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LIE SUPERALGEBRAS OF STRING THEORIES

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Abstract. We describe *simple* complex Lie superalgebras of vector fields on "supercircles" - stringy superalgebras - in intrinsic terms. This is an announcement of a classification: there are four series of such algebras and four exceptional stringy superalgebras.

We also describe Lie superalgebras close to the simple stringy ones, namely, 12 of the simple stringy Lie superalgebras are *distinguished*: only they have nontrivial central extensions and since one of the distinguished superalgebras has three nontrivial central extensions each, there exist exactly 14 superizations of the Liouville action, Schrödinger equation, KdV hierarchy, etc. We also present the three nontrivial cocycles on the $N = 4$ extended Neveu–Schwarz superalgebra in terms of primary fields.

One of these stringy superalgebras is a Kac–Moody superalgebra $\mathfrak{g}(A)$ with a nonsymmetrizable Cartan matrix A. It can not be interpreted as a central extension of a twisted loop algebra.

In the literature the stringy superalgebras are often referred to with an unfortunate term *superconformal*. We show that only three of simple stringy superalgebras are indeed conformal.

Introduction

I.1. The discovery of simple stringy Lie superalgebras. Simple and close to simple Lie superalgebras studied here appear as symmetry algebras in String Theories. For this and other reasons given below we call them *stringy superlagebras*. The discovery of simple stringy superalgebras was not easy. Even the original name of these Lie superalgebras is unfortunate. Mathematicians lately bapthised the Lie algebra of vector fields on the circle $\mathfrak{witt}$ in honor of Witt who considered it in prime characteristic. Recall that *conformal* means preserving a metric (or a more general bilinear form) up to a factor. By a theorem of Liouville $\mathfrak{witt}$ is the only infinite dimensional conformal Lie algebra. Physicists who considered the first superization of $\mathfrak{witt}$, namely $\mathfrak{k}^L(1|1)$ and $\mathfrak{k}^M(1|1)$, dubbed them "superconformal" algebras. But, as we will show, except for them, NO

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stringy *super*algebra is superconformal, or, better say, conformal in the original meaning of the word. Besides, the diversity of the stringy algebras requires often to refer to them using their "given names" that describe them more precisely, than a common term. When the common name is needed, then *stringy* is, at least, a not selfcontradictory and suggestive name. An intrinsic definition is desirable and there appeared one in [KvL]. We give another, more invariant, intrinsic description of stringy superalgebras we dug out of [Ma].

The physicists who discovered first examples of simple stringy superalgebras ([NS], [R], [Ad]) were primarily interested in unitary representations, so they started with real algebras which are more difficult to classify than complex algebras. So they gave a number of examples, not a classification. We refer to [S3] for the complete list of real forms of the stringy algebras known at that time. The real forms of the other examples will be given elsewhere.

Observe also that the physicists who studied superstrings were mainly interested in nontrivial central extensions of "superconformal" superalgebras. Only several first terms of the four series of stringy superalgebras - the 12 *distinguished* superalgebras - have such extensions, the other algebras were snubbed at. For a review of applications of distinguished stringy superalgebras in string theory, see [GSW]. For some other applications see [LX]. Observe that nondistinguished simple stringy superalgebras are also of interest, see [CLL], [GL2] and [LSh].

Ordered historically, the steps of classification are: [NS] and [R] followed by [Ad], where four series of the stringy superalgebras (without a continuous parameter) and most of the central extensions of the distinguished superalgebras were found for one real form of each algebra; in [FL] the complexifications of the algebras from [Ad] were interpreted geometrically and expressed in terms of *superfields* and two classifications were announced: (1) of simple stringy superalgebras and (2) of their central extensions. Regrettably, each classification had a gap. During the past years these gaps were partly filled by several authors, in this paper the repair is completed, thus a classification is announced here.

Poletaeva [P] in 1983 and, independently, Schoutens [Sc] in 1986, found three nontrivial central extensions of $\mathfrak{k}^{L\circ}(1|4)$, i.e., of the 4-extended Neveu–Schwarz superalgebra (the importance of [P], whose results were expressed via $H^1(\mathfrak{g};\mathfrak{g})$ rather than more conventional $H^2(\mathfrak{g})$, was not recognized in time and Poletaeva's result was never properly published).

Schwimmer and Seiberg [SS] found a deformation of the divergence-free series.

In [KvL] the completeness of the list of examples from [FL] amended with the deformation from [SS] was conjectured and the statements from [FL] and [Sc] on the nontrivial central extensions reproved; [KvL] contains the first published proof of the classification of the nontrivial central extensions of the simple superalgebras considered.

Other important steps of classification: [K], [L1] and [Sh1], [SP], where the vectorial Lie superalgebras with *polynomial* coefficients are considered, and [Ma], where a reasonable characterization of stringy algebras is given.

Observe, that after [Sc] there appeared several papers in which only two of the three central extensions of $\mathfrak{k}^{L\circ}(1|4)$ and $\mathfrak{k}^{M}(1|4)$ were recognized; the controversy is occasioned, presumably, by the insufficiently lucid description of the superalgebras involved and ensuing confusion between the exceptional simple superalgebra $\mathfrak{k}^{L\circ}(1|4)$, see below. Besides, the cocycles that the physicists need should be expressed in terms of the primary fields; so far, this was not done.

I.2. Our results. Here we define classical *stringy* Lie superalgebras (a.k.a. superconformal ones) in intrinsic terms and announce the list of all *simple* stringy superalgebras. Under certain additional assumptions the completeness of a part of our list satisfying these assumptions is proved in [K3].

We also answer a question of S. Krivonos: we replace the three cocycles found in [P] and [Sc] with cohomologic ones but expressed in terms of primary fields.

This paper, with its exceptional example based on hep-th 9702121, was preprinted in hep-th 9702120. Though later [CK] appeared, our interpretation of several exceptional *simple* stringy superalgebras, as well as the intrinsic definition of stringy ("superconformal") superlagebras, explicit presentation (description of defining relations) and the description of a distinguished stringy superalgebra with the help of a Cartan matrix are still new.

Remark. The results of this paper were obtained in Stockholm in June 1996 and delivered at the seminar of E. Ivanov, JINR, Dubna (July, 1996) and Voronezh winter school Jan. 12–18, 1997. Kac's questions concerning exceptional Lie superalgebras in his numerous letters to I. Shchepochkina in October-November 1996 culminated in [CK], where our example $\mathfrak{kas}^L$ is described in different terms. We thank Kac for the preprints [CK] and [K2], where several results from [L2] are discussed, and a kind letter to Leites that acknowledges Kac's receiving a preprint of [Sh1].

