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LINEAR EQUATIONS WITH GENERALIZED RIGHT INVERTIBLE OPERATORS

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Abstract. Let X be a linear space over a field $\mathcal{K}$ of scalars and let $R_1(X)$ be the set of all generalized right invertible operators in $L(X)$. Consider the general linear equation with generalized right invertible operator V of the form

$$\sum_{m=0}^{M}\sum_{n=0}^{N} V^m A_{mn} V^n x = y, \quad y \in X,$$

where $A_{mn} \in L_0(X)$, $A_{MN} = I$, $A_{mn}X_{M+N-n} \subset X_m$, $X_j := \operatorname{dom} V^j$. Similar equations with right invertible operators were studied by Przeworska-Rolewicz, and others (see [1], [2], [3]). In [4], N. V. Mau and N. M. Tuan constructed the generalized right invertible operators. In this paper, we present some new properties of generalized right invertible operators and then apply them to obtain all solutions of the general linear equations for the genezalized right invertible operator V with non-commutative cofficients.

1. Prelimilaries and notations

Let X be a linear space over a fielld $\mathcal{K}$ of scalars. Denote by $L(X)$ the set of all linear operators with domains and ranges in X and write

$$L_0(X) = \{A \in L(X) : \operatorname{dom} A = X\}.$$

The set of all right (left, generalized) invertible operators in $L(X)$ will be denoted by $R(X)$ (resp. $\Lambda(X)$, $W(X)$) (see [1]-[3]). Denote by $R_1(X)$ the set of all generalized right invertible operators belonging to $L(X)$. For $V \in R_1(X)$ we denote by $\mathcal{R}_V^1$ the set of all right inverses of V, i.e,

$$\mathcal{R}_V^1 = \big\{W \in L(X) : \operatorname{Im} V \subset \operatorname{dom} W,\ \operatorname{Im} W \subset \operatorname{dom} V,\ VWV = V,\ V^2W = V\big\},$$

by $\mathcal{F}_V$ the set of all right initial operators of V, i.e,

$$\mathcal{F}_V = \big\{F \in L(X) : F^2 = F,\ \operatorname{Im} F = \ker V \text{ and } \exists W \in \mathcal{R}_V^1 : FW = 0 \text{ on } \operatorname{dom} W\big\},$$

and by $\mathcal{G}_V$ the set of all left initial operators of V, i.e.

$$\mathcal{G}_V = \big\{G \in L(X) : G^2 = G,\ GV = 0 \text{ on } \operatorname{dom} V \text{ and } \exists W \in \mathcal{R}_V^1 : \operatorname{Im} G = \ker W\big\}.$$

Lemma 1.1. *For every $V \in R_1(X)$ there exists $W \in \mathcal{R}_V^1$ such that*

$$WVW = W, \quad VW^2 = W \quad \text{on } \operatorname{dom} W.$$

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Proof. Let $W_1 \in \mathcal{R}_V^1$. Write $W = VW_1^2VW_1$. We have

$$\begin{aligned} V^2W &= V^2VW_1^2VW_1 = VVW_1 = V, \\ VWV &= VVW_1^2VW_1V = VW_1VW_1V = VW_1V = V, \\ VW^2 &= VVW_1^2VW_1VW_1^2VW_1 = VW_1VW_1VW_1^2VW_1 = VW_1^2VW_1 = W, \\ WVW &= VW_1^2VW_1VVW_1^2VW_1 = VW_1^2V^2W_1^2VW_1 = VW_1^2VW_1 = W, \end{aligned}$$

which was to be proved. □

Write

$$\mathcal{R}_V^{(1)} = \{W \in \mathcal{R}_V^1 : \ WVW = W, \ VW^2 = W\}.$$

Lemma 1.2. *Suppose that* $V \in R_1(X)$, $\dim\ker V \neq 0$, $\dim\operatorname{coker} V \neq 0$ *and* $W \in \mathcal{R}_V^1$. *Then for an arbitrary positive* N, *we have*

(i) $\ker V^N = \Big\{x \in X : \ x = \sum_{k=0}^{N-1} W^k z_k, \ z_0, \dots, z_{N-1} \in \ker V\Big\}$.

(ii) $\operatorname{dom} V^N = W^N V^N X_N \bigoplus \ker V^N, \ X_N := \operatorname{dom} V^N$.

