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ENTIRE EIGENFUNCTIONS OF THE LAPLACIAN OF EXPONENTIAL TYPE WITH RESPECT TO THE LIE NORM AND THE DUAL LIE NORM

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Abstract. We consider entire functions of exponential type with respect to the Lie norm and the dual Lie norm. We characterize them by the growth condition of harmonic components in their double series expansion. Special attention will be paid to the eigenfunctions of the Laplacian.

Introduction

We consider the space of entire functions on $\tilde{\mathbf{E}} = \mathbf{C}^{n+1}$ and denote it by $\mathcal{O}(\tilde{\mathbf{E}})$. Let $F(z) = \sum_{k=0}^{\infty} F_k(z) \in \mathcal{O}(\tilde{\mathbf{E}})$ be the homogeneous expansion of F, where F_k are homogeneous polynomials of degree k. For a norm $N(z)$ on $\tilde{\mathbf{E}}$ put

$$\mathrm{Exp}\,(\tilde{\mathbf{E}}; (r, N)) = \{F \in \mathcal{O}(\tilde{\mathbf{E}}); \forall r' > r, \exists C \geq 0 \text{ s.t. } |F(z)| \leq C \mathrm{Exp}\,(r' N(z))\}$$

and $\|F\|_{C(\tilde{B}_N[1])} = \sup\{|F(z)|; N(z) \leq 1\}$. Then we know that

$$F \in \mathrm{Exp}\,(\tilde{\mathbf{E}}; (r, N)) \Longleftrightarrow \limsup_{k \to \infty} (k! \|F_k\|_{C(\tilde{B}_N[1])})^{1/k} \leq r.$$

An entire function can also be expanded into the double series with $(k-2l)$-homogeneous harmonic polynomials $F_{k,k-2l}, k = 0, 1, \ldots, l = 0, 1, \ldots, [k/2]$;

$$F(z) = \sum_{k=0}^{\infty} F_k(z) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} (z^2)^l F_{k,k-2l}(z),$$

where the convergence is uniform on compact sets in $\tilde{\mathbf{E}}$.

In this paper, we consider the case where the norm $N(z)$ is the Lie norm $L(z)$ or the dual Lie norm $L^*(z)$. First, we formulate, in terms of the growth behavior of $F_{k,k-2l}$, the necessary and sufficient conditions for an entire function F to belong to $\mathrm{Exp}\,(\tilde{\mathbf{E}}; (r, N))$. Here according to Iwahara [5] we will present the following results with improved proofs:

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Let $F(z) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} (z^2)^l F_{k,k-2l}(z)$. Then we have

$$F \in \mathrm{Exp}\,(\tilde{\mathbf{E}}; (r, L)) \iff \limsup_{k\to\infty} (k!\|F_{k,k-2l}\|_{S_1})^{1/k} \le r,$$

$$F \in \mathrm{Exp}\,(\tilde{\mathbf{E}}; (r, L^*)) \iff \limsup_{k\to\infty} (l!(k-l)!\|F_{k,k-2l}\|_{S_1})^{1/k} \le r/2,$$

where $\|F\|_{S_1} = \sup\{|F(z)|; z \in S_1\}$ and S_1 is the unit real sphere. (See Theorems 1.3 and 2.1.)

Second, we shall study the spaces of entire eigenfunctions of the Laplacian of exponential type; $\mathrm{Exp}_{\,\Delta-\lambda^2}(\tilde{\mathbf{E}}; (r, L))$ and $\mathrm{Exp}_{\,\Delta-\lambda^2}(\tilde{\mathbf{E}}; (r, L^*))$. (See Section 5.) Similar to a theorem of R. Wada (Theorem 5.3) and a theorem of A. Martineau (Theorem 1.2) on the Fourier-Borel transformation $\mathcal{F}$, we have the following topological linear isomorphism:

$$\mathcal{F} : \mathcal{O}'(\tilde{S}^*_\lambda[r]) \xrightarrow{\sim} \mathrm{Exp}_{\,\Delta-\lambda^2}(\tilde{\mathbf{E}}; (r, L)), \quad |\lambda| \le r < \infty, \tag{0.1}$$

where $\mathcal{O}'(\tilde{S}^*_\lambda[r])$ is the dual space of the space $\mathcal{O}(\tilde{S}^*_\lambda[r])$ of germs of holomorphic functions on $\tilde{S}^*_\lambda[r] = \{z \in \tilde{\mathbf{E}}; z^2 = \lambda^2, L^*(z) \le r\}$ (see Theorem 5.4).

Thanks to (0.1), we have the following relation which generalizes a result in [11]:

$$\mathrm{Exp}_{\,\Delta-\lambda^2}(\tilde{\mathbf{E}}; (r, L^*)) = \mathrm{Exp}_{\,\Delta-\lambda^2}\left(\tilde{\mathbf{E}}; \left(\frac{r^2 + |\lambda|^2}{2r}, L\right)\right), \quad |\lambda| \le r. \tag{0.2}$$

(See Theorem 5.5). These results were announced in [1].