I.3. Real forms. The 1986 result of Serganova [S3] completed description of the real forms of the distinguished and simple stringy superalgebras known at that time. Crucial there is the discovery of three, not two, types of real forms of stringy and Kac–Moody superalgebras, cf. [S1] with [K1], where only two types of real forms of Kac–Moody algebras are recognized. Another important result of Serganova pertaining here: the discovery of three basic types of unitarity, one of them with an odd form, see [S2].

0. Background

0.1. Linear algebra in superspaces. Generalities. Superization has certain subtleties, often disregarded or expressed as in [L], [L3] or [M]: too briefly. We will dwell on them a bit.

A *superspace* is a $\mathbb{Z}/2$-graded space; for a superspace $V = V_{\bar 0} \oplus V_{\bar 1}$ denote by $\Pi(V)$ another copy of the same superspace: with the shifted parity, i.e., $(\Pi(V))_{\bar i} = V_{\bar i + \bar 1}$. The *superdimension* of V is $\dim V = p + q\varepsilon$, where $\varepsilon^2 = 1$ and $p = \dim V_{\bar 0}$, $q = \dim V_{\bar 1}$. (Usually, $\dim V$ is expressed as a pair (p, q) or $p|q$; this notation obscures the fact that $\dim V\otimes W=\dim V\cdot\dim W$ which is clear with the help of ε.)

A superspace structure in V induces the superspace structure in the space $\operatorname{End}(V)$. A *basis of a superspace* is always a basis consisting of *homogeneous* vectors; let $Par=(p_1,\dots,p_{\dim V})$ be an ordered collection of their parities. We call Par the *format* of (the basis of) V. A square *supermatrix* of format (size) Par is a $\dim V\times\dim V$ matrix whose ith row and ith column are of the same parity p_i. The matrix unit E_{ij} is supposed to be of parity p_i+p_j and the bracket of supermatrices (of the same format) is defined via Sign Rule:

if something of parity p moves past something of parity q the sign $(-1)^{pq}$ accrues; the formulas defined on homogeneous elements are extended to arbitrary ones via linearity.

For example, setting $[X,Y]=XY-(-1)^{p(X)p(Y)}YX$ we get the notion of the supercommutator and the ensuing notion of the Lie superalgebra (that satisfies the superskew-commutativity and super Jacobi identity).

We do not usually use the sign $\wedge$ for differential forms on supermanifolds: in what follows we assume that the exterior differential is odd and the differential forms constitute a supercommutative superalgebra; still, we keep using it on manifolds, sometimes, not to deviate too far from conventional notations.

Usually, Par is of the form $(\bar 0,\dots,\bar 0,\bar 1,\dots,\bar 1)$. Such a format is called *standard.* In this paper we can do without nonstandard formats. But they are vital in the study of systems of simple roots that the reader might be interested in (see [GL1]) in connection with applications to q-quantization or integrable systems.

The *general linear* Lie superalgebra of all supermatrices of size Par is denoted by $\mathfrak{gl}(Par)$; usually, $\mathfrak{gl}(\bar 0,\dots,\bar 0,\bar 1,\dots,\bar 1)$ is abbreviated to $\mathfrak{gl}(\dim V_{\bar 0}|\dim V_{\bar 1})$. Any matrix from $\mathfrak{gl}(Par)$ can be expressed as the sum of its even and odd parts; in the standard format this is the following block expression:

$$\begin{pmatrix}A&B\\C&D\end{pmatrix}=\begin{pmatrix}A&0\\0&D\end{pmatrix}+\begin{pmatrix}0&B\\C&0\end{pmatrix},\quad p\left(\begin{pmatrix}A&0\\0&D\end{pmatrix}\right)=\bar 0,\ p\left(\begin{pmatrix}0&B\\C&0\end{pmatrix}\right)=\bar 1.$$

The *supertrace* is the map $\mathfrak{gl}(Par)\longrightarrow\mathbb{C}$, $(A_{ij})\mapsto\sum(-1)^{p_i}A_{ii}$. Since $\operatorname{str}[x,y]=0$, the subsuperspace of supertraceless matrices constitutes the *special linear* Lie subsuperalgebra $\mathfrak{sl}(Par)$.

Superalgebras that preserve bilinear forms: two types. To the linear map F of superspaces there corresponds the dual map F^* between the dual superspaces; if A is the supermatrix corresponding to F in a basis of format Par, then to F^* the *supertransposed* matrix A^{st} corresponds:

$$(A^{st})_{ij}=(-1)^{(p_i+p_j)(p_i+p(A))}A_{ji}.$$

The supermatrices $X\in\mathfrak{gl}(Par)$ such that

$$X^{st}B+(-1)^{p(X)p(B)}BX=0\quad\text{for a homogeneous matrix } B\in\mathfrak{gl}(Par)$$

constitute the Lie superalgebra $\mathfrak{aut}(B)$ that preserves the bilinear form on V with matrix B. Most popular is the nondegenerate supersymmetric form whose matrix in the standard format is the canonical form B_{ev} or B'_{ev}:

$$B_{ev}(m|2n) = \begin{pmatrix} 1_m & 0 \\ 0 & J_{2n} \end{pmatrix}, \quad \text{where } J_{2n} = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix},$$

or

$$B'_{ev}(m|2n) = \begin{pmatrix} \text{antidiag}(1,\dots,1) & 0 \\ 0 & J_{2n} \end{pmatrix}.$$

The usual notation for $\mathfrak{aut}(B_{ev}(m|2n))$ is $\mathfrak{osp}(m|2n)$ or $\mathfrak{osp}^{sy}(m|2n)$.

Recall that the "upsetting" map $u : Bil(V,W) \longrightarrow Bil(W,V)$ becomes for $V = W$ an involution $u : B \mapsto B^u$ which on matrices acts as follows:

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \mapsto B^u = \begin{pmatrix} B_{11}^t & (-1)^{p(B)} B_{21}^t \\ (-1)^{p(B)} B_{12}^t & B_{22}^t \end{pmatrix}.$$

This involution separates symmetric and skew-symmetric forms. The passage from V to $\Pi(V)$ sends the supersymmetric forms to superskew-symmetric ones, preserved by the "symplectico-orthogonal" Lie superalgebra $\mathfrak{osp}^{sk}(m|2n)$ which is isomorphic to $\mathfrak{osp}^{sy}(m|2n)$ but has a different matrix realization. We never use notation $\mathfrak{spo}(2n|m)$ in order not to confuse with the special Poisson superalgebra.

