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ON A GENERALIZED COX-ROSS-RUBINSTEIN OPTION MARKET MODEL

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Abstract. This paper considers a generalization of the Cox-Ross-Rubinstein model for an option market. Some limit theorems for the stock price process and their application to approximately determining the rational price and hedging strategies of standard European option are established.

1. Introduction

As well known, the simplified option market model considered by J. C. Cox, R. A. Ross, M. Rubinstein [2] and recently by A. N. Shirijaev, Yu. M. Kabanov, D. O. Kramkov, A. V. Mel'nikov [6] and by S. T. Rachev, L. Ruschendorf [5], consists of two processes:

(i) a risk free asset (for example a bank account) given by

$$B_n = B_0(1+r)^n \quad \text{or} \quad B_n = B_{n-1}(1+r),$$

where B_0 is known, $n = 1, 2, \dots, N$.

(ii) a stock price process possessing the dynamics

$$S_n = S_{n-1}(1+\rho_n), \quad n = 1, 2, \dots, N,$$

or equivalently

$$S_n = S_0 \prod_{k=1}^{n} (1+\rho_k), \quad n = 1, 2, \dots, N,$$

where S_0 is given and $\{\rho_k, k = 1, 2, \dots, N\}$ is a sequence of i.i.d. variables such that

$$\rho_k = \begin{cases} u & \text{with probability } p, \\ d & \text{with probability } q = 1-p,\ 0 < p < 1,\ -1 < d < u. \end{cases}$$

However, we observe that $1 + \rho_k = \dfrac{S_k}{S_{k-1}}$ does not always take two values $1+u$ and $1+d$ with constant probabilities p and q. For example, this is the case when S_k is the value at moment $t = \dfrac{kT}{N}$ of a diffusion price process defined by

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$dS_t = S_t(\mu_t dt + \sigma_t dW_t)$, $0 \leq t \leq T$, where W_t is a Wiener process. Therefore, a natural generalization of the structure of the stock price sequence $\{S_n\}$ can be considered as follows.

The relative increments of the stock price $\rho_k = \dfrac{(S_k - S_{k-1})}{S_{k-1}}$ is assumed to take values u_k and d_k with the respective probabilities

$$p_k = P\{\rho_k = u_k\} \quad \text{and} \quad q_k = P\{\rho_k = d_k\} = 1 - p_k, \tag{1.1}$$

$$-1 < d_k < u_k,\ \rho_k = \rho_k(N),\ u_k = u_k(N),\ p_k = p_k(N) = 1 - q_k(N). \tag{1.2}$$

In this article we will study an option market defined by the two following processes

(i) a risk free asset process given by

$$B_n = B_{n-1}(1 + r_n), \tag{1.3}$$

where B_0 is known and $r_n = r_n(N) > 0$, $n = 1, 2, \dots, N$.

(ii) a stock price process

$$S_n = S_{n-1}(1 + \rho_n), \tag{1.4}$$

where S_0 is known and ρ_n satisfies (1.1), (1.2), $n = 1, 2, \dots, N$.

The Cox-Ross-Rubinstein option market model which is also called the binomial model and its generalization defined by (1.1)-(1.4) are the rather rare cases of a complete market model of discrete time, where one can well define the fair or rational price and hedging strategy of any option contingent claim (see Section 4.1 below). However, as we can see in Sections 3 and 4, our generalized model, is a good approximation for the option pricing model of continuous time, where the stock price S_t is given by $dS_t = S_t(\alpha_t dt + \sigma_t dW_t)$, W_t being a Brownian motion.

For the sake of simplicity, the index N in the expressions of r_n, ρ_n, p_n, q_n will be deleted in the sequel.

We shall prove that under some conditions on u_k, d_k, p_k, $\ln\left(\dfrac{S_N}{S_0}\right)$ will be asymptotically normal as $N \to +\infty$. The asymptotic property of $\ln\left(\dfrac{S_N}{S_0}\right)$ will be used for pricing standard European option. The functional convergence in the space D of cadlag functions with Skorokhod's metric will be also shown. The above convergence will also be useful for hedging some contingent claim.

2. Limit distribution of $\ln\left(\dfrac{S_N}{S_0}\right)$

Suppose the price of some stock has the structure (1.1), (1.2), (1.4). Put $Z_N = \ln\left(\dfrac{S_N}{S_0}\right)$.

Lemma 2.1. *Let*

$$\alpha_N = \max_{1\le k\le N} \max(|u_k|, |d_k|) \to 0, \tag{2.1}$$

$$f_n(t) = Ee^{itZ_n}, \tag{2.2}$$

then

$$\begin{aligned}\ln f_N(t) = it\Big\{ \sum_{k=1}^{N} \big[(p_k u_k + q_k d_k) - \frac{1}{2}(p_k u_k^2 + q_k d_k^2)\big]\Big\} \\ - \frac{1}{2}t^2 \Big\{ \sum_{k=1}^{N} \big[(p_k u_k^2 + q_k d_k^2) - (p_k u_k + q_k d_k)^2\big]\Big\} \\ + \theta \max(|t|, |t^3|) \sum_{k=1}^{N} (p_k u_k^2 + q_k d_k^2)\alpha_N,\end{aligned} \tag{2.3}$$

where θ stands for a parameter bounded by some positive constant C.

Proof. Since $Z_N = \sum_{n=1}^{N} \ln(1+\rho_n)$ is the sum of independent variables taking only two values $\ln(1+u_k)$, $\ln(1+d_k)$ with respective probabilities p_k and q_k, we have

$$f_N(t) = \prod_{k=1}^{N} g_k(t) \tag{2.4}$$

with

$$\begin{aligned} g_k(t) &= E[\exp(it \ln(1+\rho_k))] \\ &= p_k \exp(it\ln(1+u_k)) + q_k \exp(it \ln(1+d_k)). \end{aligned} \tag{2.5}$$

Representing $\exp(it\ln(1+u_k))$ as a series in $\ln(1+u_k)$ we obtain

(2.6)

$$\exp(it\ln(1+u_k)) = 1 + it\ln(1+u_k) - \frac{1}{2}t^2 \ln^2(1+u_k) + \theta|t|^3(|\ln(1+u_k)|)^3.$$