Proof. (i) Suppose that $z = \sum_{k=0}^{N-1} W^k z_k$ where $z_0, \dots, z_{N-1} \in \ker V$. Then

$$V^N z = V^N \sum_{k=0}^{N-1} W^k z_k = \sum_{k=0}^{N-1} V^{N-k} z_k = 0,$$

which implies $z \in \ker V^N$. Conversely, suppose that $z \in \ker V^N$. Then the Taylor Formula (see [3]) implies that

$$z = \sum_{k=0}^{N-1} W^k F V^k z + W^N V^N z = \sum_{k=0}^{N-1} W^k z_k.$$

Write $z_k = FV^k z$ for $k = 0, \dots, N-1$. By definition, $z_0, \dots, z_{N-1} \in \ker V$. Thus z is of the required form.

(ii) Suppose that $x \in \operatorname{dom} V^N$. We can write $x = u + v$, where $u = W^N V^N x$, $v = (I - W^N V^N)x$. By definition, $u \in W^N V^N X_N$ and $V^N v = 0$. Hence $v \in \ker V^N$ and $x = u + v \in W^N V^N X_N + \ker V^N$. Suppose now that $x \in \operatorname{dom} V^N$, $z \in \ker V^N$ and $W \in \mathcal{R}_V^1$ are arbitrarily fixed. Then $y = W^N V^N x + z \in \operatorname{dom} V^N$ for $V^N y = V^N W^N V^N x + V^N z = V^N x$.

Suppose that $u \in W^N V^N X_N \cap \ker V^N$. Then there is a $v \in \operatorname{dom} V^N$ such that $u = W^N V^N v$ and $V^N u = 0$. On the other hand, $V^N v = V^N W^N V^N v = V^N u = 0$, which implies $u = W^N V^N v = 0$. This means that $\operatorname{dom} V^N$ is a direct sum of $W^N V^N X_N$ and $\ker V^N$. The proof is complete. □

Corollary 1.1. *Suppose that all assumptions of Lemma 1.2 are satisfied, then*

$$\operatorname{dom} V^N = \Big\{x \in X : \ x = W^N y + \sum_{k=0}^{N-1} W^k z_k, \ y \in X_N, \ z_k \in \ker, V\Big\}.$$

2. Equations with generalized right invertible operators

To begin with, we consider the equation

$$V^N x = y, \quad y \in X, \quad N \in \mathbb{N}. \tag{2.1}$$

Theorem 2.1. *Suppose that* $V \in R_1(X)$, $\dim \ker V \neq 0$, $\dim \operatorname{coker} V \neq 0$ *and* $W \in \mathcal{R}_V^1$. *If* $y \in \operatorname{Im} V^N$, *then all solutions of* (2.1) *are given by*

$$x = W^N y + \sum_{k=0}^{N-1} W^k z_k, \tag{2.2}$$

where $z_0, \dots, z_{N-1} \in \ker V$ *are arbitrary.*

Proof. If $y \in \operatorname{Im} V^N$, then there is $y_1 \in \operatorname{dom} V^N$ such that $y = V^N y_1$. Hence, (2.1) can be writen in the form $V^N x = V^N y_1$. Since $V^N = V^N W^N V^N$, the last equation is equivalent to $V^N(x - W^N V^N y_1) = 0$. Lemma 1.2 implies the formula (2.2). □

Now consider the equation

$$(V^N - A)x = y, \quad y \in X, \quad A \in L_0(X), \quad N \in \mathbb{N}. \tag{2.3}$$

Theorem 2.2. *Suppose that* $V \in R_1(X)$, $W \in \mathcal{R}_V^1$, $A \in L_0(X)$, $AX_N \subset \operatorname{Im} V^N$ *and* $y \in (V^N - A)X_N$.

(i) *If* $I - W^N A \in R(X)$ *and* $R_A \in \mathcal{R}_{I - W^N A}$, *then all solutions of the equation* (2.3) *are given by*

$$x = R_A\Big(W^N y + \sum_{k=0}^{N-1} W^k z_k\Big) + z, \tag{2.4}$$

where $z_0, \dots, z_{N-1} \in \ker V$, $z \in \ker(I - W^N A)$.

