm.*

3. *Q is non-negative in the sense that for all sequences $(A_k)_1^n \subset \mathcal{A}$ and all sequences $(a_k)_1^n \in X'$ we have*
$$\sum_i \sum_j (Q(A_iA_j)a_i, a_j) \geq 0.$$

Proof. 1) If A, B are disjoint then $Z(A)$ and $Z(B)$ are independent symmetric so that $[Z(A), Z(B)] = 0$. For arbitrary sets $A, B \in \mathcal{A}$ by Theorem 2.1 we have
$$\begin{aligned}[Z(A), Z(B)] &= [Z(AB) + Z(A \setminus AB), Z(AB) + Z(B \setminus AB)] \\ &= [Z(AB), Z(AB)] = Q(AB).\end{aligned}$$

2) Let $A = \bigcup_{n=1}^{\infty} A_n$ where (A_n) are disjoint sets. By Hoffmann-Jorgensen's theorem (see [3] Theorem 5.5) $Z(A) = \sum_{n=1}^{\infty} Z(A_n)$ in $L^2_X(\Omega)$. Hence by Theorem 2.1
$$\begin{aligned}Q(A) = [Z(A), Z(A)] &= \lim_n \left[\sum_{k=1}^{n} Z(A_k), \sum_{k=1}^{n} Z(A_k)\right] \\ &= \lim_n \sum_{k=1}^{n} [Z(A_k), Z(A_k)] \\ &= \sum_{k=1}^{\infty} Q(A_k) \quad \text{in} \quad N(X', X).\end{aligned}$$

3) We have
$$\sum_i \sum_j (Q(A_iA_j)a_i, a_j) = \sum_i \sum_j ([Z(A_i), Z(A_j)]a_i, a_j)$$
$$\sum_i \sum_j \int (Z(A_i), a_i)(Z(A_j), a_j) dP = E\left[\sum_{i=1}^{n} (Z(A_i), a_j)\right]^2 \geq 0.$$
□

A characterization of the class of characteristic measures of vector symmetric Gaussian random measures is given by the following theorem.

Theorem 2.3. *Let Q be a function from $\mathcal{A}$ into $G(X)$. The following assertions are equivalent:*

1. *Q is a characteristic measure of some X-valued symmetric Gaussian random measure.*
2. *Q is non-negative definite and σ-additive in the nuclear norm.*
3. *Q is non-negative definite and T-weakly σ-additive in the following sense: For each $a \in X'$ and each sequence (A_n) of disjoint sets from $\mathcal{A}$ we have*
$$\left(Q\left(\bigcup_{n=1}^{\infty} A_n\right) a, a\right) = \sum_{i=1}^{\infty} (Q(A_n)a, a).$$

Proof. By Theorem 2.2 only the implication 3) $\to$ 1) needs a proof. For $g = (A, a)$, $h = (B, b)$, where $A, B \in \mathcal{A}$, $a, b \in X'$ we define the function $K(g, h)$ by $K(g, h) = (Q(AB)a, b)$. From the assumption it follows that K is non-negative definite and symmetric. Hence there exists a Gaussian process $f(A, a)$ indexed by the set $\mathcal{A} \times X'$ such that

$$Ef(A,a)f(B,b) = (Q(AB)a, b)\,. \tag{4}$$

For each fixed $A \in \mathcal{A}$, define the mapping $T_A : X' \to L_2(\Omega)$ by $T_A a = f(A, a)$. We claim that T_A is linear. Indeed,

$$\begin{aligned} E[T_A(a+b) - T_A a - T_B b]^2 &= (Q(A)(a+b), a+b) + (Q(A)a, a) + (Q(A)b, b) \\ &\quad + 2(Q(A)a, b) - 2(Q(A)(a+b), a) \\ &\quad - 2(Q(A)(a+b), b) = 0, \end{aligned}$$

which shows that $T_A(a+b) = T_A a + T_B b$. Similarly, $T_A(ta) = tT_A a$ for each $t \in R$. Since $E|T_A a|^2 = (Q(A)a, a)$ and $Q(A) \in G(X), T_A$ is decomposed by an X-valued symmetric Gaussian r.v. denoted by $Z(A)$ i.e.