In the standard format the matrix realizations of these algebras are:

$$\mathfrak{osp}(m|2n) = \left\{ \begin{pmatrix} E & Y & X^t \\ X & A & B \\ -Y^t & C & -A^t \end{pmatrix} \right\}; \quad \mathfrak{osp}^{sk}(m|2n) = \left\{ \begin{pmatrix} A & B & X \\ C & -A^t & Y^t \\ Y & -X^t & E \end{pmatrix} \right\},$$

$$\text{where } \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix} \in \mathfrak{sp}(2n), \quad E \in \mathfrak{o}(m) \text{ and } {}^t \text{ is the usual transposition.}$$

Among nonstandard canonical forms the following ones are most important and often used:

$$B_{ev}(\bar{0},\bar{1},\dots) = \text{antidiag}(1,-1,1,-1,\dots); \quad B_{ev}(n|m|n) = \begin{pmatrix} 0 & 0 & 1_n \\ 0 & 1_m & 0 \\ -1_n & 0 & 0 \end{pmatrix}$$

A nondegenerate supersymmetric odd bilinear form $B_{odd}(n|n)$ can be reduced to a canonical form whose matrix in the standard format is J_{2n}. A canonical form of the superskew odd nondegenerate form in the standard format is $\Pi_{2n} = \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$. The usual notation for $\mathfrak{aut}(B_{odd}(Par))$ is $\mathfrak{pe}(Par)$. The passage from V to $\Pi(V)$ establishes an isomorphism $\mathfrak{pe}^{sy}(Par) \cong \mathfrak{pe}^{sk}(Par)$. This Lie superalgebra is called, as A. Weil suggested, *periplectic* one. The matrix realizations in the standard format of these superalgebras is shorthanded to:

$$\mathfrak{pe}^{sy}(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \text{ where } B = -B^t,\ C = C^t \right\};$$

$$\mathfrak{pe}^{sk}(n) = \left\{ \begin{pmatrix} A & B \\ C & -A^t \end{pmatrix}, \text{ where } B = B^t,\ C = -C^t \right\}.$$

The *special periplectic* superalgebra is $\mathfrak{spe}(n) = \{X \in \mathfrak{pe}(n) : \text{str} X = 0\}$.

0.2. Vectorial Lie superalgebras: The standard realization. The elements of the Lie algebra $\mathcal{L} = \mathfrak{der}\,\mathbb{C}[[u]]$ are considered as vector fields. The Lie algebra $\mathcal{L}$ has only one maximal subalgebra $\mathcal{L}_0$ of finite codimension (consisting of the fields that vanish at the origin). The subalgebra $\mathcal{L}_0$ determines a filtration of $\mathcal{L}$: set

$$\mathcal{L}_{-1} = \mathcal{L} \text{ and } \mathcal{L}_i = \{D \in \mathcal{L}_{i-1} : [D, \mathcal{L}] \subset \mathcal{L}_{i-1}\} \text{ for } i \geq 1.$$

The associated graded Lie algebra $L = \mathop{\oplus}\limits_{i\geq -1} L_i$, where $L_i = \mathcal{L}_i/\mathcal{L}_{i+1}$, consists of the vector fields with *polynomial* coefficients.

Superization and the passage to a subalgebras of $\mathfrak{der}\,\mathbb{C}[[u]]$ brings new phenomena. Suppose $\mathcal{L}_0 \subset \mathcal{L}$ is a maximal subalgebra of finite codimension and containing no ideals of $\mathcal{L}$. For the Lie superalgebra $\mathcal{L} = \mathfrak{der}\,\mathbb{C}[[u, \xi]]$ the minimal $\mathcal{L}_0$-invariant subspace of $\mathcal{L}$ strictly containing $\mathcal{L}_0$ coincides with $\mathcal{L}$. Not all the subalgebras of $\mathfrak{der}\,\mathbb{C}[[u, \xi]]$ have this property. Let $\mathcal{L}_{-1}$ be a minimal subspace of $\mathcal{L}$ containing $\mathcal{L}_0$, different from $\mathcal{L}_0$ and $\mathcal{L}_0$-invariant. A *Weisfeiler filtration* of $\mathcal{L}$ is determined by the formulas

$$\mathcal{L}_{-i-1} = [\mathcal{L}_{-1}, \mathcal{L}_{-i}] + \mathcal{L}_{-i} \text{ and } \mathcal{L}_i = \{D \in \mathcal{L}_{i-1} : [D, \mathcal{L}_{-1}] \subset \mathcal{L}_{i-1}\} \text{ for } i > 0.$$

Since the codimension of $\mathcal{L}_0$ is finite, the filtration takes the form

$$\mathcal{L} = \mathcal{L}_{-d} \supset \dots \mathcal{L}_0 \supset \dots \tag{0.2}$$

for some d. This d is the *depth* of $\mathcal{L}$ or of the associated graded Lie superalgebra L. We call all filtered or graded Lie superalgebras of finite depth *vectorial*, i.e., realizable with vector fields on a finite dimensional supermanifold. Considering the subspaces (0.2) as the basis of a topology, we can complete the graded or filtered Lie superalgebras L or $\mathcal{L}$; the elements of the completion are the vector fields with formal power series as coefficients. Though the structure of the graded algebras is easier to describe, in applications the completed Lie superalgebras are usually needed.

Unlike Lie algebras, simple vectorial *super*algebras possess *several* maximal subalgebras of finite codimension. We describe them, together with the corresponding gradings, in sec. 0.4.

1) General algebras. Let $x = (u_1, \dots, u_n, \theta_1, \dots, \theta_m)$, where the u_i are even indeterminates and the θ_j are odd ones. Set $\mathfrak{vect}(n|m) = \mathfrak{der}\,\mathbb{C}[x]$; it is called *the general vectorial Lie superalgebra.*

2) Special algebras. The *divergence* of the field $D = \sum\limits_i f_i \dfrac{\partial}{\partial u_i} + \sum\limits_j g_j \dfrac{\partial}{\partial \theta_j}$ is the function (in our case: a polynomial, or a series)

$$\text{div}D = \sum_i \frac{\partial f_i}{\partial u_i} + \sum_j (-1)^{p(g_j)} \frac{\partial g_i}{\partial \theta_j}.$$

• The Lie superalgebra $\mathfrak{svect}(n|m) = \{D \in \mathfrak{vect}(n|m) : \text{div}D = 0\}$ is called the *special* or *divergence-free vectorial superalgebra.*

It is clear that it is also possible to describe $\mathfrak{svect}(n|m)$ as $\{D \in \mathfrak{vect}(n|m) : L_D vol_x = 0\}$, where vol_x is the volume form with constant coefficients in coordinates x and L_D the Lie derivative with respect to D.