(ii) *If* $I - W^N A \in \Lambda(X)$ *and* $L_A \in \mathcal{L}_{I - W^N A}$, *then all solutions of the equation* (2.3) *are given by*

$$x = L_A\Big(W^N y + \sum_{k=0}^{N-1} W^k z_k\Big), \tag{2.5}$$

where $z_0, \dots, z_{N-1} \in \ker V$.

(iii) *If* $I - W^N A \in W(X)$ *and* $W_A \in \mathcal{W}_{I - W^N A}$, *then all solutions of the equation* (2.3) *are given by*

$$x = W_A\Big(W^N y + \sum_{k=0}^{N-1} W^k z_k\Big) + z, \tag{2.6}$$

where $z_0, \dots, z_{N-1} \in \ker V$, $z \in \ker(I - W^N A)$.

(iv) *If $I - W^N A$ is invertible, then all solutions of the equation* (2.3) *are given by*

$$x = \left(I - W^N A\right)^{-1}\Big(W^N y + \sum_{k=0}^{N-1} W^k z_k\Big), \tag{2.7}$$

where $z_0, \dots, z_{N-1} \in \ker V$.

Proof. Suppose that $y \in (V^N - A)(\operatorname{dom} V^N)$. Then there exists an $x \in \operatorname{dom} V^N$ such that $(V^N - A)x = y$, i.e. $V^N x = Ax + y$. By Theorem 2.1, there exist $z_0, \dots, z_{N-1} \in \ker V$ such that $x = W^N(Ax + y) + \sum\limits_{k=0}^{N-1} W^k z_k$. Since $AX_N \subset \operatorname{Im} V^N$, we have $Ax \in \operatorname{Im} V^N \subset \operatorname{dom} W^N$ and

$$(I - W^N A)x = W^N y + \sum_{k=0}^{N-1} W^k z_k. \tag{2.8}$$

By Theorem 11.2 in [3] and (2.8), we get all formulae (2.4)-(2.7). □

We shall consider now the general equation of the form

$$Q[V]x := \sum_{m=0}^{M} \sum_{n=0}^{N} V^m A_{mn} V^n x = y, \quad y \in \operatorname{Im} Q[V], \tag{2.9}$$

where $V \in R_1(X)$, $A_{mn} \in L_0(X)$, $A_{MN} = I$, $A_{mn} X_{M+N-n} \in X_m$, $(m = 0, \dots, M;\ n = 0, \dots, N;\ m + n < M + N;\ X_j := \operatorname{dom} V^j;\ j = 1, \dots, M + N)$.

Write

$$Q(V) := \sum_{j=0}^{N} B_j V^j,$$
$$Q(I, W) := \sum_{j=0}^{N} B_j W^{N-j}.$$

Lemma 2.1. *Suppose that $V \in R_1(X)$ and $W \in \mathcal{R}_V^1$. Suppose moreover that we are given $B_j \in L_0(X)$, $(j = 0, \dots, N)$ and $k \in \mathbb{N}$ such that $X_{N-j} \subset \operatorname{dom} B_j$, $B_j X_{N-j} \subset X_k$, $j = 0, \dots, N$. Then*

$$\begin{aligned} &X_N \subset \operatorname{dom} Q(V),\\ &Q(V) X_N \subset X_k,\\ &[I + W^N Q(V)] X_{N+k} \subset X_{N+k},\\ &Q(I, W) X \subset X_k,\\ &[I + Q(I, W)] X_k \subset X_k. \end{aligned}$$

Proof. Note that $V^j X_N \subset X_{N-j}$, $(j = 0, \dots, N)$.

If $j = 0$ or $j = N$, then $X_N \subset X_N$ or $V^N X_N \subset X$.

If $1 \le j \le N-1$ and $x \in X_N$, there exist $x_0 \in \operatorname{Im} V^N$, $z_0, \dots, z_{N-1} \in \ker V$ such that

$$x = W^N x_0 + \sum_{k=0}^{N-1} W^k z_k,$$

$$V^j x = W^{N-j} x_0 + \sum_{k=j+1}^{N-1} W^{k-j} z_k + VWz_j.$$

Put $l = k - j$, $VWz_j = z_0 \in \ker V$. Then

$$V^j x = W^{N-j} x_0 + \sum_{l=1}^{N-j-1} W^l z_{l+j} + z_0.$$

Thus $V^j X_N \subset X_{N-j}$. Therefore $B_j V^j X_N \subset B_j X_{N-j} \subset X_k$ $(j = 0, \dots, N)$, which implies $X_N \subset \operatorname{dom} Q(V)$ and $Q(V) X_N \subset X_k$.