Noticing that

$$\begin{aligned}\ln(1+u_k) &= u_k - \frac{u_k^2}{2} + \theta(|u_k|)^3 \approx u_k, \\ \ln^2(1+u_k) &= u_k^2 + \theta(|u_k|)^3 \approx u_k^2,\end{aligned}$$

we have

$$e^{it\ln(1+u_k)} = 1 + it\Big(u_k - \frac{u_k^2}{2}\Big) - t^2\frac{u_k^2}{2} + \theta\max(|t|, |t|^3)|u_k|^3. \tag{2.7}$$

Similarly

$$e^{it\ln(1+d_k)} = 1 + it\Big(d_k - \frac{d_k^2}{2}\Big) - t^2\frac{d_k^2}{2} + \theta\max(|t|, |t|^3)|d_k|^3. \tag{2.8}$$

It follows from (2.5), (2.7), (2.8) that

$$g_k(t) = 1 + it\big[p_k u_k + q_k d_k - \frac{1}{2}(p_k u_k^2 + q_k d_k^2)\big] - \frac{1}{2}t^2(p_k u_k^2 + q_k d_k^2) + \theta \max(|t|, |t|^3)(p_k|u_k|^3 + q_k|d_k|^3). \tag{2.9}$$

Therefore

$$\ln g_k(t) = it\big[p_k u_k + q_k d_k - \frac{1}{2}(p_k u_k^2 + q_k d_k^2)\big] - \frac{1}{2}t^2(p_k u_k^2 + q_k d_k^2) - \frac{1}{2}t^2(p_k u_k + q_k d_k)^2 + \theta \max(|t|, |t|^3)(p_k|u_k|^2 + q_k|d_k|^2)\max(|u_k|, |d_k|). \tag{2.10}$$

Finally, (2.4) and (2.10) imply that

$$\ln f_N(t) = \sum_{k=1}^{N} \ln g_k(t) \quad \text{is defined by (2.3).}$$

□

The following theorems are direct consequences of the above lemma.

Theorem 2.1. *Suppose that the following conditions are satisfied (as $N \to +\infty$):*

(i) $\alpha_N = \max_{1\le k\le N} \max(|u_k|, |d_k|) \to 0,$

(ii) $\sum_{k=1}^{N} (p_k u_k + q_k d_k) \to a,$

(iii) $\sum_{k=1}^{N} (p_k u_k + q_k d_k)^2 \to b^2 \ge 0,$

(iv) $\sum_{k=1}^{N} (p_k u_k^2 + q_k d_k^2) \to \sigma^2 > 0.$

Then

$$\lim_{N\to+\infty} \ln f_N(t) = it\Big(a - \frac{1}{2}\sigma^2\Big) - t^2\frac{(\sigma^2 - b^2)}{2}. \tag{2.11}$$

Theorem 2.2. *Put $F_N(x) = P\Big\{\ln\Big(\frac{S_N}{S_0}\Big) < x\Big\}$. Under the conditions given in Theorem 2.1 we have*

$$\lim_{N\to+\infty} \sup \Big| F_N(x) - \Phi\Big(\frac{x - a + \frac{\sigma^2}{2}}{(\sigma^2 - b^2)^{\frac{1}{2}}}\Big)\Big| = 0, \tag{2.12}$$

where

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\Big(-\frac{t^2}{2}\Big)dt.$$

Remark. If $p_k u_k + q_k d_k \ge 0$ for all $k = 1, 2, \dots, N$, it follows from (i), (ii) that $b = 0$.

3. A functional convergence theorem

Let us consider a time interval $[0,T]$ and the sequence $\{S_n\}$ defined by (1.1), (1.2), (1.4). Then we can define a process $\{S_t^+(N)\}$ as follows:

$$S_t^+(N) = S_{[\frac{Nt}{T}]}, \quad t \in [0,T], \tag{3.1}$$

where $[a]$ stands for the integer part of a. It is obvious that $\{S_t^+(N), 0 \le t \le T\}$ belongs to the space of cadlag functions and that $\{S_t^+(N), 0 \le t \le T\}$ is an independent increments process.

Let us consider the increments of the process $\ln(S_t^+(N))$:

$$\ln(S_t^+(N)) - \ln(S_s^+(N)) = \ln(S_{k(t)}) - \ln(S_{k(s)}), \tag{3.2}$$

with $k(t) = k(t,N) = \left[\frac{Nt}{T}\right]$, $k(s) = k(s,N) = \left[\frac{Ns}{T}\right]$.

Suppose that the following conditions are satisfied as $N \to +\infty$, for all $t \in [0,T]$:

(a) $\alpha_N = \max\limits_{1\le k\le N} \max(|u_k|,|d_k|) \to 0, \quad \max\limits_{1\le k\le N}(u_k - d_k)^2 \le \dfrac{\theta}{N}$,

(b) $$\sum_{i=1}^{k(t,N)} (p_i u_i + q_i d_i) \to a(t), \tag{3.3}$$

(c) $$\sum_{i=1}^{k(t,N)} (p_i u_i + q_i d_i)^2 \to b^2(t) \ge 0, \tag{3.4}$$

(d) $$\sum_{i=1}^{k(t,N)} (p_i u_i^2 + q_i d_i^2) \to \sigma^2(t) > 0. \tag{3.5}$$

Notice that $\left(\dfrac{S_i}{S_{i-1}}\right)$ takes only two values

$$1 + u_i = 1 + u_i(N),$$
$$1 + d_i = 1 + d_i(N)$$

with probabilities $p_i(N)$ and $q_i(N)$ respectively. Then by Theorem 2.2, the distribution of

$$\ln\left(\frac{S_t^+(N)}{S_0}\right) - \ln\left(\frac{S_s^+(N)}{S_0}\right) = \ln\left(\frac{S_{k(t)}}{S_0}\right) - \ln\left(\frac{S_{k(s)}}{S_0}\right)$$

converges to $N(\mu_t - \mu_s; \tilde{\sigma}_t^2 - \tilde{\sigma}_s^2)$, where

$$\mu_t = a(t) - \frac{\sigma^2(t)}{2}, \quad \tilde{\sigma}_t^2 = \sigma^2(t) - b^2(t). \tag{3.6}$$

It follows from the conditions (c) and (d), that the functions $b^2(t)$, $\sigma^2(t)$ and $\tilde{\sigma}_t^2 = \sigma^2(t) - b^2(t)$ are non-decreasing.