• The Lie superalgebra

$$\mathfrak{svect}_\lambda(0|m) = \{D \in \mathfrak{vect}(0|m) : \operatorname{div}(1 + \lambda\theta_1 \cdot \dots \cdot \theta_m)D = 0\},$$

the deform of $\mathfrak{svect}(0|m)$, is called the *deformed special* or *deformed divergence-free vectorial superalgebra*. Clearly, $\mathfrak{svect}_\lambda(0|m) \cong \mathfrak{svect}_\mu(0|m)$ for $\lambda\mu \neq 0$. Observe that $p(\lambda) \equiv m \pmod 2$, i.e., for odd m the parameter of deformation λ is odd; strictly speaking, $\mathfrak{svect}_\lambda(0|2k+1)$ is considered not over $\mathbb{C}$, but over $\mathbb{C}[\lambda]$.

Remark. Sometimes we write $\mathfrak{vect}(x)$ or even $\mathfrak{vect}(V)$ if $V = Span(x)$ and use similar notations for the subalgebras of $\mathfrak{vect}$ introduced below. Algebraists sometimes abbreviate $\mathfrak{vect}(n)$ and $\mathfrak{svect}(n)$ to W_n (in honor of Witt) and S_n, respectively.

3) The algebras that preserve Pfaff equations and differential 2-forms.

• Set $u = (t, p_1, \dots, p_n, q_1, \dots, q_n)$; let

$$\tilde{\alpha}_1 = dt + \sum_{1\leq i\leq n} (p_i dq_i - q_i dp_i) \ + \sum_{1\leq j\leq m} \theta_j d\theta_j \quad \text{and} \quad \tilde{\omega}_0 = d\tilde{\alpha}_1 \ .$$

The form $\tilde{\alpha}_1$ is called *contact*, the form $\tilde{\omega}_0$ is called *symplectic*. Sometimes it is more convenient to redenote the θ's and set

$$\begin{aligned} \xi_j &= \frac{1}{\sqrt{2}}(\theta_j - i\theta_{r+j}), \\ \eta_j &= \frac{1}{\sqrt{2}}(\theta_j + i\theta_{r+j}) \quad \text{ for } j \leq r = [m/2] \text{ (here } i^2 = -1), \\ \theta &= \theta_{2r+1}, \end{aligned}$$

and in place of $\tilde{\omega}_0$ or $\tilde{\alpha}_1$ take α and $\omega_0 = d\alpha_1$, respectively, where

$$\alpha_1 = dt + \sum_{1\leq i\leq n} (p_i dq_i - q_i dp_i) + \sum_{1\leq j\leq r} (\xi_j d\eta_j + \eta_j d\xi_j) + \begin{cases} 0 & \text{if } \ m = 2r, \\ \theta d\theta & \text{if } \ m = 2r+1. \end{cases}$$

The Lie superalgebra that preserves the *Pfaff equation* $\alpha_1 = 0$, i.e., the superalgebra

$$\mathfrak{k}(2n+1|m) = \{D \in \mathfrak{vect}(2n+1|m) : L_D\alpha_1 = f_D\alpha_1\},$$

(here $f_D \in \mathbb{C}[t, p, q, \xi]$ is a polynomial determined by D) is called the *contact superalgebra*.

• Similarly, set $u = q = (q_1, \dots, q_n)$, let $\theta = (\xi_1, \dots, \xi_n; \tau)$ be odd. Set

$$\alpha_0 = d\tau + \sum_i (\xi_i dq_i + q_i d\xi_i), \qquad \omega_1 = d\alpha_0$$

and call these forms the *odd contact* and *periplectic*, respectively.

The Lie superalgebra that preserves the Pfaff equation $\alpha_0 = 0$, i.e., the superalgebra

$$\mathfrak{m}(n) = \{D \in \mathfrak{vect}(n|n+1) : L_D\alpha_0 = f_D \cdot \alpha_0\}, \ \text{ where } \ f_D \in \mathbb{C}[q, \xi, \tau],$$

is called the *odd-contact superalgebra*.

0.3. Generating functions. A laconic way to describe $\mathfrak{k}$, $\mathfrak{m}$ and their subalgebras is via generating functions.

• Odd form α_1. For $f \in \mathbb{C}[t, p, q, \theta]$ set

$$K_f = (2 - E)(f)\frac{\partial}{\partial t} - H_f + \frac{\partial f}{\partial t}E,$$

where $E = \sum\limits_i y_i \frac{\partial}{\partial y_i}$ (here the y are all the coordinates except t) is the *Euler operator* (which counts the degree with respect to the y), and H_f is the Hamiltonian field with Hamiltonian f that preserves $d\alpha_1$:

$$H_f = \sum_{i \leq n}\left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}\right) - (-1)^{p(f)}\sum_{j \leq m}\frac{\partial f}{\partial \theta_j}\frac{\partial}{\partial \theta_j}.$$

The choice of the form α_1 instead of $\tilde{\alpha}_1$ only affects the form of H_f that we give for $m = 2k + 1$:

$$H_f = \sum_{i \leq n}\left(\frac{\partial f}{\partial p_i}\frac{\partial}{\partial q_i} - \frac{\partial f}{\partial q_i}\frac{\partial}{\partial p_i}\right) - (-1)^{p(f)}\left[\sum_{j \leq k}\left(\frac{\partial f}{\partial \xi_j}\frac{\partial}{\partial \eta_j} + \frac{\partial f}{\partial \eta_j}\frac{\partial}{\partial \xi_j}\right) + \frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta}\right].$$

• Even form α_0. For $f \in \mathbb{C}[q, \xi, \tau]$ set

$$M_f = (2 - E)(f)\frac{\partial}{\partial \tau} - Le_f - (-1)^{p(f)}\frac{\partial f}{\partial \tau}E,$$

where $E = \sum\limits_i y_i \frac{\partial}{\partial y_i}$ (here the y are all the coordinates except τ) is the Euler operator, and

$$Le_f = \sum_{i \leq n}\left(\frac{\partial f}{\partial q_i}\ \frac{\partial}{\partial \xi_i} + (-1)^{p(f)}\frac{\partial f}{\partial \xi_i}\ \frac{\partial}{\partial q_i}\right).$$

Since

$$L_{K_f}(\alpha_1) = 2\frac{\partial f}{\partial t}\alpha_1 = K_1(f)\alpha_1,$$

$$L_{M_f}(\alpha_0) = -(-1)^{p(f)}2\frac{\partial f}{\partial \tau}\alpha_0 = -(-1)^{p(f)}M_1(f)\alpha_0, \tag{0.3}$$

it follows that $K_f \in \mathfrak{k}(2n + 1|m)$ and $M_f \in \mathfrak{m}(n)$. Observe that

$$p(Le_f) = p(M_f) = p(f) + \bar{1}.$$

• To the (super)commutators $[K_f, K_g]$ or $[M_f, M_g]$ there correspond *contact brackets* of the generating functions:

$$[K_f, K_g] = K_{\{f,g\}_{k.b.}}; \qquad [M_f, M_g] = M_{\{f,g\}_{m.b.}}$$

The explicit formulas for the contact brackets are as follows. Let us first define the brackets on functions that do not depend on t (resp. τ).