Suppose that $u \in \big[I + W^N Q(V)\big] X_{N+k}$. Then there exists $v \in X_{N+k} \subset X_N$ such that $u = \big[I + W^N Q(V)\big] v$. Since $v_1 = Q(V)v \in X_k$, we conclude that $u = v + W^N v_1 \in X_{N+k}$, because $v \in X_{N+k}$ and $W^N v_1 \in X_{N+k}$. Note that $W^j X \subset X_j$. Hence, $B_j W^{N-j} X \subset B_j X_{N-j} \subset X_k$, $(j = 0, \dots, N)$, which implies $Q(I, W) X \subset X_k$.

Suppose that $y \in \big[I + Q(I, W)\big] X_k$, i.e. there exists $y_1 \in X_k$ such that $y = [I + Q(I, W)] y_1$. Since $y_2 = Q(I, W) y_1 \in X_k$, we conclude that $y = y_1 + y_2 \in X_k$. $\square$

Puting $k = N$ in Lemma 2.1 we obtain

Corollary 2.1. *Suppose that all assumptions of Lemma 2.1 are satisfied, then*

$$[I + W^N Q(V)] X_N \subset X_N.$$

Definition 2.1. An operator $A \in L(X)$ is said to be right invertible (left invertible, invertible, generalized invertible) on X_k for a given $k \in \mathbb{N}^+$ if $X_k \subset \operatorname{dom} A$, $AX_k \subset X_k$ and there exists $R_A \in \mathcal{R}_A$ (resp. $L_A \in \mathcal{L}_A$, $M_A \in \mathcal{R}_A \cap \mathcal{L}_A$, $W_A \in \mathcal{W}_A$) such that $R_A X_k \subset X_k$ (resp. $L_A X_k \subset X_k$, $M_A X_k \subset X_k$, $W_A X_k \subset X_k$).

By this definition, if A is right invertible (left invertible, invertible, generalized invertible) on X_k for $k \ge 1$ then A is right invertible (left invertible, invertible, generalized invertible).

Lemma 2.2. *Suppose that all assumptions of Lemma 2.1 are satisfied, then the operator $I + Q(I, W) - B_N G$ is right invertible (left invertible, invertible, generalized invertible) on X_k for $k \ge 1$ if only if $I + W^N Q(V)$ is right invertible (left invertible, invertible, generalized invertible) on X_{N+k}, where $G \in \mathcal{G}_V$.*

Proof. By Lemma 2.1, we have

$$I + Q(I, W) - B_N G = I + Q(V) W^N \subset L_0(X_k),$$
$$I + W^N Q(V) \in L_0(X_{N+k}).$$

(i) Suppose that $I+Q(I,W)-B_NG$ is right invertible on X_k, i.e. there exists $R_Q \in \mathcal{R}_{I+Q(V)W^N}$ such that $R_QX_k \subset X_k$ and $\big[I+Q(V)W^N\big]R_Q = I$. Write $R^Q = I - W^NR_QQ(V)$. We have to check that R^Q is well defined on X_{N+k} and $R^QX_{N+k} \subset X_{N+k}$. On X_{N+k} we have

$$\begin{aligned}\big[I+W^NQ(V)\big]R^Q &= \big[I+W^NQ(V)\big]\big[I-W^NR_QQ(V)\big]\\ &= I+W^NQ(V)-\big[I+W^NQ(V)\big]W^NR_QQ(V)\\ &= I+W^NQ(V)-W^N\big[I+Q(V)W^N\big]R_QQ(V)\\ &= I+W^NQ(V)-W^NQ(V)=I,\end{aligned}$$

which proves that $I+W^NQ(V)$ is right invertible on X_{N+k}.