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1. Preliminary I

1.1. Entire functions of exponential type

Let $N(z)$ be a norm on $\tilde{\mathbf{E}} = \mathbf{C}^{n+1}$. Its dual norm $N^*(z)$ is defined by

$$N^*(z) = \sup\{|z \cdot \zeta|; N(\zeta) \le 1\}.$$

The open and the closed N-balls of radius r with center at 0 are defined by

$$\tilde{B}_N(r) = \{z \in \tilde{\mathbf{E}}; N(z) < r\},\ r > 0, \quad \tilde{B}_N[r] = \{z \in \tilde{\mathbf{E}}; N(z) \le r\},\ r \ge 0.$$

Note that $\tilde{B}_N(\infty) = \tilde{\mathbf{E}}$. We denote by $\mathcal{O}(\tilde{B}_N(r))$ the space of holomorphic functions on $\tilde{B}_N(r)$. Put $\mathcal{O}(\tilde{B}_N[r]) = \liminf_{r'>r} \mathcal{O}(\tilde{B}_N(r'))$,

$$\begin{aligned} \mathrm{Exp}\,(\tilde{\mathbf{E}}; (r, N)) &= \{F \in \mathcal{O}(\tilde{\mathbf{E}}); \forall r' > r, \exists C \ge 0 \text{ s.t. } |F(z)| \le C \exp(r'N(z))\}, \\ \mathrm{Exp}\,(\tilde{\mathbf{E}}; [r, N]) &= \{F \in \mathcal{O}(\tilde{\mathbf{E}}); \exists r' < r, \exists C \ge 0 \text{ s.t. } |F(z)| \le C \exp(r'N(z))\}. \end{aligned}$$

Note that for any norm N on $\tilde{\mathbf{E}}$ we have $\mathrm{Exp}\,(\tilde{\mathbf{E}}; (0, N)) = \mathrm{Exp}\,(\tilde{\mathbf{E}}; (0))$.

We denote by $\mathcal{P}^k(\tilde{\mathbf{E}})$ the space of homogeneous polynomials of degree k. Define the k-homogeneous component $f_k \in \mathcal{P}^k(\tilde{\mathbf{E}})$ of $f \in \mathcal{O}(\{0\})$ by

$$f_k(z) = \frac{1}{2\pi i}\int_{|t|=\rho} \frac{f(tz)}{t^{k+1}}dt, \tag{1.1}$$

where ρ is sufficiently small. Then we know the following theorem (see, for example, [6]).

Theorem 1.1. *Let $N(z)$ be a norm on $\tilde{\mathbf{E}}$ and $F_k \in \mathcal{P}^k(\tilde{\mathbf{E}})$. Then we have*

$$F = \sum_{k=0}^{\infty} F_k(z) \in \mathrm{Exp}\,(\tilde{\mathbf{E}};(r,N)) \iff \limsup_{k\to\infty}(k!\|F_k\|_{C(\tilde{B}_N[1])})^{1/k} \le r,$$
$$F = \sum_{k=0}^{\infty} F_k(z) \in \mathrm{Exp}\,(\tilde{\mathbf{E}};[r,N]) \iff \limsup_{k\to\infty}(k!\|F_k\|_{C(\tilde{B}_N[1])})^{1/k} < r,$$

where $\|F\|_{C(\tilde{B}_N[1])} = \sup\{|F(z)|; N(z) \le 1\}$.

We denote by X' the dual space of X; for example, $\mathcal{O}'(\tilde{B}_N(r))$ means the dual space of $\mathcal{O}(\tilde{B}_N(r))$.

The Fourier-Borel transform $\mathcal{F}T$ of $T \in \mathcal{O}'(\tilde{B}_N[r])$ is defined by

$$\mathcal{F}T(\zeta) = \langle T_z, \exp(z\cdot\zeta)\rangle.$$

We call the mapping $\mathcal{F} : T \mapsto \mathcal{F}T$ the Fourier-Borel transformation.

In [6], A.Martineau proved the following theorem.

Theorem 1.2. *Let $N(z)$ be a norm on $\tilde{\mathbf{E}}$. The Fourier-Borel transformation $\mathcal{F}$ establishes the following topological linear isomorphisms*:

$$\mathcal{F}: \quad \mathcal{O}'(\tilde{B}_N[r]) \xrightarrow{\sim} \mathrm{Exp}\,(\tilde{\mathbf{E}};(r,N^*)), \quad 0 \le r < \infty,$$
$$\mathcal{F}: \quad \mathcal{O}'(\tilde{B}_N(r)) \xrightarrow{\sim} \mathrm{Exp}\,(\tilde{\mathbf{E}};[r,N^*]), \quad 0 < r \le \infty.$$

1.2. Double series expansion

We define the Lie norm $L(z)$ of $z \in \tilde{\mathbf{E}}$ by

$$L(z) = \sqrt{\|z\|^2 + \sqrt{\|z\|^4 - |z^2|^2}}. \tag{1.2}$$

Then we know that $L(z)$ is the cross norm of the Euclidean norm $\|x\|$; that is,

$$L(z) = \inf\left\{\sum_{j=1}^{m}|\lambda_j|\|x_j\|; z = \sum_{j=1}^{m}\lambda_j x_j, \lambda_j \in \mathbf{C}, x_j \in \mathbf{R}^{n+1}, m \in \mathbf{Z}_+\right\}.$$

Thus putting $\|f_k\|_{S_1} = \sup\{|f_k(x)|; x \in S_1\}$, for $f_k \in \mathcal{P}^k(\tilde{\mathbf{E}})$ we can see

$$\|f_k\|_{C(\tilde{B}_L[1])} = \|f_k\|_{S_1}.$$

Let $P_{k,n}(t)$ be the Legendre polynomial of degree k and of dimension $n+1$. The harmonic extension $\tilde{P}_{k,n}(z,w)$ of $P_{k,n}(z\cdot w)$ is given by

$$\tilde{P}_{k,n}(z,w) = (\sqrt{z^2})^k(\sqrt{w^2})^k P_{k,n}\left(\frac{z}{\sqrt{z^2}}\cdot\frac{w}{\sqrt{w^2}}\right).$$