The *Poisson bracket* $\{\cdot,\cdot\}_{P.b.}$ (in the realization with the form ω_0) is given by the formula

$$\{f,g\}_{P.b.}=\sum_{i\leq n}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}-\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right)-(-1)^{p(f)}\sum_{j\leq m}\frac{\partial f}{\partial\theta_j}\frac{\partial g}{\partial\theta_j}$$

and in the realization with the form ω_0 for $m=2k+1$ it is given by the formula

$$\begin{aligned}\{f,g\}_{P.b.}=&\sum_{i\leq n}\left(\frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i}-\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i}\right)\\&-(-1)^{p(f)}\left[\sum_{j\leq m}\left(\frac{\partial f}{\partial\xi_j}\frac{\partial g}{\partial\eta_j}+\frac{\partial f}{\partial\eta_j}\frac{\partial g}{\partial\xi_j}\right)+\frac{\partial f}{\partial\theta}\frac{\partial g}{\partial\theta}\right].\end{aligned}$$

The *Buttin bracket* $\{\cdot,\cdot\}_{B.b.}$ is given by the formula

$$\{f,g\}_{B.b.}=\sum_{i\leq n}\left(\frac{\partial f}{\partial q_i}\frac{\partial g}{\partial\xi_i}+(-1)^{p(f)}\frac{\partial f}{\partial\xi_i}\frac{\partial g}{\partial q_i}\right).$$

Remark. What we call here "Buttin bracket" was discovered in pre-super era by Schouten; Buttin was the first to prove that this bracket establishes a Lie superalgebra structure. The interpretations of the Buttin superalgebra similar to that of the Poisson algebra and of the elements of $\mathfrak{le}$ as analogs of Hamiltonian vector fields was given in [L1]. Later it gained a great deal of currency under the name "antibracket" given by Batalin and Vilkovisky who rediscovered it, cf. [GPS]. The *Schouten bracket* was originally defined on the superspace of polyvector fields on a manifold, i.e., on the superspace of sections of the exterior algebra (over the algebra $\mathcal{F}$ of functions) of the tangent bundle, $\Gamma(\Lambda^{\boldsymbol{\cdot}}(T(M)))\cong\Lambda^{\boldsymbol{\cdot}}_{\mathcal{F}}(Vect(M))$. The explicit formula of the Schouten bracket (in which the hatted slot should be ignored, as usual) is

$$\begin{aligned}&[X_1\wedge\dots\wedge\dots\wedge X_k, Y_1\wedge\dots\wedge Y_l]\\(*)\quad&=\sum_{i,j}(-1)^{i+j}[X_i,Y_j]\wedge X_1\wedge\dots\wedge\hat{X}_i\wedge\dots\wedge X_k\wedge Y_1\wedge\dots\wedge\hat{Y}_j\wedge\dots\wedge Y_l.\end{aligned}$$

With the help of Sign Rule we easily superize formula $(*)$ for the case when manifold M is replaced with supermanifold $\mathcal{M}$. Let x and ξ be the even and odd coordinates on $\mathcal{M}$. Setting $\theta_i=\Pi\Big(\frac{\partial}{\partial x_i}\Big)=\check{x}_i$, $q_j=\Pi\Big(\frac{\partial}{\partial\xi_j}\Big)=\check{\xi}_j$ we get an identification of the Schouten bracket of polyvector fields on $\mathcal{M}$ with the Buttin bracket of functions on the supermanifold $\check{\mathcal{M}}$ whose coordinates are $x,\xi;\check{x},\check{\xi}$; the transformation of x,ξ induces from that of the checked coordinates.

In terms of the Poisson and Buttin brackets, respectively, the contact brackets take the form

$$\{f,g\}_{k.b.}=(2-E)(f)\frac{\partial g}{\partial t}-\frac{\partial f}{\partial t}(2-E)(g)-\{f,g\}_{P.b.}$$

and, respectively,

$$\{f,g\}_{m.b.}=(2-E)(f)\frac{\partial g}{\partial\tau}+(-1)^{p(f)}\frac{\partial f}{\partial\tau}(2-E)(g)-\{f,g\}_{B.b.}.$$

It is not difficult to prove the following isomorphisms (as superspaces):

$$\mathfrak{k}(2n+1|m)\cong Span(K_f : f\in\mathbb{C}[t,p,q,\theta]);\quad \mathfrak{m}(n)\cong Span(M_f : f\in\mathbb{C}[\tau,q,\xi]).$$

Lie superalgebra $\mathfrak{svect}(1|n)$ has a simple ideal $\mathfrak{svect}^\circ(n)$ of codimension 1 (more exactly, (1—0) or (0—1), depending on n) defined from the exact sequence

$$0\longrightarrow\mathfrak{svect}^\circ(n)\longrightarrow\mathfrak{svect}(1|n)\longrightarrow\mathbb{C}\cdot\xi_1\dots\xi_n\frac{\partial}{\partial t}\longrightarrow 0$$

0.4. The Cartan prolongs. We will repeatedly use the Cartan prolong. So let us recall the definition and generalize it somewhat. Let $\mathfrak{g}$ be a Lie algebra, V a $\mathfrak{g}$-module, S^i the operator of the i-th symmetric power. Set $\mathfrak{g}_{-1}=V$, $\mathfrak{g}_0=\mathfrak{g}$ and for $i>0$ define the i-th *Cartan prolong* (the result of Cartan's *prolongation*) of the pair $(\mathfrak{g}_{-1},\mathfrak{g}_0)$ as

$$\begin{aligned}\mathfrak{g}_i=\{X\in\mathrm{Hom}(\mathfrak{g}_{-1},\mathfrak{g}_{i-1}) : X(v)(w,...)=X(w)(v,...)\ \text{ for any }\ v,w\in\mathfrak{g}_{-1}\}\\ =(S^i(\mathfrak{g}_{-1})^*\otimes\mathfrak{g}_0)\cap(S^{i+1}(\mathfrak{g}_{-1})^*\otimes\mathfrak{g}_{-1}).\end{aligned}$$

(Here we consider $\mathfrak{g}_0$ as a subspace in $\mathfrak{g}_{-1}^*\otimes\mathfrak{g}_{-1}$, so the intersection is well-defined.)