Further suppose that $a(t)$, $b(t)$, $\sigma^2(t)$ possess continuous derivatives and put

$$\frac{d\mu(t)}{dt} = \alpha(t); \quad \frac{d\tilde{\sigma}_t^2(t)}{dt} = \bar{\sigma}_t^2(t) > 0. \tag{3.7}$$

Theorem 3.1. *Assume that the conditions* (a), (b), (c), (d) *and* (3.7) *are satisfied. Then the process* $\ln\Big(\dfrac{S_t^+(N)}{S_0}\Big)$ *converges in distribution on the space D of cadlag functions to the process $Z(t)$ given by:*

$$dZ(t) = \alpha(t)dt + \bar{\sigma}(t)dW_t, \quad Z(0) = 0, \quad 0 \le t \le T, \tag{3.8}$$

where W_t is a standard Wiener process on $[0, T]$.

Proof. Taking in account Theorem 2.2 we see that the distribution of

$$\ln\Big(\frac{S_t^+(N)}{S_0}\Big) - \ln\Big(\frac{S_s^+(N)}{S_0}\Big)$$

converges to the normal distribution $N(\mu_t - \mu_s; \tilde{\sigma}_t^2 - \tilde{\sigma}_s^2)$ which is the distribution of $Z(t) - Z(s)$. Hence, both processes $\ln\Big(\dfrac{S_t^+(N)}{S_0}\Big)$ and $Z(t)$ are independent increments processes and all finite dimensional distributions of $\ln\Big(\dfrac{S_t^+(N)}{S_0}\Big)$ converge to the ones of $Z(t)$. Furthermore, we can prove that the sequence $\{\ln(S_t^+(N))\}$ is tight (see Appendix), and by Prohorov's Theorem (see [1]) we obtain the conclusion of Theorem 3.1. □

Remark. If $p_i u_i + q_i d_i \ge 0$ for all $i = 1, 2, \ldots, N$ then $b(t) = 0$. Infact, it follows from (a) and (c) that

$$\sum_{i=1}^{k(t,N)} (p_i u_i + q_i d_i)^2 \le \alpha_N \sum_{i=1}^{k(t,N)} (p_i u_i + q_i d_i) \to 0.$$

Remark. Putting $S(t) = S_0 \exp(Z(t))$, by Itô's formula we have

$$dS(t) = S(t)(\bar{\alpha}(t)dt + \bar{\sigma}(t)dW_t), \quad S_0(0) = S_0, \tag{3.9}$$

where $\bar{\alpha}(t) = \alpha(t) + \dfrac{(\tilde{\sigma}(t))^2}{2} = \dfrac{d(a(t) - b^2(t))}{dt}$, and it is easy to see that the process $\{S_t^+(N)\}$ converges in distribution to $S(t)$.

4. Approximately pricing for the standard European option

Let us recall some basic concepts. Consider a generalized Cox-Ross-Rubinstein market defined by two processes $\{B_n\}$ and $\{S_n\}$ given by (1.3), (1.4) where $\{\rho_n\}$ is the sequence of independent random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$ and the objective probability measure P is defined such that

$$P\{\rho_k = u_k\} = p_k;\ P\{\rho_k = d_k\} = q_k = 1 - p_k,\ (0 < p_k < 1),\ k = 1, 2, \ldots, N.$$

Put $\mathcal{F}_n = \sigma(S_k, 1 \le k \le n) = \sigma(\rho_k, 1 \le k \le n)$, $n = 1, 2, \ldots, N$. We can take $\Omega = \{d_1, u_1\} \otimes \ldots \otimes \{d_N, u_N\}$ and $\mathcal{F} = \{A : A \subset \Omega\}$.

Suppose that at each moment n $(n = 0, 1, 2, \dots, N-1)$ an agent keeps π_n^0 bank accounts of price B_n and π_n^1 shares of price S_n. Then at the moment n his asset is equal to

$$V_n^\pi = \pi_n^0 B_n + \pi_n^1 S_n, \tag{4.1}$$

where π_n^0 and π_n^1 is assumed to be $\mathcal{F}_{n-1}$-measureable.

Definition 4.1. A strategy $\pi = \{(\pi_n^0, \pi_n^1), 0 \le n \le N-1\}$ is called self-financing strategy if

$$B_{n-1}\Delta\pi_n^0 + S_{n-1}\Delta\pi_n^1 = 0, \tag{4.2}$$

where $\Delta a_n = a_n - a_{n-1}$.

It is easy to see that a strategy is self-financing if and only if

$$\Delta V_n^\pi = \pi_n^0 \Delta B_n + \pi_n^1 \Delta S_n. \tag{4.3}$$

Let us denote by SF the class of all self-financing strategies.

Definition 4.2. The quantities

$$\Delta \bar{V}_n^\pi := \frac{V_n^\pi}{B_n} = \pi_n^0 + \pi_n^1 \bar{S}_n \tag{4.4}$$

with $\bar{S}_n = \dfrac{S_n}{B_n}$ are called discounted values corresponding to π.

It is easy to see that $\pi \in SF$ if and only if

$$\Delta \bar{V}_n^\pi = \pi_n^1 \Delta \bar{S}_n = \Big(\frac{\pi_n^1 S_{n-1}}{B_n}\Big)(\rho_n - r_n). \tag{4.5}$$

In fact, by (4.2)

$$\Delta \bar{V}_n^\pi = \Delta\pi_n^0 + \bar{S}_{n-1}\Delta\pi_n^1 + \pi_n^1 \Delta\bar{S}_n = \pi_n^1 \Delta \bar{S}_n.$$

Definition 4.3. A probability measure Q is called a neutral martingale measure if $Q \sim P$ and $\bar{V}_n^\pi$ is an $(\mathcal{F}_n, Q)$-martingale for all $\pi \in SF$.