Conversely, suppose that $I+W^NQ(V)$ is right invertible on X_{N+k}, i.e. there exists $R^Q \in \mathcal{R}_{I+W^NQ(V)}$ such that $R^QX_{N+k} \subset X_{N+k}$ and $[I+W^NQ(V)]R^Q = I$ on X_{N+k}. Write $R_Q = I - Q(V)R^QW^N$. If $x \in X_k$ then $u = W^Nx \in X_{N+k}$, $y = R^Qu \in X_{N+k}$ and

$$R_Qx = \big[I-Q(V)R^QW^N\big]x = x - Q(V)y \in X_k.$$

On X_k we have

$$\begin{aligned}\big[I+Q(I,W)-B_NG\big]R_Q &= \big[I+Q(V)W^N\big]\big[I-Q(V)R^QW^N\big]\\ &= I+Q(V)W^N-\big[I+Q(V)W^N\big]Q(V)R^QW^N\\ &= I+Q(V)W^N-Q(V)\big[I+W^NQ(V)\big]R^QW^N\\ &= I+Q(V)W^N-Q(V)W^N=I,\end{aligned}$$

which proves that $I+Q(I,W)-B_NG$ is right invertible on X_k. □

In the same way, we can get proofs for the other cases. For instances, putting $k=0$ in Lemma 2.2 we obtain

Corollary 2.2. *Suppose that all assumptions of Lemma 2.1 are satisfied then the operator $I+Q(I,W)-B_NG$ is right invertible (left invertible, invertible, generalized invertible) if and only if $I+W^NQ(V)$ is right invertible (left invertible, invertible, generalized invertible) on X_N, where $G \in \mathcal{G}_V$.*

Corollary 2.3. *Suppose that all assumptions of Lemma 2.1 are satisfied. If $I+Q(I,W)-B_NG$ is invertible, then unique solution of the equation*

$$\big[I+W^NQ(V)\big]x = y, \quad y \in X_N,$$

belongs to X_N.

Theorem 2.3. *Suppose that* $V \in R_1(X)$, $W \in \mathcal{R}_V^{(1)}$ *and* $Q[V]$ *is given* (2.9). *Write*

$$Q(A) := \sum_{m=0}^{M}\sum_{n=0}^{N} W^{M-m} A_{mn} W^{N-n} - \sum_{m=0}^{M} W^{M-m}\widetilde{A}_{mN} G, \tag{2.10}$$

$$\widetilde{Q}(A) := \sum_{m=0}^{M}\sum_{n=0}^{N} W^{M-m}\widetilde{A}_{mn} V^n, \tag{2.11}$$

$$\widetilde{A}_{mn} := \begin{cases} 0 & \text{if } m = M,\ n = N, \\ A_{mn} & \text{if } m + n < M + N. \end{cases} \tag{2.12}$$

(i) *If* $Q(A)$ *is invertible, then all solutions of the equation* (2.9) *are given by*

$$x = \left[I - W^N Q^{-1}(A)\widetilde{Q}(A)\right]\Big(W^{M+N} y + \sum_{j=0}^{M+N-1} W^j z_j\Big), \tag{2.13}$$

where $z_0, \dots, z_{M+N-1} \in \ker V$ *are arbitrary.*

(ii) *If* $Q(A) \in R(X)$ *and* $R_Q \in \mathcal{R}_{Q(A)}$, *then all solutions of the equation* (2.9) *are given by*

$$x = \left[I - W^N R_Q \widetilde{Q}(A)\right]\Big(W^{M+N} y + \sum_{j=0}^{M+N-1} W^j z_j\Big) + z, \tag{2.14}$$

where $z_0, \dots, z_{M+N-1} \in \ker V$, $z \in \ker\,[I + W^N \widetilde{Q}(A)]$ *are arbitrary.*

(iii) *If* $Q(A) \in \Lambda(X)$ *and* $L_Q \in \mathcal{L}_{Q(A)}$, *then* (2.9) *is solvable if and only if there exist* $z_0, \dots, z_{M+N-1} \in \ker V$ *and* $y \in X_{M+N}$ *such that*

$$W^{M+N} y + \sum_{j=0}^{M+N-1} W^j z_j \in \left[I + W^N \widetilde{Q}(A)\right] X_{M+N}. \tag{2.15}$$