The *Cartan prolong* of the pair $(V,\mathfrak{g})$ is $(\mathfrak{g}_{-1},\mathfrak{g}_0)_*=\underset{i\geq -1}{\oplus}\mathfrak{g}_i$. (In what follows $\cdot$ in superscript denotes, as is now customary, the collection of all degrees, while $*$ is reserved for dualization; in the subscripts we retain the oldfashioned $*$ instead of $\cdot$ to avoid too close a contact with the punctuation marks.)

Suppose that the $\mathfrak{g}_0$-module $\mathfrak{g}_{-1}$ is faithful. Then, clearly,

$$\begin{aligned}(\mathfrak{g}_{-1},\mathfrak{g}_0)_*\subset\mathfrak{vect}(n)=\mathfrak{der}\ \mathbb{C}[x_1,...,x_n],\quad\text{where}\ \ n=dim\ \mathfrak{g}_{-1}\ \ \text{and}\\ \mathfrak{g}_i=\{D\in\mathfrak{vect}(n) : \deg D=i,[D,X]\in\mathfrak{g}_{i-1}\ \text{for any}\ X\in\mathfrak{g}_{-1}\}.\end{aligned}$$

It is subject to an easy verification that the Lie algebra structure on $\mathfrak{vect}(n)$ induces same on $(\mathfrak{g}_{-1},\mathfrak{g}_0)_*$.

Of the four simple vectorial Lie algebras, three are Cartan prolongs: $\mathfrak{vect}(n)=(\mathrm{id},\mathfrak{gl}(n))_*$, $\mathfrak{svect}(n)=(\mathrm{id},\mathfrak{sl}(n))_*$ and $\mathfrak{h}(2n)=(\mathrm{id},\mathfrak{sp}(n))_*$. The fourth one — $\mathfrak{k}(2n+1)$ — is also the prolong under a trifle more general construction described as follows.

A generalization of the Cartan prolong. Let $\mathfrak{g}_-=\underset{-d\leq i\leq -1}{\oplus}\mathfrak{g}_i$ be a nilpotent $\mathbb{Z}$-graded Lie algebra and $\mathfrak{g}_0\subset\mathfrak{der}_0\mathfrak{g}$ a Lie subalgebra of the $\mathbb{Z}$-grading-preserving derivations. For $i>0$ define the i-th prolong of the pair $(\mathfrak{g}_-,\ \mathfrak{g}_0)$ to be:

$$\mathfrak{g}_i=((S^\cdot(\mathfrak{g}_-)^*\otimes\mathfrak{g}_0)\cap(S^\cdot(\mathfrak{g}_-)^*\otimes\mathfrak{g}_-))_i,$$

where the subscript i in the right hand side singles out the component of degree i.

Define $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ to be $\underset{i\geq -d}{\oplus} \mathfrak{g}_i$; then, as is easy to verify, $(\mathfrak{g}_-, \mathfrak{g}_0)_*$ is a Lie algebra.

What is the Lie algebra of contact vector fields in these terms? Denote by $\mathfrak{hei}(2n)$ the Heisenberg Lie algebra: its space is $W \oplus \mathbb{C} \cdot z$, where W is a $2n$-dimensional space endowed with a nondegenerate skew-symmetric bilinear form B and the bracket in $\mathfrak{hei}(2n)$ is given by the following relations:

$$z \text{ is in the center and } [v, w] = B(v, w) \cdot z \text{ for any } v, w \in W.$$

Recall that for any $\mathfrak{g}$ we write $\mathfrak{cg} = \mathfrak{g} \oplus \mathbb{C} \cdot z$ or $\mathfrak{c}(\mathfrak{g})$ to denote the trivial central extension with the 1-dimensional even center generated by z.

Clearly,

$$\mathfrak{k}(2n+1) \cong (\mathfrak{hei}(2n), \mathfrak{csp}(2n))_*.$$

0.5. Lie superalgebras of vector fields as Cartan's prolongs. The superization of the constructions from sec. 0.4 are straightforward: via Sign Rule. We thus get infinite dimensional Lie superalgebras

$$\mathfrak{vect}(m|n) = (\mathrm{id}, \mathfrak{gl}(m|n))_*;\ \mathfrak{svect}(m|n) = (\mathrm{id}, \mathfrak{sl}(m|n))_*;$$
$$\mathfrak{h}(2m|n) = (\mathrm{id}, \mathfrak{osp}^{sk}(m|2n))_*; \mathfrak{le}(n) = (\mathrm{id}, \mathfrak{pe}^{sk}(n))_*;\ \mathfrak{sle}(n) = (\mathrm{id}, \mathfrak{spe}^{sk}(n))_*.$$

Remark. Observe that the Cartan prolongs $(\mathrm{id}, \mathfrak{osp}^{sy}(m|2n))_*$ and $(\mathrm{id}, \mathfrak{pe}^{sy}(n))_*$ are finite dimensional.

The generalization of Cartan's prolongations described in sec. 0.4 has, after superization, two analogs associated with the contact series $\mathfrak{k}$ and $\mathfrak{m}$, respectively.

• Define the Lie superalgebra $\mathfrak{hei}(2n|m)$ on the direct sum of a $(2n, m)$-dimensional superspace W endowed with a nondegenerate skew-symmetric bilinear form and a $(1, 0)$-dimensional space spanned by z.

Clearly, we have $\mathfrak{k}(2n+1|m) = (\mathfrak{hei}(2n|m), \mathfrak{cosp}^{sk}(m|2n))_*$ and, given $\mathfrak{hei}(2n|m)$ and a subalgebra $\mathfrak{g}$ of $\mathfrak{cosp}^{sk}(m|2n)$, we call $(\mathfrak{hei}(2n|m), \mathfrak{g})_*$ the *k-prolong* of $(W, \mathfrak{g})$, where W is the identity $\mathfrak{osp}^{sk}(m|2n)$-module.

• The "odd" analog of $\mathfrak{k}$ is associated with the following "odd" analog of $\mathfrak{hei}(2n|m)$. Denote by $\mathfrak{ab}(n)$ the *antibracket* Lie superalgebra: its space is $W \oplus \mathbb{C} \cdot z$, where W is an $n|n$-dimensional superspace endowed with a nondegenerate skew-symmetric odd bilinear form B; the bracket in $\mathfrak{ab}(n)$ is given by the following relations:

$$z \text{ is odd and lies in the center; } [v, w] = B(v, w) \cdot z \text{ for } v, w \in W.$$

Set $\mathfrak{m}(n) = (\mathfrak{ab}(n), \mathfrak{cpe}^{sk}(n))_*$ and, given $\mathfrak{ab}(n)$ and a subalgebra $\mathfrak{g}$ of $\mathfrak{cpe}^{sk}(n)$, we call $(\mathfrak{ab}(n), \mathfrak{g})_*$ the *m-prolong* of $(W, \mathfrak{g})$, where W is the identity $\mathfrak{pe}^{sk}(n)$-module.