Then $\tilde{P}_{k,n}(z,w)$ is a k-homogeneous harmonic polynomial in z and in w and satisfies

$$|\tilde{P}_{k,n}(z,w)| \le L(z)^k L(w)^k. \tag{1.3}$$

We denote by $\mathcal{P}^k_\Delta(\tilde{\mathbf{E}})$ the space of homogeneous harmonic polynomials of degree k. The dimension of $\mathcal{P}^k_\Delta(\tilde{\mathbf{E}})$ is known to be $(2k+n-1)(k+n-2)!/(k!(n-1)!) \equiv N(k,n)$. For $f \in \mathcal{O}(\tilde{B}(r))$, define the (k,j)-harmonic component of f by

$$f_{k,j}(z) = N(j,n)\int_{S_1} f_k(\tau)\tilde{P}_{j,n}(z,\tau)\dot{d\tau}, \tag{1.4}$$

where f_k is the k-homogeneous component of f defined by (1.1) and $\dot{d\tau}$ is the normalized invariant measure on S_1. Note that $f_{k,j}\in\mathcal{P}^j_\Delta(\tilde{\mathbf{E}})$.

When $N(z) = L(z)$, we omit the subscript; for example, we write $\tilde{B}(r)$ for $\tilde{B}_L(r)$. For a holomorphic function on $\tilde{B}(r)$ we know the following theorem:

Theorem 1.3. ([8, Theorem 3.1]) *Let $f\in\mathcal{O}(\tilde{B}(r))$. Define the k-homogeneous component of f as in* (1.1) *and define the (k,j)-harmonic component of f as in* (1.4). *Then we can expand f into the double series*:

$$f(z) = \sum_{k=0}^{\infty} f_k(z) = \sum_{k=0}^{\infty}\sum_{j=0}^{k}(\sqrt{z^2})^{k-j}f_{k,j}(z) = \sum_{k=0}^{\infty}\sum_{l=0}^{[k/2]}(z^2)^l f_{k,k-2l}(z), \tag{1.5}$$

where the convergence is uniform on compact sets in $\tilde{B}(r)$ and we have

$$\limsup_{k\to\infty}(\|f_{k,k-2l}\|_{S_1})^{1/k} \le 1/r. \tag{1.6}$$

Conversely, if we are given a double sequence $\{f_{k,k-2l}\}$ of homogeneous harmonic polynomials $f_{k,k-2l}(z)$ satisfying (1.6), *then the right-hand side of* (1.5) *converges to a holomorphic function f uniformly on compact sets in $\tilde{B}(r)$ and the $(k,k-2l)$-harmonic component of f is equal to the given $f_{k,k-2l}$.*

2. Exponential type with respect to the Lie norm

For an entire function of exponential type with respect to the Lie norm, we have the following theorem:

Theorem 2.1. ([5, Theorem 3.7]) *Let*

$$F(z) = \sum_{k=0}^{\infty}\sum_{l=0}^{[k/2]}(z^2)^l F_{k,k-2l}(z) \in \mathcal{O}(\tilde{\mathbf{E}}).$$

Then we have

$$\text{(i)} \quad F \in \operatorname{Exp}(\tilde{\mathbf{E}};(r,L)) \Longleftrightarrow \limsup_{k\to\infty} (k!\|F_{k,k-2l}\|_{S_1})^{1/k} \le r,$$

$$\text{(ii)} \quad F \in \operatorname{Exp}(\tilde{\mathbf{E}};[r,L]) \Longleftrightarrow \limsup_{k\to\infty} (k!\|F_{k,k-2l}\|_{S_1})^{1/k} < r.$$

Proof. We prove only (i).

Let $F(z) \in \operatorname{Exp}(\tilde{\mathbf{E}};(r,L))$. By the definition, for any $r' > r$ there exists a constant $C \ge 0$ such that $|F(z)| \le C\exp(r'L(z))$. Then by (1.3), (1.1) and (1.4), we have

$$\begin{aligned} |F_{k,k-2l}(z)| &= \left| N(k-2l,n) \int_{S_1} \tilde{P}_{k-2l,n}(z,\tau) \frac{1}{2\pi i} \int_{|t|=\rho} \frac{F(t\tau)}{t^{k+1}} dt\dot{d\tau} \right| \\ &\le \frac{N(k-2l,n)}{2\pi} \int_{S_1} \int_{|t|=\rho} \left| \tilde{P}_{k-2l,n}(z,\tau) \frac{F(t\tau)}{t^{k+1}} \right| dt\dot{d\tau} \\ &\le C\frac{N(k-2l,n)}{\rho^k} \exp(r'\rho) L(z)^{k-2l}. \end{aligned}$$

Because this inequality holds for any $\rho > 0$, we have

$$|F_{k,k-2l}(z)| \le CN(k-2l,n)\left(\frac{r'}{k}\right)^k \exp(k) L(z)^{k-2l}$$

for $k = 1,2,\cdots$. By the Stirling formula, with another constant $C' > 0$,

$$|F_{k,k-2l}(z)| \le C'N(k-2l,n)\frac{(r')^k\sqrt{k}}{k!} L(z)^{k-2l}.$$

Thus for $k = 1,2,\cdots$, and $l = 0,1,\cdots,[k/2]$,

$$k!\|F_{k,k-2l}\|_{S_1} \le C'N(k-2l,n)\left(r'\right)^k\sqrt{k} \le C'N(k,n)\left(r'\right)^k\sqrt{k}.$$

Therefore, we have

$$\limsup_{k\to\infty} (k!\|F_{k,k-2l}\|_{S_1})^{1/k} \le \limsup_{k\to\infty} \left(C'N(k,n)\left(r'\right)^k\sqrt{k}\right)^{1/k} \le r'.$$