If this condition is satisfied, then all solutions of the equation (2.9) *are given by*

$$x = \left[I - W^N L_Q \widetilde{Q}(A)\right]\Big(W^{M+N} y + \sum_{j=0}^{M+N-1} W^j z_j\Big), \tag{2.16}$$

where $z_0, \dots, z_{M+N-1} \in \ker V$ *are arbitrary.*

(iv) *If* $Q(A) \in W(X)$ *and* $W_Q \in \mathcal{W}_{Q(A)}$, *then* (2.9) *is solvable if and only if condition* (2.15) *is satisfied and then all solutions of the equation* (2.9) *are given by*

$$x = \left[I - W^N W_Q \widetilde{Q}(A)\right]\Big(W^{M+N} y + \sum_{j=0}^{M+N-1} W^j z_j\Big) + z, \tag{2.17}$$

where $z_0, \dots, z_{M+N-1} \in \ker V$, $z \in \ker\,[I + W^N \widetilde{Q}(A)]$ *are arbitrary.*

Proof. We have

$$\Big(\sum_{m=0}^{M}\sum_{n=0}^{N} V^m A_{mn} V^n\Big)x = y,$$

$$\Big(V^{M+N} + \sum_{m=0}^{M}\sum_{n=0}^{N} V^m \widetilde{A}_{mn} V^n\Big)x = y,$$

$$\Big[V^{M+N}\Big(I + \sum_{m=1}^{M}\sum_{n=0}^{N} W^{M+N-m} \widetilde{A}_{mn} V^n\Big) + \sum_{n=0}^{N} \widetilde{A}_{0N} V^n\Big]x = y.$$

By Theorem 2.2, we imply

$$\Big(I + \sum_{m=1}^{M}\sum_{n=0}^{N} W^{M+N-m} \widetilde{A}_{mn} V^n\Big)x = W^{M+N}\Big(y - \sum_{n=0}^{N} \widetilde{A}_{0N} V^n x\Big) + \sum_{j=0}^{M+N-1} W^j z_j$$

$$\Big(I + W^N \sum_{m=0}^{M}\sum_{n=0}^{N} W^{M-m} \widetilde{A}_{mn} V^n\Big)x = W^{M+N} y + \sum_{j=0}^{M+N-1} W^j z_j$$

$$\Big[I + W^N \widetilde{Q}(A)\Big]x = W^{M+N} y + \sum_{j=0}^{M+N-1} W^j z_j. \tag{2.18}$$

It is easy to see that

$$Q(A) = I + \widetilde{Q}(A) W^N. \tag{2.19}$$

(i) If $Q(A)$ is invertible, then $Q(A)$ is invertible on X_M. Lemma 2.2 and (2.19) together imply that

$$\big[I + W^N \widetilde{Q}(A)\big]^{-1} := I - W^N Q^{-1}(A) \widetilde{Q}(A).$$

This and (2.18) imply (2.13).

(ii) If $Q(A)$ is right invertible, then $Q(A)$ is right invertible on X_M. Lemma 2.2 and (2.19) together imply that $I + W^N \widetilde{Q}(A)$ is right invertible on X_{M+N}. Moreover, $R_{\widetilde{Q}} := I - W^N R_Q \widetilde{Q}(A)$ is a right inverse of $I + W^N \widetilde{Q}(A)$ and $R_{\widetilde{Q}} X_{M+N} \subset X_{M+N}$. This and (2.18) together imply (2.14).

(iii) If $Q(A)$ is left invertible, then $Q(A)$ is left invertible on X_M. Lemma 2.2 and (2.19) together imply $I + W^N \widetilde{Q}(A)$ is left invertible on X_{M+N}. Moreover, $L_{\widetilde{Q}} := I - W^N L_Q \widetilde{Q}(A)$ is a left inverse of $I + W^N \widetilde{Q}(A)$ and $L_{\widetilde{Q}} X_{M+N} \subset X_{M+N}$. This and (2.18) imply that (2.9) has solutions if only if the condition (2.15) is satisfied. If this is the case, all solutions are of the form (2.16).