Generally, given a nondegenerate form B on a superspace W and a superalgebra $\mathfrak{g}$ that preserves B, we refer to the above generalized prolongations as to *mk-prolongation* of the pair $(W, \mathfrak{g})$.

A partial Cartan prolong or the prolong of a positive part. Take a $\mathfrak{g}_0$-submodule $\mathfrak{h}_1$ in $\mathfrak{g}_1$ such that $[\mathfrak{g}_{-1}, \mathfrak{h}_1] = \mathfrak{g}_0$, not a subalgebra of $\mathfrak{g}_0$. If such $\mathfrak{h}_1$ exists, define the 2nd prolongation of $(\mathop{\oplus}\limits_{i\leq 0} \mathfrak{g}_i, \mathfrak{h}_1)$ to be $\mathfrak{h}_2 = \{D \in \mathfrak{g}_2 : [D, \mathfrak{g}_{-1}] \in \mathfrak{h}_1\}$. The terms $\mathfrak{h}_i$, $i > 2$, are similarly defined. Set $\mathfrak{h}_i = \mathfrak{g}_i$ for $i \leq 0$ and $\mathfrak{h}_* = \sum \mathfrak{h}_i$.

Examples. $\mathfrak{vect}(1|n; n)$ is a subalgebra of $\mathfrak{k}(1|2n; n)$. The former is obtained as Cartan's prolong of the same nonpositive part as $\mathfrak{k}(1|2n; n)$ and a submodule of $\mathfrak{k}(1|2n; n)_1$, cf. Table 0.7. The simple exceptional superalgebra $\mathfrak{kas}$ introduced in 0.7 is another example.

0.6. The modules of tensor fields. To advance further, we have to recall the definition of the modules of tensor fields over $\mathfrak{vect}(m|n)$ and its subalgebras, see [BL]. For any other $\mathbb{Z}$-graded vectorial Lie superalgebra the construction is identical.

Let $\mathfrak{g} = \mathfrak{vect}(m|n)$ and $\mathfrak{g}_{\geq} = \mathop{\oplus}\limits_{i\geq 0} \mathfrak{g}_i$. Clearly, $\mathfrak{vect}_0(m|n) \cong \mathfrak{gl}(m|n)$. Let V be the $\mathfrak{gl}(m|n)$-module with the *lowest* weight $\lambda = \text{lwt}(V)$. Make V into a $\mathfrak{g}_{\geq}$-module setting $\mathfrak{g}_+ \cdot V = 0$ for $\mathfrak{g}_+ = \mathop{\oplus}\limits_{i>0} \mathfrak{g}_i$. Let us realize $\mathfrak{g}$ by vector fields on the $m|n$-dimensional linear supermanifold $\mathcal{C}^{m|n}$ with coordinates $x = (u, \xi)$. The superspace $T(V) = \text{Hom}_{U(\mathfrak{g}_{\geq})}(U(\mathfrak{g}), V)$ is isomorphic, due to the Poincaré–Birkhoff–Witt theorem, to $\mathbb{C}[[x]] \otimes V$. Its elements have a natural interpretation as formal *tensor fields of type* V. When $\lambda = (a, \dots, a)$ we will simply write $T(\vec{a})$ instead of $T(\lambda)$. We usually consider irreducible $\mathfrak{g}_0$-modules.

Examples. $T(\vec{0})$ is the superspace of functions; $Vol(m|n) = T(1, \dots, 1; -1, \dots, -1)$ (the semicolon separates the first m coordinates of the weight with respect to the matrix units E_{ii} of $\mathfrak{gl}(m|n)$) is the superspace of *densities* or *volume forms*. We denote the generator of $Vol(m|n)$ corresponding to the ordered set of coordinates x by $vol(x)$. The space of λ-densities is $Vol^\lambda(m|n) = T(\lambda, \dots, \lambda; -\lambda, \dots, -\lambda)$. In particular, $Vol^\lambda(m|0) = T(\vec{\lambda})$ but $Vol^\lambda(0|n) = T(\overrightarrow{-\lambda})$.

More examples: $\mathfrak{vect}(m|n)$ as $\mathfrak{vect}(m|n)$- and $\mathfrak{svect}(m|n)$-modules is $T(\text{id})$; see also sec. 1.3.

Remark. To view the volume element as "$d^m u d^n \xi$" is totally wrong: the superdeterminant can never appear as a factor under the changes of variables. We still can try to use the usual notations of differentials provided *all* the differentials anticommute. Then linear transformations that do not intermix the even u's with the odd ξ's the volume element $vol(x)$ viewed as the fraction $\frac{du_1 \cdot \dots \cdot du_m}{d\xi_1 \cdot \dots \cdot d\xi_n}$ is multiplied by the Berezinian of the transformation. But how could we justify this? Let $X = (x, \xi)$. If we consider the usual, exterior, differential forms, then the dX_i's super anti-commute, hence, the $d\xi_i$ commute; whereas if we consider the *symmetric* product of the differentials, as in the metrics, then the dX_i's supercommute, hence, the dx_i commute. However, the $\frac{\partial}{\partial \xi_i}$ anticommute and, from transformation point of view, $\frac{\partial}{\partial\xi_i}=\frac{1}{d\xi_i}$. The notation, $du_1\cdot\ldots\cdot du_m\cdot\frac{\partial}{\partial\xi_1}\cdot\ldots\cdot\frac{\partial}{\partial\xi_n}$, is, nevertheless, still wrong: almost any transformation $A:(u,\xi)\mapsto(v,\eta)$ sends $du_1\cdot\ldots\cdot du_m\cdot\frac{\partial}{\partial\xi_1}\cdot\ldots\cdot\frac{\partial}{\partial\xi_n}$ to the correct element, $\mathrm{ber}(A)(dv^m\cdot\frac{\partial}{\partial\eta_1}\cdot\ldots\cdot\frac{\partial}{\partial\eta_n})$, plus extra terms. Indeed, the fraction $du_1\cdot\ldots\cdot du_m\cdot\frac{\partial}{\partial\xi_1}\cdot\ldots\cdot\frac{\partial}{\partial\xi_n}$ is the highest weight vector of an *indecomposable* $\mathfrak{gl}(m|n)$-module and $vol(x)$ is the notation of the image of this vector in the 1-dimensional quotient module modulo the invariant submodule that consists precisely of all the extra terms.