Since $r' > r$ is arbitrary, we have

$$\limsup_{k\to\infty} (k!\|F_{k,k-2l}\|_{S_1})^{1/k} \le r. \tag{2.1}$$

Conversely suppose that $F \in \mathcal{O}(\tilde{\mathbf{E}})$ satisfies (2.1). Then for any $\delta > 0$ there exists $C \ge 0$ such that

$$\|F_{k,k-2l}\|_{S_1} \le C\frac{r^k(1+\delta)^k}{k!}.$$

Thus we have

$$|F_{k,k-2l}(z)| \le L(z)^{k-2l}\|F_{k,k-2l}\|_{S_1} \le L(z)^{k-2l}C\frac{r^k(1+\delta)^k}{k!}.$$

Therefore, we have

$$\begin{aligned}|F(z)| &= \left|\sum_{k=0}^{\infty}\sum_{l=0}^{[k/2]}(z^2)^l F_{k,k-2l}(z)\right| \\ &\leq \sum_{k=0}^{\infty}\sum_{l=0}^{[k/2]} L(z)^{2l}L(z)^{k-2l}C\frac{r^k(1+\delta)^k}{k!} \\ &\leq C\sum_{k=0}^{\infty}L(z)^k\frac{r^k(1+\delta)^k([k/2]+1)}{k!}.\end{aligned}$$

With another constant $C' > 0$ we have

$$|F(z)| \leq C'\sum_{k=0}^{\infty}\frac{L(z)^k r^k(1+2\delta)^k}{k!} = C'\exp((1+2\delta)rL(z)).$$

Since $\delta > 0$ is arbitrary, $F \in \operatorname{Exp}(\tilde{\mathbf{E}}; (r, L))$. □

3. Preliminary II

3.1. Lie sphere

The Shilov boundary of $\tilde{B}[r]$ is the Lie sphere Σ_r:

$$\Sigma_r = \{re^{i\theta}\omega; 0 \leq \theta < 2\pi,\ \omega \in S_1\} = \{e^{i\theta}\omega; 0 \leq \theta < 2\pi,\ \omega \in S_r\}.$$

Note that $-xe^{i(\theta+\pi)} = xe^{i\theta}$ and $\Sigma_r = (\mathbf{R}/(2\pi\mathbf{Z}) \times S_r)/\sim$, where $\sim$ is the equivalence relation defined by $(\theta, x) \sim (\theta+\pi, -x)$, and that for $f \in \mathcal{O}(\tilde{B}[r])$ we have $\sup\{|f(z)|; z \in \tilde{B}[r]\} = \sup\{|f(z)|; z \in \Sigma_r\}$.

We define the invariant integral over Σ_r by

$$\int_{\Sigma_r} f(z)\dot{d}z = \frac{1}{2\pi}\int_0^{2\pi}\int_{S_1} f(re^{i\theta}\omega)\dot{d}\omega d\theta.$$

For $f, g \in \mathcal{O}(\tilde{B}[r])$, the integral $\int_{\Sigma_r} f(z)\overline{g(z)}\dot{d}z$ is well-defined. Since

$$(3.1)\quad \begin{aligned}(f,g)_{\Sigma_r} \equiv \int_{\Sigma_r} f(z)\overline{g(z)}\dot{d}z &= \sum_{k=0}^{\infty} r^{2k}\int_{S_1} f_k(\omega)\overline{g_k(\omega)}\dot{d}\omega \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{[k/2]} r^{2k}\int_{S_1} f_{k,k-2l}(\omega)\overline{g_{k,k-2l}(\omega)}\dot{d}\omega,\end{aligned}$$

$(\ ,\ )_{\Sigma_r}$ is an inner product on $\mathcal{O}(\tilde{B}[r])$. If $f \in \mathcal{O}(\tilde{B}[r])$ and $g \in \mathcal{O}(\tilde{B}(r))$ (resp. $f \in \mathcal{O}(\tilde{B}(r))$ and $g \in \mathcal{O}(\tilde{B}[r])$), then for $s > 1$ sufficiently close to 1, the integral $\int_{\Sigma_r} f(z/s)\overline{g(sz)}\dot{d}z$ (resp. $\int_{\Sigma_r} f(sz)\overline{g(z/s)}\dot{d}z$) is well-defined and does not depend on

s by (3.1). Thus for $f \in \mathcal{O}(\tilde{B}[r])$ and $g \in \mathcal{O}(\tilde{B}(r))$ or for $f \in \mathcal{O}(\tilde{B}(r))$ and $g \in \mathcal{O}(\tilde{B}[r])$ we write

$$\int_{\Sigma_r} f(z/s)\overline{g(sz)}\dot{d}z = s.\int_{\Sigma_r} f(z)\overline{g(z)}\dot{d}z.$$

3.2. Cauchy-Hua transformation

The Cauchy-Hua kernel $H_r(z,w)$ is defined by

$$H_r(z,w) = H_1(z/r, w/r), \quad H_1(z,w) = \frac{1}{(1 - 2z \cdot \overline{w} + z^2\overline{w}^2)^{(n+1)/2}}\,.$$

Then $H_r(z,\overline{w})$ is holomorphic on $\{(z,w) \in \tilde{\mathbf{E}} \times \tilde{\mathbf{E}}; L(z)L(w) < r^2\}$. Note that $H_r(z,w) = \overline{H_r(w,z)}$ and $H_1(z,\overline{w})$ is expanded as follows;

$$\begin{aligned} H_1(z,\overline{w}) &= \sum_{k=0}^{\infty} \frac{N(k,n+2)(n+1)}{2k+n+1}\tilde{P}_{k,n+2}(z,w) \\ &= \sum_{k=0}^{\infty}\sum_{l=0}^{[k/2]} N(k-2l,n)(z^2)^l(w^2)^l\tilde{P}_{k-2l,n}(z,w). \end{aligned}$$

For $f \in \mathcal{O}(\tilde{B}(r))$, we have the following integral representation:

$$f(z) = s.\int_{\Sigma_r} H_r(z,w)f(w)\dot{d}w.$$

(See, for example, Theorem 5.7 in [9].)