$$\forall a \in X' \qquad T_A a = (Z(A), a) \quad \text{a.s.}$$

Now we show that the mapping $A \mapsto Z(A)$ yields an X-valued symmetric Gaussian random measure with the characterization measure Q. Indeed,we have

$$[Z(A), Z(A)] = Q(A).$$

Further, for disjoint sets $(A_n) \subset \mathcal{A}$ and arbitrary elements $(a_n) \subset X'$ by (4) the Gaussian r.v.'s $(Z(A_k), a_k)$ are uncorrelated so that they are independent. Hence so are the r.v.'s $Z(A_n)$. For each $a \in X'$ we have

$$\left[(Z(A), a) - \sum_{k=1}^{n} (Z(A_k), a)\right]^2 = (Q(A)a, a) - \sum_{k=1}^{n} (Q(A_k)a, a)$$

which tends to 0 as $n \to \infty$. Consequently, $(Z(A)a, a) = \sum_{k=1}^{n} (Z(A_k), a)$ in $L_2(\Omega)$. According to the Ito-Nisio theorem we conclude that

$$Z(A) = \sum_{k=1}^{\infty} Z(A_k) \quad \text{in} \quad L_X^0(\Omega).$$

□

Example 2.1. Let $H : T \to L^+(X', X)$ be a function on T with values in $L^+(X', X)$ such that H is T-weakly integrable w.r.t. a finite positive measure μ on $(T, \mathcal{A})$ in the sense that for each $A \in \mathcal{A}$ there exists an operator $H_A \in L^+(X', X)$ such that

$$(H_A a, a) = \int_A (H(t)a, a)\, d\mu, \qquad \forall a \in X'.$$

We shall prove that if $H_T \in G(X)$ then there exists an X-valued symmetric Gaussian random measure Z with the characteristic measure Q given by $Q(A) = H_A$. Indeed, since $(H_A a, a) \le (H_T a, a)$, $\forall a \in X'$, by the key property of Gaussian covariance operators (see [4]) it follows that $H_A \in G(X)$. Put $Q(A) = H_A$. For $(A_k)_{k=1}^n \subset \mathcal{A}$ and $(a_k)_{k=1}^n \subset X'$ we have

$$\sum_{i,j=1}^{n} (Q(A_i A_j)a_i, a_j) = \sum_{i,j=1}^{n} \int\limits_T \left(H(t)I_{A_i}(t)a_i, I_{A_j}(t)a_j\right) d\mu(t)$$
$$= \int\limits_t (H(t)a(t), a(t))\, d\mu(t) \geq 0,$$

where $a(t) = \sum\limits_{i=1}^{n} I_{A_i}(t)a_i$. Hence Q is non-negative definite. The T-weak σ-additivity of Q follows from the T-weak integrability of H. Thus the assertion follows from Theorem 2.3.

Example 2.2. (Vector Wiener random measures) Given an operator $R \in G(X)$, consider a function H given by $H(t) = R, \ \ \forall t \in T$. Clearly, H is T-weakly integrable and $H_A = \mu(A)R$. By Example 2.1 there exists an X-valued symmetric Gaussian random measure W such that for each $A \in \mathcal{A}$, the covariance operator of $W(A)$ is $\mu(A)R$. We call W the X-valued Wiener random measure with the parameters (μ, R).

Next, we consider the case $0 < p < 2$. Let S be the unit sphere of X, endowed with the metric generated by the norm of X. Let $\mathcal{M}(S)$ denote the set of all real-valued measures of bounded variation on S and $\mathcal{M}^+(S)$ denote the set of finite non-negative symmetric measures on S. $\mathcal{M}(S)$ is a Banach space under the usual operations of addition and multiplication by numbers. The norm of a measure $\lambda \in \mathcal{M}(S)$ is given by $\|\lambda\|_{\mathcal{M}} = |\lambda|(S)$, where $|\lambda|$ stands for the variation of λ.

It is known (see [4], Th. 6.4.4) that for each X-valued symmetric p-stable r.v. ξ there exists a unique finite symmetric measure $\Gamma_\xi \in \mathcal{M}^+(S)$ such that the characteristic function of ξ is given by

$$E \exp\left[i(\xi, a)\right] = \exp\left\{-\int_S |(x,a)|^p d\Gamma_\xi\right\}.$$

From now on, Γ_ξ is called the spectral measure of ξ.

