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OPTIMALITY CONDITIONS FOR CONTROLS ACTING AS COEFFICIENTS OF A NONLINEAR ORDINARY DIFFERENTIAL EQUATION OF SECOND ORDER

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ABSTRACT. We study in this paper a control problem associated to a semilinear second order ordinary differential equation with pointwise state constraints. The control acts as a coefficient of the state equation. For this problem, we prove the existence of optimal controls and obtain a necessary optimality condition. This condition looks somehow like Pontryagin's maximum principle. We end this work by giving illustrative examples where we apply our results.

1. INTRODUCTION AND STATEMENT OF THE PROBLEM

The purpose of this paper is to study the following control problem where the controls play as coefficients of a nonlinear second order differential equation. The nonlinear character of this equation is given by the action of a Nemytskij operator. To be precise, we are concerned by finding:

$$\inf \mathcal{J}(y), \quad \mathcal{J}(y) = \int_0^1 (y(x) - h(x))^2 \, dx,$$

where the state y verifies the equation:

$$\frac{d}{dx}\Big(\frac{y'(x)}{u(x)}\Big) + \theta(y(x)) = 0, \quad x \in (0,1), \quad y(0) = 0, \quad y'(1) = 0, \tag{1}$$

under the constraints

$$0 \le y(x) \le a \quad \forall x \in [0,1], \tag{2}$$

where h is a fixed continuous function on $[0,1]$, and $a > 0$ is a fixed number. The control u is belonging to a compact subset Σ_{ad} of the Banach space $C([0,1])$ which is contained in $C^1([0,1])$, and verifying $\omega \le u(x) \le \Omega$ for all $x \in [0,1]$, where ω and Ω are fixed numbers in $]0, \infty[$. For example, we may take

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$$\begin{aligned}\Sigma_{ad} = \big\{u \in C^1([0,1]) : \omega \le u(x) \le \Omega \quad \forall x \in [0,1], \\ |u(x_1) - u(x_2)| \le k|x_1 - x_2|, \quad \forall x_1, x_2 \in [0,1]\big\}.\end{aligned} \tag{3}$$

where $k \in]0, \infty[$. The function $\theta \in C^1([0,a])$ is supposed to verify $0 \le \theta(x) \le b$ for all $x \in [0,a]$ and

$$|\theta(x_1) - \theta(x_2)| \le l|x_1 - x_2|, \quad \forall x_1, x_2 \in [0,a],$$

where b, l are two fixed numbers in $]0, \infty[$.

We shall denote our control problem by $\mathbf{P}_\theta$. We see that the controls are acting in this setting as coefficients for the state equation associated to $\mathbf{P}_\theta$. Similar problems were considered in [2], [5], and [4], by completely different methods. General remarks concerning coefficient control problems in both ordinary and partial differential equations can be found in [6]. We notice that this paper is a sequel of the papers [3], [1] where investigations are made for smooth and nonsmooth optimal Lipschitz control for problems governed by semilinear second order differential equations in which the nonlinear part is given by the action of a Nemytskij operator.

2. Solutions of the state equation and existence of optimal controls

In this section, we give some sufficient conditions ensuring existence for solutions to our problem. These conditions are expressed by some inequalities involving the fixed parameters ω, Ω, a, b, k, l listed above.

Existence and uniqueness of the solution of the problem P_θ

Let $u \in \Sigma_{ad}$, and let $G_u = G_u(x, \xi)$ be the uniquely determined Green's function to the next boundary problem

$$\frac{d}{dx}\Big[\frac{y'(x)}{u(x)}\Big] = 0, \quad x \in (0,1), \quad y(0) = 0, \quad y'(1) = 0, \tag{4}$$

An easy computation shows that G_u is given by

$$\begin{cases} G_u(x,\xi) &= -\int_0^\xi u(s)\,ds \quad \text{for} \quad 0 \le \xi \le x \le 1, \\ G_u(x,\xi) &= -\int_0^x u(s)\,ds \quad \text{for} \quad 0 \le x \le \xi \le 1. \end{cases}$$