Let $T \in \mathcal{O}'(\tilde{B}[r])$. If $w \in \tilde{B}(r)$, then the mapping $z \mapsto H_r(z,w)$ belongs to $\mathcal{O}(\tilde{B}[r])$. Thus we can define the Cauchy-Hua transform $\mathcal{C}T$ of T by

$$\mathcal{C}T(w) = \overline{\langle T_z, H_r(z,w)\rangle}, \quad w \in \tilde{B}(r).$$

We call the mapping $\mathcal{C} : T \mapsto \mathcal{C}T$ the Cauchy-Hua transformation.

Theorem 3.1. *Let $r > 0$. The Cauchy-Hua transformation $\mathcal{C}$ establishes the following topological antilinear isomorphisms:*

$$\begin{aligned} \mathcal{C} &: \mathcal{O}'(\tilde{B}[r]) \xrightarrow{\sim} \mathcal{O}(\tilde{B}(r)), \\ \mathcal{C} &: \mathcal{O}'(\tilde{B}(r)) \xrightarrow{\sim} \mathcal{O}(\tilde{B}[r]). \end{aligned}$$

Further, we have

$$\langle T, g\rangle = s.\int_{\Sigma_r} g(w)\overline{\mathcal{C}T(w)}\dot{d}w$$

for $T \in \mathcal{O}'(\tilde{B}[r])$ and $g \in \mathcal{O}(\tilde{B}[r])$ or for $T \in \mathcal{O}'(\tilde{B}(r))$ and $g \in \mathcal{O}(\tilde{B}(r))$, which gives the inverse of $\mathcal{C}$.

(For a proof see, for example, Theorem 5.9 in [9].)

3.3. Fourier transformation

Composing the Fourier-Borel transformation $\mathcal{F}$ and the Cauchy-Hua transformation $\mathcal{C}$ on $\mathcal{O}'(\tilde{B}[r])$, we can consider the Fourier transformation $\mathcal{Q}$ on $\mathcal{O}(\tilde{B}(r))$ as $\mathcal{Q} = \mathcal{F} \circ \mathcal{C}^{-1}$. Then by Theorems 3.1 and 1.2, for $f \in \mathcal{O}(\tilde{B}(r))$ we have

$$\mathcal{Q}f(\zeta) = s. \int_{\Sigma_r} \exp(z \cdot \zeta) \overline{f(z)} \dot{d}z.$$

By the definition of $\mathcal{Q}$, Theorems 3.1 and 1.2 imply the following corollary.

Corollary 3.1. *Let $r > 0$. The Fourier transformation $\mathcal{Q}$ establishes the following topological antilinear isomorphisms*:

$$\mathcal{Q}: \quad \mathcal{O}(\tilde{B}(r)) \xrightarrow{\sim} \mathrm{Exp}\,(\tilde{\mathbf{E}}; (r, L^*)),$$
$$\mathcal{Q}: \quad \mathcal{O}(\tilde{B}[r]) \xrightarrow{\sim} \mathrm{Exp}\,(\tilde{\mathbf{E}}; [r, L^*]).$$

Since we know the double series expansion of exponential function;

$$\exp(z \cdot w) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} \frac{\Gamma((n+1)/2) N(k-2l, n)}{2^k l! \Gamma(k-l+(n+1)/2)} (z^2)^l (w^2)^l \tilde{P}_{k-2l,n}(z, w),$$

by simple calculation we can determine the image $\mathcal{Q}f$ of $f \in \mathcal{O}(\tilde{B}(r))$, concretely as follows.

Lemma 3.1. *Let*

$$f(z) = \sum_{k=0}^{\infty} f_k(z) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} (z^2)^l f_{k,k-2l}(z) \in \mathcal{O}(\tilde{B}(r)), \;\; f_{k,k-2l} \in \mathcal{P}_{\Delta}^{k-2l}(\tilde{\mathbf{E}}).$$

Then we have

$$\mathcal{Q}f(\zeta) \;\;=\;\; \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} \frac{r^{2k} \Gamma(\frac{n+1}{2})}{2^k l! \Gamma(k-l+\frac{n+1}{2})} (\zeta^2)^l \overline{f_{k,k-2l}}(\zeta),$$

where we write $\overline{f}(z) = \overline{f(\overline{z})}$.