Denote by $\mathcal{S}_p(X)$ the set of spectral measures of X-valued symmetric p-stable r.v.'s. We have the inclusion $\mathcal{S}_p(X) \subset \mathcal{M}^+(S)$. Moreover, it is known that (see [4]) $\mathcal{S}_p(X)$ coincides with $\mathcal{M}^+(S)$ if and only if X is of stable type p.

Some useful properties of the correspondence $\xi \mapsto \Gamma_\xi$ are listed in the following theorem.

Theorem 2.4.

1. $\Gamma_{t\xi} = |t|^p \Gamma_\xi \quad \forall t \in R$.
2. *There exists a constant* $C > 0$ *depending only on* r, p, $(0 < r < p)$ *such that*

$$\|\Gamma_\xi\| \leq C \left\{E\|\xi\|^r\right\}^{p/r}.$$

Moreover, if X is of stable type p then there exists a constant $K > 0$ depending only on r, p such that

$$\{E\|\xi\|^r\}^{p/r} \leq K\|\Gamma_\xi\|.$$

3. *Let (ξ_n) be a sequence of X-valued symmetric independent p-stable r.v.'s such that the series $\sum_{n=1}^{\infty} \xi_n$ converges to a X-valued r.v. ξ in $L_X^0(\Omega)$. Then we have $\Gamma_\xi = \sum_{n=1}^{\infty} \Gamma_{\xi_n}$ in $\mathcal{M}^(S)$ and we have*

$$\|\Gamma_\xi\| = \sum_{n=1}^{\infty} \|\Gamma_{\xi_n}\|.$$

Proof. The assertion 1) is easy to prove. The assertion 2) is an immediate consequence of Corollary 7.3.5 and Proposition 7.5.4 in [4]. Now we prove 3). Put $x_n = \sum_{i=1}^{n} \xi_i$. It is easy to see that $\Gamma_{x_n} = \sum_{i=1}^{n} \Gamma_{\xi_i}$. Since $\Gamma_{\xi_i} \in \mathcal{M}^+(S)$ we have

$$\|\Gamma_{x_n}\| = \sum_{i=1}^{n} \|\Gamma_{\xi_i}\|, \tag{5}$$

$$\|\Gamma_{x_n} - \Gamma_{x_m}\| = \sum_{i=m+1}^{n} \|\Gamma_{\xi_i}\|. \tag{6}$$

Since x_n converges to ξ in probability by Proposition 6.6.5 in [4] Γ_{x_n} converges weakly to Γ_ξ which implies that $\|\Gamma_{x_n}\|$ converges to$\|\Gamma_\xi\|$ i.e.

$$\|\Gamma_\xi\| = \sum_{n=1}^{\infty} \|\Gamma_{\xi_n}\|.$$

From this and (6) it follows that Γ_{x_n} converges in $\mathcal{M}(S)$. The limit must be Γ_ξ since Γ_{x_n} converges weakly to Γ_ξ. □

Definition 2.2. Let Z_p be an X-valued symmetric p-stable random measure. A function on $\mathcal{A}$, whose value on a set $A \in \mathcal{A}$ is the spectral measure of $Z_p(A)$ is called a characteristic measure of Z_p and it is denoted by Q_p.

Theorem 2.5. *The characteristic measure Q_p of Z_p is a mapping from $\mathcal{A}$ into $\mathcal{S}_p(X)$ possessing the following properties:*

1. *Q_p is σ-additive in the norm of $\mathcal{M}(S)$. Thus Q_p is a vector measure with values in the Banach space $\mathcal{M}(S)$.*
2. *Q_p is of bounded variation and the variation $|Q_p|$ is given by*

$$|Q_p|(A) = \|Q_p(A)\|_{\mathcal{M}}.$$

3. *If X is of stable type p then there exists a constant $K > 0$ such that*

$$P\{\|Z_p(A)\| > t\} \leq Kt^{-p}|Q_p|(A), \quad \forall A \in \mathcal{A}, \quad \forall t \in R.$$

Proof. The assertion 1) follows easily from Theorem 2.4. In order to prove 2) let $\{B_1, B_2, \dots, B_n\}$ be a finite partition of A. Then

$$\sum_{i=1}^{m} \|Q_p(B_i)\| = \Big\| \sum_{i=1}^{m} Q_p(B_i) \Big\| = \|Q_p(A)\|.$$