G_u is continuous and symmetric on $[0,1] \times [0,1]$ and the following estimations hold

$$\begin{cases} 0 & \leq -G_u(x,\xi) \leq \Omega \quad \forall x,\xi \in [0,1], \quad \text{and} \\ 0 & \leq -\int\limits_0^1 G_u(x,\xi)\,d\xi \leq \int\limits_0^1 (1-s)u(s)\,ds \leq \dfrac{\Omega}{2} \quad \forall x \in [0,1]. \end{cases}$$

In all what follows, we suppose that $b\Omega \leq 2a$, and that $l\Omega < 2$. The Banach space $C([0,1])$ will be endowed with its usual norm denoted by $\|.\|_{C([0,1])}$. We consider the subset $B_+(a)$ of $C([0,1])$ defined by

$$B_+(a) := \{y \in C([0,1]) : 0 \leq y(x) \leq a \quad \forall x \in [0,1]\}$$

and take a fixed control $u \in \Sigma_{ad}$. It is clear that an element $y \in C^2([0,1]) \cap B_+(a)$ is a solution to the nonlinear boundary value problem (1) associated to $\mathbf{P}_\theta$ if and only if $y \in B_+(a)$ and y is a solution to the Hammerstein integral equation

$$y(x) = -\int\limits_0^1 G_u(x,\xi)\theta(y(\xi))\,d\xi \quad \forall x \in [0,1]. \tag{5}$$

This consideration will enable us to state and prove our first result.

Theorem 2.1. *For each control $u \in \Sigma_{ad}$, the boundary value problem* (1) *associated to $\mathbf{P}_\theta$ has a unique solution y_u. Moreover, this solution belongs to $C^2([0,1]) \cap B_+(a)$.*

Proof. Let $u \in \Sigma_{ad}$ be fixed and associate to it the map T_u defined from $B_+(a)$ to $C([0,1])$ by

$$T_u(y)(x) := -\int\limits_0^1 G_u(x,\xi)\theta(y(\xi))\,d\xi \quad \forall x \in [0,1]. \tag{6}$$

An easy computation will show that for all $y, z \in B_+(a)$ we have

$$\|T_u(y) - T_u(z)\|_{C([0,1])} \leq \frac{l\Omega}{2}\|y - z\|_{C([0,1])}. \tag{7}$$

According to the fact that $b\Omega \leq 2a$, we see that $T_u(B_+(a)) \subset B_+(a)$. Moreover, since by asssumption we have $l\Omega < 2$, then the map T_u must be a contraction from $B_+(a)$ to itself. Since the set $B_+(a)$ is a closed (convex) subset of the Banach space $C([0,1])$, we can use the Banach fixed point theorem and assert that T_u has a unique fixed point $y_u \in B_+(a)$. This ensures the solvability of (1) and the uniqueness of the solution to this problem. □

Existence of optimal controls for the problem P_θ

To each control $u \in \Sigma_{ad}$ we associate the unique solution $S(u) = y_u$ to the problem (1) (under condition (2)). One can see that S is a Lipschitz continuous map from the compact convex subset Σ_{ad} of $C([0,1])$ to the Banach space

$C([0,1])$. Indeed, for all controls $u, v \in \Sigma_{ad}$, an easy computation will give the following estimation

$$\|y_u - y_v\|_{C([0,1])} \leq \frac{2b}{2 - l\Omega}\|u - v\|_{C([0,a])}. \tag{8}$$

Now, we can prove the existence of optimal controls for our problem $\mathbf{P}_\theta$.

Theorem 2.2. *The optimal control problem* $\mathbf{P}_\theta$ *has an optimal solution* $u_0 \in \Sigma_{ad}$.

Proof. For each control $u \in \Sigma_{ad}$ we set $j(u) := \mathcal{J}(y_u)$. We obtain by easy computation the following estimation

$$|j(u) - j(v)| \leq \frac{4b(a + \|h\|_{C([0,1])})}{2 - l\Omega}\|u - v\|_{C([0,1])}. \tag{9}$$

This inequality says that the map j is Lipschitz continuous from the compact subset Σ_{ad} of the Banach space $C([0,1])$ to the set of real numbers. Hence, using the classical Weierstrass theorem, we see that there exists at least one optimal control to our problem (P_θ). □