4. Exponential type with respect to the dual Lie norm

The dual Lie norm $L^*(z) = \sup\{|z \cdot w|; L(w) \leq 1\}$ is given by

$$L^*(z) = \sqrt{(\|z\|^2 + |z^2|)/2}. \tag{4.1}$$

By (1.2) and (4.1) we have

$$L^*(z) = \frac{1}{2}\left(L(z) + \frac{|z^2|}{L(z)}\right) \tag{4.2}$$

and

$$L(z) = L^*(z) + \sqrt{L^*(z)^2 - |z^2|}. \tag{4.3}$$

Since $|\sqrt{z^2}| \leq L^*(z) \leq \|z\| \leq L(z) \leq 2L^*(z)$, we have

$$\tilde{B}_L[r] \subset \tilde{B}_{L^*}[r] \subset \tilde{B}_L[2r].$$

Because $\tilde{B}_L[r]$ and $\tilde{B}_{L^*}[r]$ are convex sets, we have

$$\mathcal{O}'(\tilde{B}_L[r]) \underset{\neq}{\subset} \mathcal{O}'(\tilde{B}_{L^*}[r]) \underset{\neq}{\subset} \mathcal{O}'(\tilde{B}_L[2r]).$$

Applying Theorem 1.2 we have

$$\operatorname{Exp}(\tilde{\mathbf{E}};(r,L^*)) \underset{\neq}{\subset} \operatorname{Exp}(\tilde{\mathbf{E}};(r,L)) \underset{\neq}{\subset} \operatorname{Exp}(\tilde{\mathbf{E}};(2r,L^*)). \tag{4.4}$$

Similar to Theorem 2.1, for the dual Lie norm $L^*(z)$, we have the following theorem:

Theorem 4.1. ([5, Theorem 5.2]) *Let*

$$F(z) = \sum_{k=0}^{\infty} \sum_{j=0}^{[k/2]} (z^2)^l F_{k,k-2l}(z) \in \mathcal{O}(\tilde{\mathbf{E}}).$$

Then we have

(i) $F \in \operatorname{Exp}(\tilde{\mathbf{E}};(r,L^*)) \Longleftrightarrow \limsup_{k\to\infty} (l!(k-l)!\|F_{k,k-2l}\|_{S_1})^{\frac{1}{k}} \leq r/2,$

(ii) $F \in \operatorname{Exp}(\tilde{\mathbf{E}};[r,L^*]) \Longleftrightarrow \limsup_{k\to\infty} (l!(k-l)!\|F_{k,k-2l}\|_{S_1})^{\frac{1}{k}} < r/2.$

Proof. We prove only (i). Let

$$F(\zeta) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} (\zeta^2)^l F_{k,k-2l}(\zeta) \in \operatorname{Exp}(\tilde{\mathbf{E}};(r,L^*)).$$

By Corollary 3.1, there exists $f \in \mathcal{O}(\tilde{B}(r))$ such that $F(\zeta) = \mathcal{Q}f(\zeta) \in \operatorname{Exp}(\tilde{\mathbf{E}};(r,L^*))$. By Lemma 3.1, for $f(z) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} (z^2)^l f_{k,k-2l}(z)$, $f_{k,k-2l} \in \mathcal{P}_\Delta^{k-2l}(\tilde{\mathbf{E}})$, we have

$$F(\zeta) = \sum_{k=0}^{\infty} \sum_{l=0}^{[k/2]} \frac{r^{2k}\Gamma\left(\frac{n+1}{2}\right)}{2^k l!\Gamma\left(k-l+\frac{n+1}{2}\right)} (\zeta^2)^l \overline{f_{k,k-2l}}(\zeta).$$

Thus we have

$$F_{k,k-2l}(\zeta) = \frac{r^{2k}\Gamma\left(\frac{n+1}{2}\right)}{2^k l!\Gamma\left(k-l+\frac{n+1}{2}\right)} \overline{f_{k,k-2l}}(\zeta).$$

Since $f \in \mathcal{O}(\tilde{B}(r))$, by Theorem 1.3, we have

$$\limsup_{k\to\infty} (\|f_{k,k-2l}\|_{S_1})^{1/k} \leq 1/r.$$

Therefore, we have

$$\limsup_{k\to\infty}\left(\frac{l!\Gamma\big(k-l+\dfrac{n+1}{2}\big)}{\Gamma\big(\dfrac{n+1}{2}\big)}\|F_{k,k-2l}\|_{S_1}\right)^{1/k}\le r/2,$$

which is equivalent to

$$\limsup_{k\to\infty}\left(l!(k-l)!\|F_{k,k-2l}\|_{S_1}\right)^{\frac{1}{k}}\le r/2. \tag{4.5}$$

Conversely, assume that a sequence $\{F_{k,k-2l}\}$ of $(k-2l)$-homogeneous harmonic polynomials satisfies (4.5). Then for any $\delta>0$ there exists $C\ge 0$ such that

$$\|F_{k,k-2l}\|_{S_1}\le C\frac{(1+\delta)^k r^k}{2^k l!(k-l)!}. \tag{4.6}$$

Put

$$f_{k,k-2l}(z)=\frac{2^k l!\Gamma\big(k-l+\dfrac{n+1}{2}\big)}{r^{2k}\Gamma\big(\dfrac{n+1}{2}\big)}\overline{F_{k,k-2l}}(z). \tag{4.7}$$

Noting that $\lim\limits_{p\to\infty}\Big(\dfrac{\Gamma(p+q)}{\Gamma(p)}\Big)^{1/p}=1$ for any constant $q\in\mathbf{R}$, by (4.6), we have

$$\limsup_{k\to\infty}\left(\frac{2^k l!\Gamma\big(k-l+\dfrac{n+1}{2}\big)}{\Gamma\big(\dfrac{n+1}{2}\big)r^k}\|F_{k,k-2l}\|_{S_1}\right)^{1/k}\le 1+\delta.$$

Since $\delta>0$ is arbitrary, we have $\limsup\limits_{k\to\infty}\left(\|f_{k,k-2l}\|_{S_1}\right)^{1/k}\le 1/r$. Therefore, the function $f(z)=\sum\limits_{k=0}^{\infty}\sum\limits_{l=0}^{[k/2]}(z^2)^l f_{k,k-2l}(z)$ belongs to $\mathcal{O}(\tilde{B}(r))$ by Theorem 1.3, and $\mathcal{Q}f(\zeta)=\sum\limits_{k=0}^{\infty}\sum\limits_{l=0}^{[k/2]}(\zeta^2)^l F_{k,k-2l}(\zeta)$ by Lemma 3.1 and (4.7). Further by Corollary 3.1, we have

$$F(\zeta)=\sum_{k=0}^{\infty}\sum_{l=0}^{[k/2]}(\zeta^2)^l F_{k,k-2l}(\zeta)\in \mathrm{Exp}\,(\tilde{\mathbf{E}};(r,L^*)).$$

$\square$

Similarly, we can characterize holomorphic functions on the dual Lie ball (see [2] and [4]).