Proposition 4.1. *Q is a neutral martingale measure if and only if $\bar{S}_n = \dfrac{S_n}{B_n}$ is an $(\mathcal{F}_n, Q)$-martingale.*

Proof. According to (4.5) and under the assumption that π_n^1 is $\mathcal{F}_{n-1}$-measureable, $E_Q(\Delta\bar{V}_n^\pi / \mathcal{F}_n) = 0$ if and only if $E_Q(\Delta\bar{S}_n/\mathcal{F}_n) = 0$. In other words, $\bar{V}_N^\pi$-is an $(\mathcal{F}_n, Q)$-martingale if and only if $\bar{S}_n$ is an $(\mathcal{F}_n, Q)$-martingale. □

Proposition 4.2. *In the market $(B, S) = \{(B_n, S_n), n = 1, 2, \dots, N\}$ with $-1 < d_n < r_n < u_n$, there exists a unique neutral martingale measure Q such that*

$$Q\{\rho_n = u_n\} = p_n^*; \quad Q\{\rho_n = d_n\} = q_n^* = 1 - p_n^*,$$

where p_n^*, q_n^* are defined by

$$E_Q(\rho_n) = p_n^* u_n + q_n^* d_n = r_n \tag{4.6}$$

or equivalently

$$p_n^* = \frac{r_n - d_n}{u_n - d_n}; \qquad q_n^* = \frac{u_n - r_n}{u_n - d_n} \tag{4.7}$$

Proof. By (4.5) we have

$$E_Q(\Delta \bar{V}_n^\pi / \mathcal{F}_n) = \frac{\pi_n^1 S_{n-1}}{B_n} E_Q(\rho_n - r_n) = 0$$

if and only if $E_Q(\rho_n) = r_n$ for all $n = 1, 2, \ldots, N$. □

Definition 4.4. The value

$$C(H_N) = \inf\{V_0 : \exists \pi \in SF, V_0^\pi = V_0, V_n^\pi \geq H_N\} \tag{4.8}$$

is called the rational cost or price corresponding to the claim H_N.

The problem is to define $C(H_N)$ and to find a strategy π such that $V_0^\pi = V_0$ and $V_n^\pi \geq H_N$. The following theorem gives an answer:

Theorem 4.1. *In the generalized Cox-Ross-Rubinstein market* (B, S) *we have:*

(i) *For any* $\mathcal{F}_n$*-measureable claim* H_N

$$C(H_N) = E_Q(\bar{H}_N) \quad \textit{with} \quad \bar{H}_N = \frac{H_N}{B_N}; \tag{4.9}$$

(ii) *With the initial capital* $V_0 = C(H_N)$ *there exitsts the so called minimum hedging strategy* π^* *such that*

$$V_0^{\pi^*} = C(H_N); \quad \bar{V}_n^{\pi^*} = E_Q(\bar{H}_N/\mathcal{F}_n); \quad V_N^{\pi^*} = H_N. \tag{4.10}$$

This theorem is an analogy of Theorem 1 in [6] for a binomial option market model.

Remark. The claim $H_N = \max(S_N - K, 0) := (S_N - K)_+$ or $H_N = (S_N - K)_+$ concerns the problem of pricing a standard European call option (S.E.C.O) or standard European put option (S.E.P.O), respectively.

Definition 4.5.

(1) A strategy $\pi \in SF$ is said to be arbitrage if $V_0^\pi = 0$, $V_N^\pi \geq 0$ and $P\{V_N^\pi > 0\} > 0$.

(2) The market (B,S) is said to be arbitrage free if there is no arbitrage self-financing strategy.

Remark. A market of arbitrage is essentially a mechanism for making money.

Definition 4.6. A market (B, S) is said to be complete if any contingent claim H_N is attainable, i.e., there exists an initial capital V_0 and $\pi \in SF$ such that $V_0^\pi = V_0$, $V_N^\pi = H_N$.

Remark. The market (B, S) with $-1 < d_k < r_k < u_k$ is arbitrage free and complete. This assertion follows from Proposition 4.2 and from [4], where it is stated that a market (B, S) is arbitrage free and complete if and only if there exists a unique martingale measure.

Let (B, S) be the option market defined above. Let Q be the neutral martingale measure (whose existence is assured by Proposition 4.2). Suppose that

$$u_k = r_k + \sigma_k,\ d_k = r_k - \tau_k;\ \tau_k > 0,\ \sigma_k > 0 \text{ for all } k = 1, 2, \dots, N. \tag{4.11}$$

Put

$$F_N^*(x) = Q\Big\{\ln\big(\frac{S_N}{S_0}\big) < x\Big\}. \tag{4.12}$$

Theorem 4.2. *Assume that the following conditions are fulfilled:*

(a) $\max_{1\le k\le N} \max(r_k, \sigma_k, \tau_k) \to 0,$

(b) $\sum_{k=1}^{N} r_k \to a \ge 0;\ \ \sum_{k=1}^{N} \sigma_k\tau_k \to \sigma^2 > 0$ *as* $N \to +\infty$.

Then

$$\lim_{N\to+\infty} \sup_x \Big| F_N^*(x) - \Phi\Big(\frac{x - a + \frac{\sigma^2}{2}}{\sigma}\Big)\Big| = 0. \tag{4.13}$$

Proof. Let us verify the conditions of Theorem 2.1 with p_k replaced by p_k^*. At first, the condition $\alpha_N \to 0$ follows from (a) and the condition (ii) is equivalent to

$$\sum_{k=1}^{N} E_Q(\rho_k) = \sum_{k=1}^{N} r_k \to a.$$

Since $p_k^* u_k + q_k^* d_k = r_k > 0$, the conditions (iii) and (iv) follow from the first remark of Section 3 and the fact that

$$\begin{aligned}\sum_{k=1}^{N}(p_k^* u_k^2 + q_k^* d_k^2) &= \sum_{k=1}^{N} E_Q(\rho_k^2) = \sum_{k=1}^{N} \big[E_Q(\rho_k - r_k)^2 + r_k^2\big] \\ &= \sum_{k=1}^{N} \big[p_k^*(u_k - r_k)^2 + q_k^*(d_k - r_k)^2\big] + \sum_{k=1}^{N} r_k^2 \\ &= \sum_{k=1}^{N} \sigma_k\tau_k + \sum_{k=1}^{N} r_k^2 \to \sigma^2 \quad \text{(by } (b)\text{)}.\end{aligned}$$