3. Necessary optimality conditions for $\mathbf{P}_\theta$

Let $u_0 \in \Sigma_{ad}$ be an optimal control to the problem $\mathbf{P}_\theta$ and $u \in \Sigma_{ad}$ another admissible control. The respective states are denoted by $y_0 = S(u_0)$ and $y_u = S(u)$. For any $\lambda \in [0,1]$ we set

$$u_\lambda := u_0 + \lambda(u - u_0) \in \Sigma_{ad}, \quad \text{and} \quad y_\lambda := S(u_\lambda).$$

From (8) we see that

$$\left\|y_\lambda - y_0\right\|_{C([0,1])} \leq \frac{2b\lambda}{2 - l\Omega}\left\|u - u_0\right\|_{C([0,1])}, \quad \forall \lambda \in [0,1]. \tag{10}$$

We set $q(x) := \int_0^1 G_u(x,\xi)\theta(y_0(\xi))\,d\xi$ for all $x \in [0,1]$, and consider the map Υ from $C([0,1])$ to itself defined for all $z \in C([0,1])$ by

$$\Upsilon(z)(x) := -y_0(x) - q(x) - \int_0^1 G_{u_0}(x,\xi)\theta'(y_0(\xi))z(\xi)\,d\xi \quad \forall x \in [0,1].$$

We see that q and Υ depend on the controls u_0, and u. By an easy computation we obtain for all $z_1, z_2 \in C([0,1])$ the following estimation

$$\left\|\Upsilon(z_1) - \Upsilon(z_2)\right\|_{C([0,1])} \leq \frac{l\Omega}{2}\left\|z_1 - z_2\right\|_{C([0,1])}. \tag{11}$$

Inequality (11) shows that the map Υ is Lipschitz continuous and we can apply the Banach fixed point theorem to assert that there exists a unique element $\tilde{y} \in C([0,1])$ such that $\Upsilon(\tilde{y}) = \tilde{y}$.

Now with these notations and considerations, we are ready to state and prove our main result.

Theorem 3.1. *Let $u_0 \in \Sigma_{ad}$ be an optimal and let $u \in \Sigma_{ad}$. For all $\lambda \in]0,1]$, we set*

$$\Delta_\lambda := \frac{y_\lambda - y_0}{\lambda}\cdot$$

Then the following assertions hold true

(1) $\Delta_\lambda \longrightarrow \tilde{y}$ *in the space* $C([0,1])$ *when* $\lambda \longrightarrow 0$.

(2) $\int_0^1 (y_0(x) - h(x))\tilde{y}(x)\,dx \geq 0$. *(This is the necessary optimality condition)*

Proof. (1) Let $\tilde{y}$ be the unique fixed point of the map Υ, and set $H_\lambda(x) := \Delta_\lambda(x) - \tilde{y}(x)$ for all $\lambda \in]0,1]$ and all $x \in [0,1]$. Then we can write $H_\lambda = A_\lambda + B_\lambda$, where $A_\lambda(x)$ and $B_\lambda(x)$ are given for all $x \in [0,1]$ by

$$\begin{aligned} A_\lambda(x) = &\int_0^1 G_{u_0}(x,\xi)\tilde{y}(\xi)\theta'(y_0(\xi))\,d\xi \\ &- \frac{1}{\lambda}\int_0^1 (\theta(y_\lambda(\xi)) - \theta(y_0(\xi)))G_{u_0}(x,\xi)\,d\xi, \quad \text{and} \end{aligned}$$

$$\begin{aligned} B_\lambda(x) = y_0(x) + &\int_0^1 G_u(x,\xi)\theta(y_0(\xi))\,d\xi \\ &- \frac{1}{\lambda}\int_0^1 (G_{u_\lambda}(x,\xi) - G_{u_0}(x,\xi))\theta(y_\lambda(\xi))\,d\xi. \end{aligned}$$

By the classical mean value theorem, for every $\xi \in [0,1]$ there exists a real number $\omega(\xi,\lambda) \in (0,1)$ such that

$$\frac{\theta(y_\lambda(\xi)) - \theta(y_0(\xi))}{\lambda} = \theta'\big(y_0(\xi) + \omega(\xi,\lambda)(y_\lambda(\xi) - y_0(\xi))\big)\frac{y_\lambda(\xi) - y_0(\xi)}{\lambda}.$$