Hence $|Q_p|(A) = \|Q_p(A)\|$ as claimed. Finally, by Propositions 7.5.4 and 7.3.1 in [4] we get

$$P\{\|Z_p(A)\| > t\} \leq Kt^{-p}\|Q_p(A)\| = Kt^{-p}|Q_p|(A),$$

where K is a constant. □

3. Random integral for real-valued deterministic functions

The stochastic integral of real-valued deterministic functions w.r.t. vector random measures was investigated in [9]. Let us recall the definition. Let M be an X-valued symmetric random measure with the control measure μ. If $f : T \to R$ is a simple function, $f = \sum\limits_{i=1}^{n} t_i I_{A_i}$ then the random integral of f w.r.t. M is $\int f dM = \sum\limits_{i=1}^{n} t_i M(A_i)$. A function f is said to be M-integrable if there exists a sequence of simple functions (f_n) such that $\lim f_n(t) = f(t) \quad \mu -$ a.s. and the sequence $\{\int f_n dM\}$ converges in $L^0_X(\Omega)$. If f is M-integrable then we put

$$\int\limits_T f\, dM = p - \lim \int\limits_T f_n\, dM.$$

The set of M-integrable functions is denoted by $\mathcal{L}_X(M)$.

Let Z be an X-valued symmetric Gaussian random measure with the characteristic measure Q. We notice that there exists a control measure for Z. Indeed, by Bartle-Dunford-Schartz's theorem (see [2], Corollary 6) there is a finite non-negative measure μ such that $Q(A) = 0$ whenever $\mu(A) = 0$. Clearly, μ is a control measure for Z.

Theorem 3.1.

1. *If functions f and g are Z-integrable then fg is Q-integrable and for each $A \in \mathcal{A}$ we have*

$$\left[\int\limits_A f dZ, \int\limits_A g dZ\right] = \int\limits_A fg dQ.$$

2. *A function f is Z-integrable if and only if the function $|f|^2$ is Q-integrable and*

$$\int |f|^2\, dQ \in G(X).$$

Proof. 1) By definition there exist simple functions (f_n) and (g_n) such that $\lim f_n = f$ and $\lim g_n = g$ for μ-almost all t and

$$p-\lim \int_A f_n dZ = \int_A f dZ, \quad p-\lim \int_A g_n dZ = \int_A g dZ.$$

Since $\int_A f_n dZ$ and $\int_A g_n dZ$ are Gaussian symmetric r.v.'s, they also converge in $L^2_X(\Omega)$ (see [3]). It is easy to check that

$$\left[\int_A f_n dZ, \int_A g_n dZ\right] = \int_A f_n g_n dQ.$$

By Theorem 2.1 it follows that $\int_A f_n g_n dQ$ converges to $\left[\int_A f dZ, \int_A g dZ\right]$ in $N(X', X)$. Sin $f_n g_n$ converges to $fg \quad \mu-$a.e. we conclude that fg is Q-integrable and for each $A \in \mathcal{A}$ we have

$$\left[\int_A f dZ, \int_A g dZ\right] = \int_A fg dQ.$$

2) The necessity follows from what has been proved. Conversely, suppose $|f|^2$ is Q-integrable and $\inf |f|^2 dQ \in G(X)$. Let

$$A_n = \{t : |f(t)| \leq n\}, B_n = A_n \setminus A_{n-1}, f_n = I_{A_n} f.$$

Since f_n is bounded it is Z-integrable (see [9] Theorem 2.4). Put $x_n = \int f_n dZ$, $S_n = \int g_n dZ = \sum_{i=1}^{n} x_i$ where $g_n = I_{A_n} f$. Since (B_n) are disjoint the r.v.'s (x_n) are independent and symmetric. The characteristic function of S_n is

$$\exp\left\{-\sum_{i=1}^{n}\left([\int |f_k|^2 dQ]a, a\right)\right\} = \exp\left\{-\left([\int_{A_n} |f|^2 dQ]a, a\right)\right\},$$

which converges to $\exp\left\{-\left([\int |f|^2 dQ]a, a\right)\right\}$ when $n \to \infty$. Therefore, by the Ito-Nisio theorem the sequence (S_n) converges in $L^0_X(\Omega)$. Since $\lim g_n = f \ \ \mu$-a.e. from Theorem 2.3 in [9] we conclude that f is Z-integrable. □