By virtue of Theorem 2.2 we obtain (4.13). □

Theorem 4.3. *Under the conditions of Theorem 4.2 the rational price of S.E.C.O is approximately given by*

$$C_C = E_Q\Big[\frac{(S_N - K)_+}{B_N}\Big] \approx S_0\Phi(d_+) - Ke^{-a}\Phi(d_-), \tag{4.14}$$

where

$$d_{\pm} = \frac{\ln\big(\dfrac{S_0}{K}\big) + a \pm \dfrac{\sigma^2}{2}}{\sigma},$$

and the rational price of S.E.P.O is approximately given by

$$C_P = E_Q\Big[\frac{(K - S_N)_+}{B_N}\Big] \approx S_0 - Ke^{-a} - C_C. \tag{4.15}$$

Proof. According to (4.9), with $H_N = (S_N - K)_+$ we have

$$C_C = E_Q\Big[\frac{(S_N - K)_+}{B_N}\Big] = \frac{S_0}{B_N}E_Q\Big[\Big(\frac{S_N}{S_0} - \frac{K}{S_0}\Big)_+\Big], \tag{4.16}$$

(4.17)

$$B_N = \prod_{k=1}^{N}(1 + r_k) = \exp\Big(\sum_{k=1}^{N}\ln(1 + r_k)\Big) = \exp\Big(\sum_{k=1}^{N} r_k + \theta\sum_{k=1}^{N} r_k^2\Big) \to e^a.$$

Further, since F_N^* converges weakly to $\Phi\Big(\dfrac{x - a + \dfrac{\sigma^2}{2}}{\sigma}\Big)$ we have

$$\int_{-\infty}^{\ln(\frac{K}{S_0})} \Big[e^x - \frac{K}{S_0}\Big]dF_N^*(x) \to \int_{-\infty}^{\ln(\frac{K}{S_0})} \Big[e^x - \frac{K}{S_0}\Big]d\Phi\Big(\frac{x - a + \dfrac{\sigma^2}{2}}{\sigma}\Big)$$

by taking into account of the continuity and the boundedness of the function $\min\big\{\exp(x) - \dfrac{K}{S_0}, 0\big\}$ on $(-\infty, +\infty)$). On the other hand,

$$\int_{\ln(\frac{K}{S_0})}^{+\infty} \Big[e^x - \frac{K}{S_0}\Big]dF_N^*(x) = \int_{-\infty}^{+\infty} \Big[e^x - \frac{K}{S_0}\Big]dF_N^*(x) - \int_{-\infty}^{\ln(\frac{K}{S_0})} \Big[e^x - \frac{K}{S_0}\Big]dF_N^*(x)$$

$$= E_Q e^{\ln(\frac{S_N}{S_0})} - \frac{K}{S_0} - \int_{-\infty}^{\ln(\frac{K}{S_0})} \Big[e^x - \frac{K}{S_0}\Big]dF_N^*(x)$$

$$= E_Q\prod_{i=1}^{N}(1 + \rho_i) - \frac{K}{S_0} - \int_{-\infty}^{\ln(\frac{K}{S_0})} \Big[e^x - \frac{K}{S_0}\Big]dF_N^*(x) =$$

$$= \prod_{i=1}^{N}(1+r_i) - \frac{K}{S_0} - \int_{-\infty}^{\ln(\frac{K}{S_0})} \Big[e^x - \frac{K}{S_0}\Big] dF_N^*(x)$$

$$\to e^a - \frac{K}{S_0} - \int_{-\infty}^{\ln(\frac{K}{S_0})} \Big[e^x - \frac{K}{S_0}\Big] d\Phi\Big(\frac{x-a+\frac{\sigma^2}{2}}{\sigma}\Big)$$

$$= e^a - \frac{K}{S_0} - \Big\{\int_{-\infty}^{+\infty} \Big[e^x - \frac{K}{S_0}\Big] d\Phi\Big(\frac{x-a+\frac{\sigma^2}{2}}{\sigma}\Big) - \int_{\ln(\frac{K}{S_0})}^{+\infty} \Big[e^x - \frac{K}{S_0}\Big] d\Phi\Big(\frac{x-a+\frac{\sigma^2}{2}}{\sigma}\Big)\Big\}$$

$$= e^a - \int_{-\infty}^{+\infty} e^x d\Phi\Big(\frac{x-a+\frac{\sigma^2}{2}}{\sigma}\Big) + \int_{\ln(\frac{K}{S_0})}^{+\infty} \Big[e^x - \frac{K}{S_0}\Big] d\Phi\Big(\frac{x-a+\frac{\sigma^2}{2}}{\sigma}\Big)$$

$$= \int_{\ln(\frac{K}{S_0})}^{+\infty} \Big[e^x - \frac{K}{S_0}\Big] d\Phi\Big(\frac{x-a+\frac{\sigma^2}{2}}{\sigma}\Big).$$

Therefore
(4.18)

$$E_Q\Big(\frac{S_N}{S_0} - \frac{K}{S_0}\Big)_+ = E_Q\Big[\exp\Big(\ln\big(\frac{S_N}{S_0}\big)\Big) - \frac{K}{S_0}\Big]_+$$

$$= \int_{\ln(\frac{K}{S_0})}^{+\infty} \Big[e^x - \frac{K}{S_0}\Big] dF_N^*(x)$$

$$\approx \int_{\ln(\frac{K}{S_0})}^{+\infty} \Big[e^x - \frac{K}{S_0}\Big] d\Phi\Big(\frac{x-a+\frac{\sigma^2}{2}}{\sigma}\Big)$$

$$= \int_{\ln(\frac{K}{S_0})}^{+\infty} e^x d\Phi\Big(\frac{x-a+\frac{\sigma^2}{2}}{\sigma}\Big) - \frac{K}{S_0}\Big[1 - \Phi\Big(\frac{\ln\big(\frac{K}{S_0}\big) - a + \frac{\sigma^2}{2}}{\sigma}\Big)\Big]$$

$$= e^a\Phi(d_+) - \frac{K}{S_0}\Phi(d_-).$$

It follows from (4.16)-(4.18) that

$$C_C \approx S_0 e^{-a}\Big\{e^a\Phi(d_+) - \frac{K}{S_0}\Phi(d_-)\Big\}$$

which is equivalent to (4.14). To prove (4.15) we notice that

$$(S_N - K)_+ - (K - S_N)_+ = S_N - K$$

and $\dfrac{S_n}{B_n}$ is an $(\mathcal{F}_n, Q)$-martingale. Hence,

$$\begin{aligned}\frac{E_Q(S_N - K)_+ - E_Q(K - S_N)_+}{B_N} &= E_Q\Big(\frac{S_N}{B_N}\Big) - \frac{K}{B_N}\\ &= S_0 - \frac{K}{B_N} \approx S_0 - Ke^{-a},\end{aligned}$$

or $C_P \approx S_0 - Ke^{-a} - C_C$. □

Remark. If $d_k = r_k - \sigma_k,\ \ u_k = r_k + \sigma_k$ then the condition (b) of Theorem 4.2 is replaced by (b')

$$\sum_{k=1}^{N} r_k \to a \geq 0, \quad \sum_{k=1}^{N} \sigma_k^2 \to \sigma^2 > 0.$$