Then we get

$$\begin{aligned} A_\lambda(x) &= \int_0^1 \Big[\tilde{y}(\xi)\theta'(y_0(\xi)) - \frac{\theta(y_\lambda(\xi)) - \theta(y_0(\xi))}{\lambda}\Big]G_{u_0}(x,\xi)\,d\xi \\ &= \int_0^1 G_{u_0}(x,\xi)\big[\tilde{y}(\xi)\theta'(y_0(\xi)) - \Delta_\lambda(\xi)[\theta'\big(y_0(\xi) + \omega(\xi,\lambda)(y_\lambda(\xi) - y_0(\xi))\big)]\big]\,d\xi \end{aligned}$$

$$= \int_0^1 G_{u_0}(x,\xi)\big[\tilde{y}(x) - \Delta_\lambda(\xi)\big]\theta'(y_0(\xi)\, d\xi$$

$$+ \int_0^1 G_{u_0}(x,\xi)\Delta_\lambda(\xi)\big(\theta'(y_0(\xi)) - \theta'(y_0(\xi) + \omega(\xi,\lambda)(y_\lambda(\xi) - y_0(\xi))\big)d\xi.$$

The last equality implies the following estimation

$$\big\|A_\lambda\big\|_{C([0,1])} \leq \frac{l\Omega}{2}\big\|\Delta_\lambda - \tilde{y}\big\|_{C([0,1])}$$

$$+ \frac{lb\Omega}{2 - l\Omega}\big\|u - u_0\big\|_{C([0,1])} \int_0^1 |\theta'(y_0(\xi)) - \theta'[y_0(\xi) + \omega(\xi,\lambda)(y_\lambda(\xi) - y_0(\xi))]|d\xi.$$

On the other hand, for all x, $\xi \in [0,1]$, we have

$$\frac{1}{\lambda}(G_{u_\lambda}(x,\xi) - G_{u_0}(x,\xi)) = G_u(x,\xi) - G_{u_0}(x,\xi).$$

Therefore, for $B_\lambda(x)$ we have the espression

$$B_\lambda(x) = \int_0^1 G_u(x,\xi)\big[\theta(y_0(\xi)) - \theta(y_\lambda(\xi))\big]\, d\xi$$

$$+ \int_0^1 G_{u_0}(x,\xi)\big[\theta(y_\lambda(\xi)) - \theta(y_0(\xi))\big]\, d\xi,$$

which yields the following inequality

$$\big\|B_\lambda\big\|_{C([0,1])} \leq l\Omega\big\|y_\lambda - y_0\big\|_{C([0,1])}.$$

Now, we reach the final conclusion. Indeed, from the estimates obtained above we get the following inequality

$$\big(1 - \frac{l\Omega}{2}\big)\big\|\Delta_\lambda - \tilde{y}\big\|_{C([0,1])} \leq l\Omega\big\|y_\lambda - y_0\big\|_{C([0,1])}$$

$$+ \frac{lb\Omega}{2 - l\Omega}\big\|u - u_0\big\|_{C([0,1])} \int_0^1 |\theta'(y_0(\xi)) - \theta'[y_0(\xi) + \omega(\xi,\lambda)(y_\lambda(\xi) - y_0(\xi))]|d\xi.$$

This inequality implies that Δ_λ converges to $\tilde{y}$ in the space $C([0,1])$ when $\lambda \longrightarrow 0$. Thus our claim (1) is proved.

(2) The optimality condition is easily obtained from assertion (1) and the following inequality

$$0 \leq \frac{j(u_\lambda) - j(u_0)}{\lambda} = 2\int_0^1 \frac{y_\lambda - y_0}{\lambda}(y_0 - h)dx + \frac{1}{\lambda}\int_0^1 (y_\lambda - y_0)^2 dx, \tag{12}$$

which holds for all control $u \in \Sigma_{ad}$, and all $\lambda \in]0,1]$. □

4. Illustrative examples

Example 4.1. We take θ identically equal to one on $[0,a]$. We take $\Sigma_{ad} := [\omega,\Omega]$, which means that we take constant controls on the interval $[0,1]$. We take $b = k = l = 1$. We let a to be any positive constant such that $a \geq \frac{\Omega}{2}\cdot$ The decision function h is taken to be an element of $L^2([0,1])$ satisfying

$$\frac{15}{2}\omega \leq \int_0^1 h(x)\psi(x)\,dx \leq \frac{15}{2}\Omega,$$

where $\psi(x) = -\frac{x^2}{2} + x$ for all $x \in [0,1]$. It is easy to see that all the conditions and assumptions of our theory are satisfied. It is easy to see that for each control $u \in [\omega,\Omega]$, the solution of equation (1) is given by $y_u = u\psi$. Therefore, the cost functional is given by

$$\mathcal{J}(u) = Au^2 - 2Bu + C,$$

where

$$A = \int_0^1 \psi(x)^2\,dx = \frac{2}{15}, \quad B = \int_0^1 h(x)\psi(x)dx, \quad \text{and} \quad C = \int_0^1 h(x)^2 dx.$$