(iv) If $Q(A)$ is generalized invertible, then $Q(A)$ is generalized invertible on X_M. Lemma 2.2 and (2.19) together imply that $I + W^N \widetilde{Q}(A)$ is generalized invertible on X_{M+N}. Moreover, $W_{\widetilde{Q}} = I - W^N W_Q \widetilde{Q}(A)$ is a generalized inverse of $I + W^N \widetilde{Q}(A)$ and $W_{\widetilde{Q}} X_{M+N} \subset X_{M+N}$. This and (2.18) imply that (2.9) has solutions if only if the condition (2.15) is satisfied. If this is the case, all solutions are of the form (2.17). The theorem is complete. □

Putting $A_{mn} = 0$, $(m = 0, \ldots, M-1;\ n = 0, \ldots, N)$ and $A_{Mn} = A_n$, $(n = 0, \ldots, N)$ in Theorem 2.3, we obtain

Corollary 2.4. *Suppose that $V \in R_1(X)$ and $W \in \mathcal{R}_V^{(1)}$. Write*

$$Q(V) := \sum_{n=0}^{N} A_n V^n, \quad P(V) := V^M Q(V),$$

$$Q_1 := \sum_{n=0}^{N} A_n W^{N-n}, \quad \widetilde{Q}_1 := \sum_{n=0}^{N-1} A_n V^n.$$

(i) *If Q_1 is invertible, then all solutions of the equation*

$$P(V)x = y, \quad y \in \operatorname{Im} P(V) \tag{2.20}$$

are given by

$$x = \big[I - W^N Q_1^{-1} \widetilde{Q}_1\big]\Big(W^{M+N} y + \sum_{j=0}^{M+N-1} W^j z_j\Big),$$

where $z_0, \ldots, z_{M+N-1} \in \ker V$ are arbritrary.

(ii) *If $Q_1 \in R(X)$ and $R_{Q_1} \in \mathcal{R}_{Q_1}$, then all solutions of the equation* (2.20) *are given by*

$$x = \big[I - W^N R_{Q_1} \widetilde{Q}_1\big]\Big(W^{M+N} y + \sum_{j=0}^{M+N-1} W^j z_j\Big) + z,$$

where $z_0, \ldots, z_{M+N-1} \in \ker V$, $z \in \ker \big[I + W^N \widetilde{Q}_1\big]$ are arbitrary.

(iii) *If $Q_1 \in \Lambda(X)$ and $L_{Q_1} \in \mathcal{L}_{Q_1}$, then* (2.20) *is solvable if and only if there exist $z_0, \ldots, z_{M+N-1} \in \ker V$ and $y \in X_{M+N}$ such that*

$$W^{M+N} y + \sum_{j=0}^{M+N-1} W^j z_j \in \big(I + W^N \widetilde{Q}_1\big) X_{M+N}. \tag{2.21}$$

If this condition is satisfied, then all solutions of the equation (2.20) *are given by*

$$x = \big[I - W^N L_Q \widetilde{Q}_1\big]\Big(W^{M+N} y + \sum_{j=0}^{M+N-1} W^j z_j\Big),$$

where $z_0, \ldots, z_{M+N-1} \in \ker V$ are arbitrary.

(iv) *If $Q_1 \in W(X)$ and $W_{Q_1} \in \mathcal{W}_{Q_1}$, then* (2.20) *is solvable if and only if condition* (2.21) *is satisfied, then all solutions of the equation* (2.20) *are given by*

$$x = \big[I - W^N W_Q \widetilde{Q}_1\big]\Big(W^{M+N} y + \sum_{j=0}^{M+N-1} W^j z_j\Big) + z,$$

where $z_0, \dots, z_{M+N-1} \in \ker V$, $z \in \ker [I + W^N \widetilde{Q}_1]$ *are arbitrary.*

Putting $A_{mn} = 0$, $(m = 0, \dots, M;\ n = 0, \dots, N-1)$ and $A_{mN} = A_m$, $(m = 0, \dots, M)$ in Theorem 2.3, we obtain

Corollary 2.5. *Suppose that* $V \in R_1(X)$, $W \in \mathcal{R}_V^{(1)}$, *and* $G \in \mathcal{G}_V$. *Write*

$$Q\langle V\rangle := \sum_{m=0}^{M} V^m A_m, \quad P\langle V\rangle := Q\langle V\rangle V^N,$$

$$Q_2 := \sum_{m=0}^{M} W^{M-m} A_m - \sum_{m=0}^{M-1} W^{M-m} A_m G, \quad \widetilde{Q}_2 := \sum_{m=0}^{M-1} W^{M-m} A_m V^N.$$