Corollary 3.1. *Suppose that X is of type 2. Then*

1. *A function f is Z-integrable if and only if $|f|^2$-is Q-integrable.*
2. *The inclusion $L_2(T, \mathcal{A}, |Q|) \subset \mathcal{L}_X(Z)$ holds. Moreover, there exists a constant $K > 0$ such that*

$$E\|\int f dZ\|^2 \leq K \int |f|^2 d|Q|$$

for all $f \in L_2(T, \mathcal{A}, |Q|)$, where $|Q|$ stands for the variation of Q.

Proof. The assertion 1) is a direct consequence of Theorem 3.1 and the fact that $G(X) = N^+(X', X)$ provided X is of type 2. We prove the assertion 2). Let f be a simple function, $f = \sum t_i I_{A_i}$. Since X is of type 2 we have

$$E\|\int f dZ\|^2 = E\|\sum t_i Z(A_i)\|^2 \leq C_1 \sum |t_i|^2 E\|Z(A_i)\|^2,$$

where C_1 is a constant. By Theorem 2.1 it follows that

$$E\|Z(A_i)\|^2 \leq C_2\|Q(A_i)\|_{nuc} \leq C_2|Q|(A_i),$$

where C_2 is a constant. From this we obtain

$$E\|\int f dZ\|^2 \leq C_1C_2 \sum |t_i|^2|Q|(A_i) = K\int |f|^2 d|Q|,$$

where $K = C_1C_2$. Since the set of simple functions is dense in $L_2(T, \mathcal{A}, |Q|)$ the assertion 2) follows. □

Theorem 3.2. *Let W be a X-valued Wiener random measure with the parameters (μ, R) (Example 2.2). A function f is W-integrable if and only if $f \in L_2(T, \mathcal{A}, \mu)$. Moreover, for any orthonormal basis (e_n) in $L_2(T, \mathcal{A}, \mu)$ we have*

1. *$\{\int e_n dW\}$ is a sequence of X-valued independent symmetric Gaussian r.v.'s with the same covariance operator R.*
2. *For each $f \in L_2(T, \mathcal{A}, \mu)$ we have*

$$\int f dW = \sum_{n=1}^{\infty}(f, e_n)\int e_n\, dW \tag{7}$$

where $(.,.)$ stands for the scalar product in $L_2(T, \mathcal{A}, \mu)$. The series (7) converges a.s. in the norm topology of X.

Proof. Only the assertions 1) and 2) need to be proved. We have by Theorem 3.1 $\left[\int e_n dW, \int e_m dW\right] = (e_m, e_n)R$. Thus the assertion 1) is proved. Now, for each $a \in X'$

$$E\left[\Big(\int f dW, a\Big) - \sum (f, e_k)\Big(\int e_k dW, a\Big)\right]^2 = (Ra, a)\left[\int |f|^2 d\mu - \sum_{k=1}^{n}|(f, e_k)|^2\right]$$

which converges to 0 as $n \to \infty$. Hence by the Ito-Nisio theorem the expansion (7) is proved. □

As an application of the above theorem, let us investigate the possibility of representing an X-valued symmetric Gaussian process as a Gaussian random series. Let $\xi(u), u \in I$, be an X-valued symmetric Gaussian process indexed by the parameter I. By Theorem 2.7 in [11] we get the following statement:

There exists a sequence (α_n) of real-valued Gaussian i.i.d. random variables and a sequence $(f_n(u))$ of X-valued deterministic functions defined on I such that for each $u \in I$ we have

$$\xi(u) = \sum_{n=1}^{\infty} \alpha_n f_n(u) \qquad a.s.$$

in the norm of X.

Now it is natural to ask if there exists a sequence (α_n) of X-valued Gaussian i.i.d. random variables and a sequence $(f_n(u))$ of real-valued deterministic functions defined on I such that for each $u \in I$ we have

$$\xi(u) = \sum_{n=1}^{\infty} \alpha_n f_n(u) \qquad \text{a.s.} \tag{8}$$

in norm of X.