In this case, it is clear that the optimal control exists and is given by

$$u_* = \frac{15}{2}\int_0^1 h(x)\psi(x)dx.$$

Let us determine this optimal control by using Theorem 3.1. Let u_0 be an optimal control of this problem. We know that u_0 must satisfy condition (2) of Theorem 3.1. For each $\lambda \in]0,1[$, we have $\Delta_\lambda = (u-u_0)\psi = \tilde{y}$. Therefore, we must have

$$[u-u_0]\int_0^1 [y_{u_0}(x) - h(x)]\psi(x)\,dx \geq 0, \quad \forall u \in [\omega,\Omega]. \tag{13}$$

(13) is equivalent to say that

$$[u-u_0]\Big[u_0 - \frac{B}{A}\Big] \geq 0, \quad \forall u \in [\omega,\Omega].$$

This inequality implies that

$$u_0 = \frac{B}{A} = \frac{15}{2}\int_0^1 h(x)\psi(x)\,dx = u_*.$$

Thus our results help to determine the optimal control of this problem.

Example 4.2. We take θ identically equal to one on $[0, a]$. We take

$$\Sigma_{ad} := \Big\{ u_{\alpha,\beta} : \omega \le \alpha \le \frac{\Omega}{2},\ 0 \le \beta \le \frac{\Omega}{2} \Big\},$$

where $u_{\alpha,\beta}(x) := \alpha + \beta x$ for all $x \in [0,1]$. It is clear that Σ_{ad} is a compact subset of $C([0,1])$ contained in $C^1([0,1])$ and satisfying the Lipshitz condition with $k = \frac{\Omega}{2}$. We can take $b = l = 1$, and choose a positive constant a such that $\Omega \le 3a$. For each control $u = u_{\alpha,\beta}$, the corresponding state y_u is given by

$$y_u(x) = -\frac{\beta}{3}x^3 + \frac{(\beta-\alpha)}{2}x^2 + \alpha x,$$

and satisfies $0 \le y_u(x) \le a$ for all $x \in [0,1]$. In this case, our problem is to find

$$\min \Big\{ \int_0^1 \Big[-\frac{\beta}{3}x^3 + \frac{(\beta-\alpha)}{2}x^2 + \alpha x - h(x) \Big]^2 dx : (\alpha,\beta) \in \big[\omega, \frac{\Omega}{2}\big] \times \big[0, \frac{\Omega}{2}\big] \Big\},$$

where $h \in L^2([0,1])$ is the decision function. To solve this problem, one can use the gradient and find by classical methods the optimal controls. But we shall try to use the necessary optimality condition found in our Theorem 3.1 in order to determine the solutions of this optimal control problem. Our general theory ensures the existence of optimal controls. So, let $u_0(x) = \alpha_0 + \beta_0 x$ be an optimal control. An easy computation will show that for any $u \in \Sigma_{ad}$ the state $\tilde{y} = \tilde{y}(u_{\alpha,\beta}, u_0)$ is given dor all $x \in [0,1]$ by

$$\tilde{y}(x) = -(\beta - \beta_0)\frac{x^3}{3} + (\beta - \beta_0 + \alpha_0 - \alpha)\frac{x^2}{2} + (\alpha - \alpha_0)x.$$

This implies that $\tilde{y} = y_u - y_{u_0}$. Then, by Theorem 3.1, the optimal control u_0 satisfies

$$\int_0^1 [y_{u_0}(x) - h(x)][y_u(x) - y_{u_0}(x)]\,dx \ge 0, \quad \forall u \in \Sigma_{ad}. \tag{14}$$