Remark. In the option market model considered above there are too many unknown parameters: r_k, u_k, d_k, $k = 1, 2, \ldots, N$. However even if these quantities change while $\sum_{k=1}^{N} r_k$, $\sum_{k=1}^{N} \sigma_k \tau_k$ remain constant we can still apply Theorem 4.3 to calculate the rational prices of S.E.P.O and S.E.C.O.

In this part we shall prove an assertion similar to Theorem 3.1, where P is replaced by Q.

Let us return to the sequence of the prices $\{S_k, k = 1, 2, \ldots, N\}$ given by (3.1) and to the price process $\{S_t^+(N), t \in [0, T]\}$ defined by (3.1) of Section 3. We consider the following conditions:

(a$_1$) $-1 < d_k < r_k < u_k$, $k = 1, 2, \ldots, N$,

(b$_1$) $\max\limits_{1\leq k\leq N} \max(r_k, \sigma_k, \tau_k) \to 0, \ \ \max\limits_{1\leq k\leq N} \sigma_k \tau_k \leq \dfrac{\theta}{N}$,

(c$_1$) $\sum\limits_{i=1}^{k(t,N)} r_i \to a(t) \geq 0$, for all $t \in [0, T]$,

(d$_1$) $\sum\limits_{i=1}^{k(t,N)} \sigma_i \tau_i \to \sigma^2(t)$, for all $t \in [0, T]$,

where $\sigma_k = u_k - r_k$, $\tau_k = r_k - d_k$, $k(t, N) = [\dfrac{Nt}{T}]$,

(e$_1$) the functions $a(t)$, $\sigma^2(t)$ possess the continuous derivatives

$$\frac{da(t)}{dt} := \alpha_1(t), \quad \frac{d\sigma^2(t)}{dt} := \sigma_1^2(t) > 0.$$

Let Q be the neutral martingale measure defined by

$$Q\{\rho_k = u_k\} = p_k^*, \quad Q\{\rho_k = d_k\} = q_k^* = 1 - p_k^*,$$

where p_k^*, q_k^* are defined by

$$E_Q(\rho_k) = p_k^* u_k + q_k^* d_k = r_k \quad \text{(see 4.6)}. \tag{4.19}$$

Theorem 4.4. *Suppose that the conditions* (a$_1$) - (e$_1$) *are fulfilled. Putting*

$$V_k = E_Q\{(S_N - K)_+ / \mathcal{F}_k\}$$

for $k = \left[\frac{Nt}{T}\right]$ *we have*

$$V_k \approx S_k \Phi(d_+(S_k)) - K \exp[-(a(T) - a(t))]\Phi(d_-(S_k)) \tag{4.20}$$

where

$$d_\pm(S_k) = \frac{\ln\big(\frac{K}{S_k}\big) + a(T) - a(t) \pm \frac{\sigma^2(T) - \sigma^2(t)}{2}}{(\sigma^2(T) - \sigma^2(t))^{\frac{1}{2}}}. \tag{4.21}$$

Proof. We have

$$(S_N - K)_+ = S_k\Big(\frac{S_N}{S_k} - \frac{K}{S_k}\Big)_+.$$

Further, it follows from (c$_1$) and (d$_1$) that for $k = k(t, N) = \left[\frac{tN}{T}\right]$

$$\sum_{i=1}^{N} r_i \to a(T) - a(t) \geq 0, \quad \text{for all } t \in [0, T],$$
$$\sum_{i=1}^{N} \sigma_i \tau_i \to \sigma^2(T) - \sigma^2(t), \quad \text{for all } t \in [0, T].$$

Therefore, by Theorem 4.2 $\ln\left(\frac{S_N}{S_k}\right)$ is asymptotically normal $N(\beta(t); \delta^2(t))$ where

$$\beta(t) = a(T) - a(t) - \frac{\sigma^2(T) - \sigma^2(t)}{2},$$
$$\delta^2(t) = \sigma^2(T) - \sigma^2(t) > 0.$$

Finally, the expression (4.20) is established in a similar way as in the proof of Theorem 4.3. □

Theorem 4.5. *Suppose that the condition* (a$_1$) - (e$_1$) *are fulfilled. Then under* Q*, the process* $\ln\big(\frac{S_t^+(N)}{S_0}\big)$ *with* $\{S_t^+(N), t \in [0, T]\}$*, defined by* (3.1) *converges in distribution on the space* D *to the process* $Z_1(t)$ *given by the following stochastic differential equation:*

$$dZ_t^1 = (\alpha_1(t) - \frac{1}{2}\sigma_1^2(t))dt + \sigma_1(t)dW_t, \quad Z_0^1 = 0. \tag{4.22}$$

The proof of Theorem 4.5 is similar to that of Theorem 3.1.

Remark. Put $S_t^1 = S_0 \exp(Z_t^1)$. Then S_t^1 satisfies the following SDE:

$$dS_t^1 = S_t^1(\alpha_1(t)dt + \sigma_1(t)dW_t), \quad S_0^1 = S_0, \tag{4.23}$$

and by Theorem 4.5, $\{S_t^+(N), 0 \le t \le T\}$ converges in distribution on D to $\{S_t^1, 0 \le t \le T\}$.

Finally, let us consider the market $\{(B_k, S_k),\ k = 1, 2, \dots, N\}$ where $B_k = \prod_{i=1}^{k}(1 + r_i)$ with r_k is the value of a short interest rate process at time points $\frac{kT}{N}$ of the interval $[0, T]$ and S_k is the value of some stock price process at time point $\frac{kT}{N}$ of $[0, T]$.