5. Entire eigenfunctions of the Laplacian

Let λ be a complex number. We denote the space of eigenfunctions of the Laplacian by $\mathcal{O}_{\Delta-\lambda^2}(\tilde{B}(r)) = \{f \in \mathcal{O}(\tilde{B}(r)); (\Delta_z - \lambda^2)f(z) = 0\}$, where Δ_z is the complex Laplacian: $\Delta_z = \dfrac{\partial^2}{\partial z_1^2} + \dfrac{\partial^2}{\partial z_2^2} + \cdots + \dfrac{\partial^2}{\partial z_{n+1}^2}$.

Lemma 5.1. ([12, Theorem 2.1]) *Let $f \in \mathcal{O}(\tilde{B}(r))$ and $f_{k,k-2l}$ be the $(k, k-2l)$-harmonic component of f defined by* (1.4). *Then we have*

$$f \in \mathcal{O}_{\Delta-\lambda^2}(\tilde{B}(r)) \Longleftrightarrow f_{k,k-2l} = \frac{(\lambda/2)^{2l}\Gamma\big(k-2l+\frac{n+1}{2}\big)}{\Gamma(l+1)\Gamma\big(k-l+\frac{n+1}{2}\big)} f_{k-2l,k-2l}$$

for $l = 0, 1, 2, \cdots, [k/2]$ and $k = 0, 1, 2, \cdots$.

If $f \in \mathcal{O}_{\Delta-\lambda^2}(\tilde{B}(r))$, by Lemma 5.1 the double series (1.5) reduces to

$$f(z) = \sum_{k=0}^{\infty}\sum_{l=0}^{[k/2]}(z^2)^l f_{k,k-2l}(z) = \sum_{k=0}^{\infty}\tilde{j}_k(i\lambda\sqrt{z^2}) f_{k,k}(z),$$

where $\tilde{j}_k(t)$ is the entire Bessel function:

$$\tilde{j}_k(t) = \tilde{J}_{k+(n-1)/2}(t) = \Gamma(k+(n+1)/2)(t/2)^{-(k+\frac{n-1}{2})}J_{k+\frac{n-1}{2}}(t).$$

Then the (k,k)-harmonic component of $f \in \mathcal{O}_{\Delta-\lambda^2}(\tilde{B}(r))$ is given by

$$f_{k,k}(z) = (\tilde{j}_k(i\lambda))^{-1}N(k,n)\int_{S_1}\tilde{P}_{k,n}(z,\tau)f(\tau)\dot{d\tau}.$$

Note that the k-homogeneous component f_k of f is $f_k(z) = \tilde{j}_k(i\lambda)f_{k,k}(z)$ and that $\limsup_{k\to\infty}|f_k|^{1/k} = \limsup_{k\to\infty}|\tilde{j}_k(i\lambda)f_{k,k}|^{1/k}$ because $\lim_{\mu\to\infty}|\tilde{J}_\mu(t)| = 1$ for $t \in \mathbf{C}$.

For a norm on $N(z)$ on $\tilde{\mathbf{E}}$, we put

$$\mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; (r, N)) = \mathrm{Exp}(\tilde{\mathbf{E}}; (r, N)) \cap \mathcal{O}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}).$$

We have the following theorem:

Theorem 5.1. ([12, Theorem 2.1]) *Let*

$$F(z) = \sum_{k=0}^{\infty}\tilde{j}_k(i\lambda\sqrt{z^2})F_{k,k}(z) \in \mathcal{O}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}).$$

Then we have

$$F \in \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; (r, L^*)) \iff \limsup_{k\to\infty}(k!\|F_{k,k}\|_{S_1})^{1/k} \le \frac{r}{2}.$$

We define the complex sphere $\tilde{S}_\lambda$ of complex radius λ with center at 0 by

$$\tilde{S}_\lambda = \{z \in \tilde{\mathbf{E}}; z^2 = \lambda^2\}.$$

Put

$$\tilde{S}_\lambda(r) = \tilde{S}_\lambda \cap \tilde{B}(r), \quad |\lambda| < r, \quad \tilde{S}_\lambda[r] = \tilde{S}_\lambda \cap \tilde{B}[r], \quad |\lambda| \le r.$$

Define the k-spherical harmonic component f_k of $f \in \mathcal{O}(\tilde{S}_\lambda(r))$ by

$$f_k(z) = N(k,n) \int_{S_1} \tilde{P}_{k,n}(z,\tau) f(\tau) \dot{d\tau}$$

and the k-spherical harmonic component T_k of $T \in \mathcal{O}'(\tilde{S}_\lambda[r])$ by

$$T_k(w) = N(k,n) \langle T_z, \tilde{P}_{k,n}(z,w)\rangle.$$

Then we have the following theorem:

Theorem 5.2. ([7, Theorems 5.1 and 6.1])

$$f = \sum_{k=0}^\infty f_k \in \mathcal{O}(\tilde{S}_\lambda(r)) \iff \limsup_{k\to\infty} (\|f_k\|_{S_1})^{1/k} \le \frac{1}{r},$$

$$T = \sum_{k=0}^\infty T_k \in \mathcal{O}'(\tilde{S}_\lambda[r]) \iff \limsup_{k\to\infty} (\|T_k\|_{S_1})^{1/k} \le r.$$