(i) *If* Q_2 *is invertible, then all solutions of the equation*

$$P\langle V\rangle x = y, \quad y \in \operatorname{Im} P\langle V\rangle, \tag{2.22}$$

are given by

$$x = \big[I - W^N Q_2^{-1} \widetilde{Q}_2\big]\Big(W^{M+N} y + \sum_{j=0}^{M+N-1} W^j z_j\Big),$$

where $z_0, \dots, z_{M+N-1} \in \ker V$ *are arbitrary.*

(ii) *If* $Q_2 \in R(X)$ *and* $R_{Q_2} \in \mathcal{R}_{Q_2}$, *then all solutions of the equation* (2.22) *are given by*

$$x = \big[I - W^N R_{Q_2} \widetilde{Q}_2\big]\Big(W^{M+N} y + \sum_{j=0}^{M+N-1} W^j z_j\Big) + z,$$

where $z_0, \dots, z_{M+N-1} \in \ker V$, $z \in \ker \big(I + W^N \widetilde{Q}_2\big)$ *are arbitrary.*

(iii) *If* $Q_2 \in \Lambda(X)$ *and* $L_{Q_2} \in \mathcal{L}_{Q_2}$, *then* (2.22) *is solvable if and only if there exist* $z_0, \dots, z_{M+N-1} \in \ker V$ *and* $y \in X_{M+N}$ *such that*

$$W^{M+N} y + \sum_{j=0}^{M+N-1} W^j z_j \in \big(I + W^N \widetilde{Q}_2\big) X_{M+N}. \tag{2.23}$$

If this condition is satisfied, then all solutions of the equation (2.22) *are given by*

$$x = \big[I - W^N L_Q Q_2\big]\Big(W^{M+N} y + \sum_{j=0}^{M+N-1} W^j z_j\Big),$$

where $z_0, \dots, z_{M+N-1} \in \ker V$ *are arbitrary.*

(iv) *If* $Q_2 \in W(X)$ *and* $W_{Q_2} \in \mathcal{W}_{Q_2}$, *then* (2.22) *is solvable if and only if condition* (2.23) *is satisfied, then all solutions of the equation* (2.22) *are given by*

$$x = \big[I - W^N W_Q Q_2\big]\Big(W^{M+N} y + \sum_{j=0}^{M+N-1} W^j z_j\Big) + z,$$

where $z_0, \dots, z_{M+N-1} \in \ker V$, $z \in \ker \big[I + W^N \widetilde{Q}_2\big]$ *are arbitrary.*

Example 2.6. Let X be a linear space, let $V \in R_1(X)$, $\dim\ker V \neq 0$, $\dim\operatorname{coker} V \neq 0$, $W \in \mathcal{R}_V^{(1)}$ and let $A, B \in L_0(X)$, $AX \subset \operatorname{dom} V$. Consider the equation

$$(VAV + B)x = y, \quad y \in \operatorname{Im}(VAV + B). \tag{2.24}$$

It can be written as $V^2\big[I + W(AV - V)\big]x = y - Bx$, which is equivalent to

$$\big[I + W(AV - V + WB)\big]x = W^2y + Wz_1 + z_0.$$

Write $Q(A,B) = I + (AV - V + WB)W = G + A(I - G) + WBW$ for $G \in \mathcal{G}_V$.

(i) If $Q(A,B)$ is invertible, then all solutions of the equation (2.24) are given by

$$x = \big[I - WQ^{-1}(A,B)(AV - V + WB)\big]\big(W^2y + Wz_1 + z_0\big).$$

(ii) If $Q(A,B) \in R(X)$ and $R_Q \in \mathcal{R}_{Q(A,B)}$, then all solutions of the equation (2.24) are given by

$$x = \big[I - WR_Q(AV - V + WB)\big]\big(W^2y + Wz_1 + z_0\big) + z.$$

(iii) If $Q(A,B) \in \Lambda(X)$ and $L_Q \in \mathcal{L}_{Q(A,B)}$, then all solutions of the equation (2.24) are given by

$$x = \big[I - WL_Q(AV - V + WB)\big]\big(W^2y + Wz_1 + z_0\big).$$

(iv) If $Q(A,B) \in W(X)$ and $W_Q \in \mathcal{W}_{Q(A,B)}$, then all solutions of the equation (2.24) are given by

$$x = \big[I - WW_Q(AV - V + WB)\big]\big(W^2y + Wz_1 + z_0,\big) + z,$$

where $z_0, z_1 \in \ker V$, $z \in \ker\,[I + W(AV - V + WB)]$ are arbitrary.

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