According to Theorem 4.1, for an initial capital $V_0 = C_C$ defined by (4.14), there always exists a SF strategy π such that $V_0^\pi = C_C$ and $V_N^\pi = (S_N - K)_+$. A similar assertion is also valid for $V_0 = C_P$ and the claim $(K - S_N)_+$ of S.E.P.O. C_C, C_P and the corresponding hedging strategies can be approximately calculated by applying Theorem 4.3 and Theorem 4.4. In fact, by (4.20), for $k = \left[\frac{tN}{T}\right]$ we have

$$V_k \approx F_k(S_k) = F_k(S_{k-1}(1 + \rho_k))$$

where $F_k(S_k)$ stands for the right side of (4.20). Further

$$V_k = \pi_k^0 B_k + \pi_k^1 S_{k-1}(1 + \rho_k) \approx F_k(S_{k-1}(1 + \rho_k)) \tag{4.24}$$

and for $\rho_k = d_k$, $\rho_k = u_k$ we have

$$\pi_k^0 B_k + \pi_k^1 S_{k-1}(1 + u_k) \approx F_k(S_{k-1}(1 + u_k))$$
$$\pi_k^0 B_k + \pi_k^1 S_{k-1}(1 + d_k) \approx F_k(S_{k-1}(1 + d_k)).$$

Subtracting the above two equalities we obtain

$$\pi_k^1 S_{k-1}(u_k - d_k) \approx F_k(S_{k-1}(1 + u_k)) - F_k(S_{k-1}(1 + d_k))$$

or

$$\pi_k^1 \approx \frac{F_k(S_{k-1}(1 + u_k)) - F_k(S_{k-1}(1 + d_k))}{S_{k-1}(u_k - d_k)}. \tag{4.25}$$

For the case where $u_k = r_k + \sigma_k$, $d_k = r_k - \sigma_k$, π_k^1 can be approximately calculated by

$$\pi_k^1 \approx \frac{F_k(S_{k-1}(1 + r_k + \sigma_k)) - F_k(S_{k-1}(1 + r_k - \sigma_k))}{2\sigma_k S_{k-1}}$$

since

$$\pi_k^1 \approx F_k'(S_{k-1}(1 + r_k)), \quad \text{as } \sigma_k \to 0. \tag{4.26}$$

After defining π_k^1 from (4.26) we can define π_k^0 from (4.4) i.e.

$$\pi_k^0 = \bar{V}_n^\pi - \pi_n^1 \bar{S}_n.$$

The remaining part of Proposition 4.3, the claim $H_N = (K - S_N)_+$, can be proved similarly.

Appendix

On the tightness of the sequence

$$\Big\{\ln\Big(\frac{S_t^+(N)}{S_0}\Big),\ 0 \le t \le T,\ N = N_0, N_0+1, \dots,\Big\}$$

Let us consider the sequence of the processes

$$\ln\Big(\frac{S_t^+(N)}{S_0}\Big) = \sum_{i=1}^{k(t,N)} \ln(1+\rho_i) \tag{1}$$

with $k(t,N) = \Big[\frac{tN}{T}\Big]$, $0 \le t \le T$, $N = N_0, N_0+1, \dots$, where N_0 is some possible integer (see (3.1) - (3.6) of Section 3).

Lemma. *Under the conditions* (a) - (d) *and* (3.7) *of Theorem 3.1, the sequence* $\Big\{\ln \frac{S_t^+(N)}{S_0}, 0 \le t \le T, N = N_0, N_0+1, \dots\Big\}$ *is tight.*

Proof. Put

$$M_t(N) = E \ln\Big(\frac{S_t^+(N)}{S_0}\Big) = \sum_{i=1}^{k(t,N)} E(\ln(1+\rho_i)). \tag{2}$$

Then

$$\begin{aligned} M_t(N) &= \sum_{i=1}^{k(t,N)} \big[p_i \ln(1+u_i) + q_i \ln(1+d_i)\big] \\ &= \sum_{i=1}^{k(t,N)} \Big[p_i\Big(u_i - \frac{u_i^2}{2}\Big) + q_i\Big(d_i - \frac{d_i^2}{2}\Big)\Big] + \theta\big(p_i|u_i|^3 + q_i|d_i|^3\big) \\ &= \sum_{i=1}^{k(t,N)} \Big[(p_i u_i + q_i d_i) - \Big(\frac{p_i u_i^2}{2} + \frac{q_i d_i^2}{2}\Big)\Big] \\ &\quad + \theta \max(|u_i| + |d_i|) \sum_{i=1}^{k(t,N)} (p_i u_i^2 + q_i d_i^2). \end{aligned}$$

By (3.3) and (3.5) we see that

$$M_t(N) \to a(t) - \frac{\sigma^2(t)}{2} \quad \text{as } N \to +\infty. \tag{3}$$

Putting

$$X_t(N) = \ln\Big(\frac{S_t^+(N)}{S_0}\Big) - M_t(N), \tag{4}$$

it is easy to see that all finite dimensional distributions of $\{X_t(N); 0 \le t \le N\}$ converge weakly to the distribution of

$$X_t = \int_0^t \bar{\sigma}(s) dW_s \tag{5}$$

where $\bar{\sigma}^2(s) = \dfrac{d(\sigma^2(s) - b^2(s))}{ds}$. It follows from (3), (4), (5) that $\ln\Big(\dfrac{S_t^+(N)}{S_0}\Big)$ converges in distribution to $a(t) - \sigma^2(t) + X_t$ if and only if $X_t(N) \to X_t$ in distribution and hence $\Big\{\ln\Big(\dfrac{S_t^+(N)}{S_0}\Big); 0 \le t \le T; N = N_0, N_0+1, \dots\Big\}$ is tight if and only if $\{X_t(N); 0 \le t \le T; N = N_0, N_0+1, \dots\}$ is tight.

In order to prove that $\{X_t(N)\}$ is tight, according to Theorem 15.6 in [1], it suffices to show that there exists a positive constant C such that

$$E\{[X_{t_1}(N) - X_t(N)]^2[X_{t_2}(N) - X_t(N)]^2\} \le C(t_2 - t_1)^2 \tag{6}$$

for all $0 \le t_1 \le t \le t_2 \le T$ and $N \ge N_0$.