Put

$$\tilde{S}^*_\lambda(r) = \tilde{S}_\lambda \cap \tilde{B}_{L^*}(r), \quad |\lambda| < r, \quad \tilde{S}^*_\lambda[r] = \tilde{S}_\lambda \cap \tilde{B}_{L^*}[r], \quad |\lambda| \le r.$$

By (4.2) and (4.3), we have $\tilde{S}_\lambda(r) = \tilde{S}^*_\lambda(\dfrac{r^2+|\lambda|^2}{2r})$ and $\tilde{S}^*_\lambda(r) = \tilde{S}_\lambda(r+\sqrt{r^2-|\lambda|^2})$. Thus we have the following corollary:

Corollary 5.1.

$$f = \sum_{k=0}^\infty f_k \in \mathcal{O}(\tilde{S}^*_\lambda(r)) \iff \limsup_{k\to\infty} (\|f_k\|_{S_1})^{1/k} \le \frac{1}{r+\sqrt{r^2-|\lambda|^2}} \quad ,$$

$$T = \sum_{k=0}^\infty T_k \in \mathcal{O}'(\tilde{S}^*_\lambda[r]) \iff \limsup_{k\to\infty} (\|T_k\|_{S_1})^{1/k} \le r+\sqrt{r^2-|\lambda|^2} \quad .$$

Restrict the Fourier-Borel transformation on $\mathcal{O}'(\tilde{B}_N(r))$ to $\mathcal{O}'(\tilde{S}_\lambda \cap \tilde{B}_N(r))$. Then R.Wada proved the following theorem:

Theorem 5.3. ([10, Theorem 3.1]) *The Fourier-Borel transformation $\mathcal{F}$ establishes the following topological linear isomorphisms*:

$$\mathcal{F}: \quad \mathcal{O}'(\tilde{S}_\lambda[r]) \xrightarrow{\sim} \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; (r, L^*)), \quad 0 \le r < \infty,$$

$$\mathcal{F}: \quad \mathcal{O}'(\tilde{S}_\lambda(r)) \xrightarrow{\sim} \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}}; [r, L^*]), \quad 0 < r \le \infty.$$

To this theorem, in [3], we gave a proof different from R. Wada's by using the cohomology theory. By the same idea in [3] we can prove the following theorem:

Theorem 5.4. *The Fourier-Borel transformation $\mathcal{F}$ establishes the following topological linear isomorphisms*:

$$\mathcal{F}: \quad \mathcal{O}'(\tilde{S}^*_\lambda[r]) \xrightarrow{\sim} \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}};(r,L)), \quad 0 \le r < \infty,$$
$$\mathcal{F}: \quad \mathcal{O}'(\tilde{S}^*_\lambda(r)) \xrightarrow{\sim} \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}};[r,L]), \quad 0 < r \le \infty.$$

By Theorems 5.3 and 5.4 we have the following theorem.

Theorem 5.5. *For $|\lambda| \le r$, we have*

$$\mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}};(r,L^*)) = \mathrm{Exp}_{\Delta-\lambda^2}\left(\tilde{\mathbf{E}};\left(\frac{r^2+|\lambda|^2}{2r},L\right)\right)$$

or,

$$\mathrm{Exp}_{\Delta-\lambda^2}\left(\tilde{\mathbf{E}};(r,L)\right) = \mathrm{Exp}_{\Delta-\lambda^2}\left(\tilde{\mathbf{E}};(r+\sqrt{r^2-|\lambda|^2},L^*)\right).$$

This generalizes a result in [11];

$$\mathrm{Exp}_{\Delta}(\tilde{\mathbf{E}};(r,L^*)) = \mathrm{Exp}_{\Delta}\left(\tilde{\mathbf{E}};(\frac{r}{2},L)\right), \quad r \ge 0.$$

Moreover, if $|\lambda| = r$, then $\mathrm{Exp}_{\Delta-\lambda^2}\left(\tilde{\mathbf{E}};(r,L^*)\right) = \mathrm{Exp}_{\Delta-\lambda^2}\left(\tilde{\mathbf{E}};(r,L)\right)$.

From Theorems 5.1 and 5.5 we have the following corollary.

Corollary 5.2. *Let*

$$F(z) = \sum_{k=0}^{\infty} \tilde{j}_k(i\lambda\sqrt{z^2})F_{k,k}(z) \in \mathrm{Exp}_{\Delta-\lambda^2}(\tilde{\mathbf{E}};(r,L)), \quad |\lambda| \le r.$$

Then we have

$$\limsup_{k\to\infty} (k!\|F_{k,k}\|_{S_1})^{1/k} \le \frac{r+\sqrt{r^2-|\lambda|^2}}{2}.$$

Conversely, if we are given a sequence $\{F_{k,k}\}$ of k-homogeneous harmonic polynomials $F_{k,k}(z)$ satisfying

$$\limsup_{k\to\infty} (k!\|F_{k,k}\|_{S_1})^{1/k} \le r,$$

then $\sum_{k=0}^{\infty} \tilde{j}_k(i\lambda\sqrt{z^2})F_{k,k}(z)$ converges to $F \in \mathrm{Exp}_{\Delta-\lambda^2}\left(\tilde{\mathbf{E}};(r+\frac{|\lambda|^2}{4r},L)\right)$ and the (k,k)-harmonic component of F is equal to the given $F_{k,k}$.

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