Let $K(u,v) = [\xi(u), \xi(v)]$ be the covariance function of ξ. $K(u,v)$ takes values in $N^+(X', X)$ and non-negative (Theorem 2.2). It is easy to check that if ξ can be represented in the form (8) then

$$K(u,v) = k(u,v)R, \tag{9}$$

where $k(u,v)$ is a real-valued non-negative function, and R is a covariance operator. Conversely we have

Theorem 3.3. *If the covariance function $K(u,v)$ of ξ is of the form (9) then there exist a sequence (α_n) of X-valued Gaussian i.i.d. random variables and a sequence $(f_n(u))$ of real-valued deterministic functions defined on I such that ξ is equivalent to the process η defined by*

$$\eta(u) = \sum_{n=1}^{\infty} \alpha_n f_n(u) \qquad \textit{a.s.}$$

Proof. Since $k(u,v)$ is non-negative there exist a measurable $(T, \mathcal{A}, \mu)$ and a family $(h_u)\,, u \in I$ of functions in $L_2(T, \mathcal{A}, \mu)$ such that

$$k(u,v) = \int_T h_u(t) h_v(t)\, d\mu(t).$$

Let W be an X-valued Wiener random measure with the parameter (μ, R). Define an X-valued Gaussian process η by

$$\eta(u) = \int_T h_u(t)\, dW(t).$$

Then by Theorem 3.1

$$[\eta(u), \eta(v)] = \Big(\int h_u h_v\, d\mu\Big) R = k(u,v)R = K(u,v).$$

Two Gaussian processes η and ξ the same covariance function so they have same finite dimensional distributions. From (7) we get

$$\eta(u) = \int h_u dW = \sum_{n=1}^{\infty} (h_u, e_n) \int e_n\, dW \quad \text{a.s.}$$

Put $\alpha_n = \int e_n dW,\ f_n(u) = (h_u, e_n)$ we obtain the desired claim. □

Now we consider the case $0 < p < 2$. Clearly, the variation $|Q_p|$ of the characteristic measure Q_p of Z_p is a control measure for Z_p.

Theorem 3.4. *A function f is Z_p-integrable if and only if the function $|f|^p$ is Q_p-integrable and*

$$\int |f|^p \, dQ_p \in \mathcal{S}_p(X).$$

In this case, $\int |f|^p \, dQ_p$ is exactly the spectral measure of the X-valued symmetric p-stable r.v. $\int f dZ_p$.

To prove Theorem 3.4 we shall need the following lemmas.

Lemma 1. *Let (ξ_n) be a sequence of X-valued symmetric p-stable r.v.'s such that ξ_n converges to ξ in $L^0_X(X)$. Then for each $r < p$ the sequence (ξ_n) also converges to ξ in $L^r_X(\Omega)$.*

Proof. It follows from Proposition 7.3.11 in [4]. □

Lemma 2. *Let g be non-negative, $|Q_p|$-integrable. Then g is Q_p-integrable and we have*

$$\Big\| \int g dQ_p \Big\|_{\mathcal{M}} = \int g \, d|Q_p|.$$

Proof. It is easy to see that the lemma holds for simple functions g. Now let g be non-negative, $|Q_p|$-integrable. There exists an increasing sequence (g_n) of simple functions converging to g everywhere and $\int (g - g_n) d|Q_p| \to 0$. Then

$$\Big\| \int\limits_A g_n dQ_p - \int\limits_A g_m dQ_p \|_{\mathcal{M}} = \int_A |g_n - g_m| d|Q_p| \to 0 \quad \text{as} \quad m, n \to 0.$$

Hence by the definition of the integral w.r.t. vector measure we infer that g is Q_p-integrable and

$$\Big\| \int g dQ_p \Big\|_{\mathcal{M}} = \lim \Big\| \int g_n dQ_p \Big\|_{\mathcal{M}} = \lim \int g_n d|Q_p| = \int g \, d|Q_p|.$$ □

Proof of Theorem 3.4. Suppose that f is a simple function $f = \sum t_i I_{A_i}$. Put $\xi_i = Z_p(A_i)$, $\xi = \int f dZ_p = \sum t_i \xi_i$. By Theorem 2.4 the spectral measure of ξ is

$$\Gamma_\xi = \sum \Gamma_{t_i \xi_i} = \sum |t_i|^p \Gamma_{x i_i} = \sum |t_i|^p Q_p(A_i) = \int \|f|^p dQ_p. \tag{10}$$

In view of Lemma 2 and Theorem 2.4 we get

$$\int |f|^p d|Q_p| = \|\Gamma_\xi\| \leq C\{E\|\xi\|^r\}^{p/r} = C\{E\| \int f dZ_p\|^r\}^{p/r}, \tag{11}$$

where C is a constant.