Let us prove (6). Since $\{X_t(N); 0 \le t \le T\}$ is an independent increments process we have

$$\begin{aligned} &E\{[X_{t_1}(N) - X_t(N)]^2[X_{t_2}(N) - X_t(N)]^2\} \\ &= E\{[X_{t_1}(N) - X_t(N)]^2\}E\{[X_{t_2}(N) - X_t(N)]^2\}. \end{aligned} \tag{7}$$

On the other hand, we have

$$\begin{aligned} &E\{[X_{t_1}(N) - X_t(N)]^2\} \\ &= E\Big\{\Big[\ln\big(\frac{S_{t_1}^+(N)}{S_0}\big) - M_{t_1}(N) - \ln\big(\frac{S_t^+(N)}{S_0}\big) + M_t(N)\Big]^2\Big\} \\ &= E\Big\{\sum_{k(t_1,N)}^{k(t,N)} \big[\ln(1+\rho_i) - E(\ln(1+\rho_i))\big]\Big\}^2 \\ &= \sum_{k(t_1,N)}^{k(t,N)} E\big[\ln(1+\rho_i) - E(\ln(1+\rho_i))\big]^2. \end{aligned} \tag{8}$$

Noticing that

$$\begin{aligned} \ln(1+\rho_i) - E(\ln(1+\rho_i)) &= \ln(1+\rho_i) - p_i\ln(1+u_i) - q_i\ln(1+u_i) \\ &= p_i\big[\ln(1+\rho_i) - \ln(1+u_i)\big] \\ &\quad + q_i\big[\ln(1+\rho_i) - \ln(1+d_i)\big]. \end{aligned} \tag{9}$$

and putting

$$\alpha_i = \frac{\ln(1+u_i) - \ln(1+d_i)}{u_i - d_i}, \tag{10}$$

we can easily verify the following equalities

$$\begin{aligned} \ln(1+\rho_i) - \ln(1+u_i) &= \alpha_i(\rho_i - u_i), \\ (1+\rho_i) - \ln(1+d_i) &= \alpha_i(\rho_i - d_i). \end{aligned} \tag{11}$$

It follows from (9) - (11) that

$$\ln(1+\rho_i) - E\ln(1+\rho_i) = \alpha_i[p_i(\rho_i - u_i) + q_i(\rho_i - d_i)]$$

and hence

$$\begin{aligned} E\{\ln(1+\rho_i) - E\ln(1+\rho_i)\}^2 &= \alpha_i^2 E\{p_i(\rho_i - u_i) + q_i(\rho_i - d_i)\}^2 \\ &= \alpha_i^2[p_i q_i^2(u_i - d_i)^2 + q_i p_i^2(d_i - u_i)^2] \\ &= \alpha_i^2 p_i q_i (u_i - d_i)^2. \end{aligned} \tag{12}$$

Futhermore, from (8), (12) we obtain

$$E[X_N(t) - X_N(t_1)]^2 = \sum_{i=k(t_1,N)}^{k(t,N)} \alpha_i^2 p_i q_i (u_i - d_i)^2. \tag{13}$$

We notice also that

$$p_i q_i \le \frac{1}{4} \quad \text{and } \alpha_i^2 \le 4 \quad \text{if } \max(|u_i|, |d_i|) \le \frac{1}{2}\cdot \tag{14}$$

Hence (10) yields

$$\alpha_i = \frac{1}{u_i - d_i}\Big[\frac{u_i - d_i}{1+d_i} - \frac{1}{2}\Big(\frac{u_i - d_i}{1+d_i}\Big)^2 + \delta\Big|\frac{u_i - d_i}{1+d_i}\Big|^3\Big] \le \frac{1}{1+d_i} \le 2, \quad (0 < \delta < 1).$$

From (13), (14) we obtain

$$E[X_{t_1}(N) - X_t(N)]^2 \le \sum_{i=k(t_1,N)}^{k(t,N)} (u_i - d_i)^2. \tag{15}$$

Similarly

$$E[X_t(N) - X_{t_2}(N)]^2 \le \sum_{i=k(t,N)}^{k(t_2,N)} (u_i - d_i)^2. \tag{16}$$

According to condition (a) of Theorem 3.1 the relations (7), (15), (16) imply that

$$\begin{aligned} E\{[X_{t_1}(N) - X_t(N)]^2[X_{t_2}(N) - X_t(N)]^2\} &\le \Big\{\sum_{k(t_1,N)}^{k(t_2,N)} (u_i - d_i)^2\Big\}^2 \\ &\le \frac{\theta^2}{N^2}\big[k(t_2,N) - k(t_1,N)\big]^2. \end{aligned} \tag{17}$$

If $t_2 - t_1 < \dfrac{T}{N}$ then either $t, t_1 \in \Big[\dfrac{iT}{N}, \dfrac{(i+1)T}{N}\Big]$ or $t, t_2 \in \Big[\dfrac{iT}{N}, \dfrac{(i+1)T}{N}\Big]$ for some integer i and in this case either $X_t(N) - X_{t_1}(N) = 0$ or $X_t(N) - X_{t_2}(N) = 0$, whereas if $t_2 - t_1 \geq \dfrac{T}{N}$ then

$$\frac{k(t_2,N) - k(t_1,N)}{N} = \frac{\Big[\dfrac{t_2 N}{T}\Big] - \Big[\dfrac{t_1 N}{T}\Big]}{N} \leq 2\frac{t_2 - t_1}{T}\,.$$

Finally, by (17), we always have

$$E\{[X_{t_1}(N) - X_t(N)]^2[X_{t_2}(N) - X_t(N)]^2\} \leq \frac{4\theta^2}{T^2}(t_2 - t_1)^2 \leq C(t_2 - t_1)^2 \tag{18}$$

with $C = 4\Big(\dfrac{\theta}{T}\Big)^2$.

Thus (6) holds and this proves the tightness of $\{X_t(N)\}$. □

Remark. Under Q we have to replace p_i by p_i^*, where $p_i^* = \dfrac{r_i - d_i}{u_i - d_i}$, in this case

$$p_i^* q_i^* (u_i - d_i)^2 = (r_i - d_i)(u_i - r_i) = \tau_i \sigma_i.$$

Hence, if $\max\limits_{1 \leq t \leq N} \tau_i \sigma_i \leq \dfrac{\theta}{N}$ we obtain immediately (17), and the above lemma remains valid when P is replaced by Q.

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