(i) Suppose that h takes its values in $]-\infty, 0]$. Then necessarily we have $\alpha_0 = \omega$. Indeed, if $\alpha_0 > \omega$, we consider the control $u_\varepsilon(x) := \alpha_0 - \varepsilon + \beta_0 x$, (for all $x \in [0,1]$) with $0 < \varepsilon < \alpha_0 - \omega$. A short computation will show that

$$y_{u_\varepsilon}(x) - y_{u_0}(x) = \varepsilon x\big(\frac{x}{2} - 1\big) \le 0, \quad \text{for all } x \in [0,1].$$

By applying (14) to u_ε, we get

$$\int_0^1 x(x-2)(y_{u_0}(x) - h(x))\,dx = 0. \tag{15}$$

Since $y_{u_0}(x) - h(x) \ge 0$, we deduce from (15) that $y_{u_0}(x) = h(x)$ for all $x \in]0,1]$. This will imply that y_{u_0} vanishes on the whole interval $]0,1]$. This is a contradiction. Therefore, $\alpha_0 = \omega$. Now, we shall prove that $\beta_0 = 0$. Take any $\beta \in \big[0, \frac{\Omega}{2}\big]$, and consider the control $u_{\omega,\beta}(x) := \omega + \beta x$, (for all $x \in [0,1]$). A short computation will show that

$$y_{u_{\omega,\beta}}(x) - y_{\omega,0}(x) = \frac{\beta}{6}x^2(3-2x) \geq 0$$

for all $x \in [0,1]$. Another computation will show that

$$J(u_{\omega,\beta}) - J(u_{\omega,0}) = \frac{\beta}{6}\int_0^1 x^2(3-2x)[y_{u_{\omega,\beta}}(x) + y_{u_{\omega,0}}(x) - 2h(x)]dx \geq 0. \tag{16}$$

From (16) we deduce that $u_{\omega,0} = \omega$ is (the unique) optimal control.

(ii) Suppose that h takes its values in $[a, \infty[$. Then necesarily we have $\alpha_0 = \frac{\Omega}{2}$. Indeed, if $\alpha_0 < \frac{\Omega}{2}$, we consider the control $u^\varepsilon(x) := \alpha_0 + \varepsilon + \beta_0 x$, (for all $x \in [0,1]$) with $0 < \varepsilon < \frac{\Omega}{2} - \alpha_0$. A short computation will show that

$$y_{u^\varepsilon}(x) - y_{u_0}(x) = \varepsilon x\big(1 - \frac{x}{2}\big) \geq 0$$

for all $x \in [0,1]$. By assumption, $y_{u_0} - h$ is negative on the interval $[0,1]$. Therefore an aplication of (14) to u^ε gives

$$\int_0^1 x(x-2)(h(x) - y_{u_0}(x))\,dx = 0. \tag{17}$$

Equality (17) implies that $h(x) = y_{u_0}(x)$ for all $x \in [0,1]$. This is possible only when h and u_0 are identically zero, a contradiction. Therefore, we must have $\alpha_0 = \frac{\Omega}{2}$. Now, we shall prove that $\beta_0 = \frac{\Omega}{2}$. Take any $\beta \in \big[0, \frac{\Omega}{2}\big]$, and consider the control $u^\beta(x) := \frac{\Omega}{2} + \beta x$, (for all $x \in [0,1]$). A short computation will show that

$$y_{u^\beta}(x) - y_{u_{\frac{\Omega}{2},\frac{\Omega}{2}}}(x) = \frac{1}{6}\big(\beta - \frac{\Omega}{2}\big)x^2(3-2x) \leq 0$$

for all $x \in [0,1]$. Another computation will show that

$$J(u^\beta) - J(u_{\frac{\Omega}{2},\frac{\Omega}{2}}) = \frac{1}{6}\big(\beta - \frac{\Omega}{2}\big)\int_0^1 x^2(3-2x)[y_{u^\beta}(x) + y_{u_{\frac{\Omega}{2},\frac{\Omega}{2}}}(x) - 2h(x)]dx \geq 0. \tag{18}$$

From (18) we deduce that $u_{\frac{\Omega}{2},\frac{\Omega}{2}}$ is (the unique) optimal control.

In all these examples, Theorem 3.1 helps to determine the optimal controls. It is interesting to provide other examples where our theory should be applied.

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