Let f be an arbitrary Z_p-integrable function. There exist simple functions (f_n) such that $\lim f_n = f \quad |Q_p|$-a.e and $\int f_n dZ_p$ converges to $\int f dZ_p$ in $L^0_X(\Omega)$. By Lemma 1, the inequality (11) and Fatou's Lemma we get

$$\int |f|^p d|Q_p| \leq C \Big\{ E \| \int f dZ_p \|^r \Big\}^{p/r}. \tag{12}$$

Thus $|f|^p$ is $|Q_p|$-integrable so that it is Q_p-integrable by Lemma 2.

Finally, we shall show that $\int |f|^p dQ_p$ is the spectral measure of $\int f dZ_p$. Indeed, there exist simple functions (f_n) such that $|f_n(t)| \leq |f(t)|$ and $\lim f_n(t) = f(t)$ for all t. By the dominated convergence theorem for vector measure we have

$$\lim \int |f_n|^p dQ_p = \int |f|^p \, dQ_p \quad \text{in} \quad \mathcal{M}(S). \tag{13}$$

On the other hand, by the dominated convergence theorem for vector random measure (see [9] Cololllary 3.4) we get

$$\lim \int f_n dZ_p = \int f \, dZ_p \quad \text{in } L^0_X(\Omega).$$

Thus the spectral measure of $\int f_n dZ_p$ converges weakly to that of $\int f dZ_p$. From this and together with (10) and (13) it follows that $\int |f_n|^p dQ_p$ is precisely the spectral measure of $\int f dZ_p$.

In order to prove the converse, let

$$A_n = \{t : |f(t) \leq n\}, \quad B_n = A_n \setminus A_{n-1}, \quad f_n = I_{B_n} f.$$

Since f_n is bounded,it is Z_p-integrable (see [9]). Put

$$x_n = \int f_n dZ_p, \quad S_n = \int g_n dZ_p = \sum_{i=1}^{n} x_i$$

where $g_n = I_{A_n} f$. Since (B_n) are disjoint the r.v.'s (x_n) are independent and symmetric. The characteristic function of S_n is $\exp\left\{-\int_S |(x,a)|^p d\Gamma_n\right\}$ where $\Gamma_n = \int_{A_n} |f|^p dQ_p$ which converges to $\exp\left\{-\int_S |(x,a)|^p d\Gamma\right\}$ where $\Gamma = \int |f|^p dQ_p$. The rest of the proof is identical to the last part of the proof of Theorem 3.1. □

Theorem 3.5. *Suppose that X is of stable type p. Then*

1. *A function f is Z_p-integrable if and only if $|f|^p$ is Q_p-integrable.*
2. *A function f is Z_p-integrable if and only if $|f|^p$ is $|Q_p|$-integrable. Moreover, there exist a constant $K > 0$ such that*

$$P\left\{\| \int f dZ_p \| > t\right\} \leq K t^{-p} \int |f|^p d|Q_p|$$

for all $f \in L_p(T, \mathcal{A}, |Q_p|)$ and all $t \in R$.

Proof. The assertion 1) is a direct consequence of Theorem 3.4 and the fact that $\mathcal{S}_p(X) = \mathcal{M}^+(S)$ provided that X is of stable type p. Now we prove 2). If $f \in L_p(T, \mathcal{A}, |Q_p|)$ then $|f|^p$ is Q_p-integrable so f is Z_p-integrable. The converse follows from (12). Finally, by using Lemma 2 of Theorem 3.4, together Proposition 7.5.4 and 7.3.1 in [4] we get

$$P\left\{\| \int f dZ_p \| > t\right\} \leq K t^{-p} \left\| \int |f|^p dQ_p \right\|_{\mathcal{M}} = K t^{-p} \int |f|^p d|Q_p|,$$

where K is a constant. □

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