0.7. The exceptional Lie subsuperalgebra $\mathfrak{kas}$ of $\mathfrak{k}(1|6)$. The Lie superalgebra $\mathfrak{g}=\mathfrak{k}(1|2n)$ is generated by the functions from $\mathbb{C}[t,\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_n]$. The standard $\mathbb{Z}$-grading of $\mathfrak{g}$ is induced by the $\mathbb{Z}$-grading of $\mathbb{C}[t,\xi,\eta]$ given by $\deg t=2$, $\deg\xi_i=\deg\eta_i=1$; namely, $\deg K_f=\deg f-2$. Clearly, in this grading $\mathfrak{g}$ is of depth 2. Let us consider the functions that generate several first homogeneous components of $\mathfrak{g}=\underset{i\geq-2}{\oplus}\mathfrak{g}_i$:

component	$\mathfrak{g}_{-2}$	$\mathfrak{g}_{-1}$	$\mathfrak{g}_0$	$\mathfrak{g}_1$
its generators	1	$\Lambda^1(\xi,\eta)$	$\Lambda^2(\xi,\eta)\oplus\mathbb{C}\cdot t$	$\Lambda^3(\xi,\eta)\oplus t\Lambda^1(\xi,\eta)$

As one can prove directly, the component $\mathfrak{g}_1$ generates the whole subalgebra $\mathfrak{g}_+$ of elements of positive degree. The component $\mathfrak{g}_1$ splits into two $\mathfrak{g}_0$-modules $\mathfrak{g}_{11}=\Lambda^3$ and $\mathfrak{g}_{12}=t\Lambda^1$. It is obvious that $\mathfrak{g}_{12}$ is always irreducible and the component $\mathfrak{g}_{11}$ is trivial for $n=1$.

Recall that if the operator d that determines a $\mathbb{Z}$-grading of the Lie superalgebra $\mathfrak{g}$ does not belong to $\mathfrak{g}$, we denote the Lie superalgebra $\mathfrak{g}\oplus\mathbb{C}\cdot d$ by $\mathfrak{d}\mathfrak{g}$. Recall also that $\mathfrak{c}(\mathfrak{g})$ or just $\mathfrak{c}\mathfrak{g}$ denotes the trivial 1-dimensional central extension of $\mathfrak{g}$ with the even center.

The Cartan prolongations from these components are well-known:

$$(\mathfrak{g}_-\oplus\mathfrak{g}_0,\mathfrak{g}_{11})^{mk}_*\cong\mathfrak{po}(0|2n)\oplus\mathbb{C}\cdot K_t\cong\mathfrak{d}(\mathfrak{po}(0|2n));$$
$$(\mathfrak{g}_-\oplus\mathfrak{g}_0,\mathfrak{g}_{12})^{mk}_*=\mathfrak{g}_{-2}\oplus\mathfrak{g}_{-1}\oplus\mathfrak{g}_0\oplus\mathfrak{g}_{12}\oplus\mathbb{C}\cdot K_{t^2}\cong\mathfrak{osp}(2n|2).$$

Observe a remarkable property of $\mathfrak{k}(1|6)$. For $n>1$ and $n\neq3$ the component $\mathfrak{g}_{11}$ is irreducible. For $n=3$ it splits into 2 irreducible conjugate modules that we will denote $\mathfrak{g}^{\xi}_{11}$ and $\mathfrak{g}^{\eta}_{11}$. Observe further, that $\mathfrak{g}_0=\mathfrak{co}(6)\cong\mathfrak{gl}(4)$. As $\mathfrak{gl}(4)$-modules, $\mathfrak{g}^{\xi}_{11}$ and $\mathfrak{g}^{\eta}_{11}$ are the symmetric squares $S^2(\mathrm{id})$ and $S^2(\mathrm{id}^*)$ of the identity 4-dimensional representation and its dual, respectively.

Theorem 1. *The Cartan prolong $(\mathfrak{g}_-\oplus\mathfrak{g}_0,\mathfrak{g}^{\xi}_{11}\oplus\mathfrak{g}_{12})^{mk}_*$ is infinite dimensional and simple. It is isomorphic to $(\mathfrak{g}_-\oplus\mathfrak{g}_0,\mathfrak{g}^{\eta}_{11}\oplus\mathfrak{g}_{12})^{mk}_*$.*

We will denote the simple exceptional vectorial Lie superalgebras described in Theorem 0.7 by $\mathfrak{kas}^{\xi}$ and $\mathfrak{kas}^{\eta}$, respectively; or just by $\mathfrak{kas}$.

1. Stringy superalgebras

These superalgebras are particular cases of the Lie superalgebras of vector fields, namely, the ones that preserve a structure on a what physicists call superstring, i.e., the supermanifold associated with a vector bundle on the circle. These superalgebras themselves are "stringy" indeed: as modules over the Witt algebra $\mathfrak{witt} = \mathfrak{der}\ \mathbb{C}[t^{-1}, t]$ they are direct sums of several "strings", the modules $\mathcal{F}_\lambda$ described in sec. 1.3.

This description, sometimes taken for definition of the stringy superalgebra $\mathfrak{g}$, depends on the embedding $\mathfrak{witt} \longrightarrow \mathfrak{g}$ and the spectrum of $\mathfrak{witt}$-modules constituting $\mathfrak{g}$ might vary hampering recognition. Rigorous and deep is the definition of a *deep* superalgebra due to Mathieu. He separates the deep algebras of which stringy is a particular case Lie algebras from affine Kac–Moody ones. Both are of infinite depth (see 0.2) but for the loop algebras all real root vectors act locally nilpotently, whereas $\mathfrak{g}$ is *stringy* if $\mathfrak{g}$ is of polynomial growth and

(1.0) there exists a root vector which does not act locally nilpotently.

(Roughly speaking, the stringy superalgebras have the root vector $\frac{d}{dt}$).

Similarly, we say that a Lie superalgebra $\mathfrak{g}$ of infinite depth and of polynomial growth is of the *stringy type* if it satisfies (1.0) and of *loop* type otherwise. Observe, that a stringy superalgebra of polynomial growth can be a Kac–Moody superalgebra, i.e., have a Cartan matrix, but it can not be a (twisted) loop superalgebra.

1.1. Let φ be the angle parameter on the circle, $t = exp(i\varphi)$. The only stringy Lie algebra is $\mathfrak{witt} := \mathfrak{der}\ \mathbb{C}[t^{-1}, t]$.

Examples of *stringy Lie superalgebras* are certain subalgebras of the Lie superalgebra of superderivations of either of the two supercommutative superalgebras

$$R^L(1|n) = \mathbb{C}[t^{-1}, t, \theta_1, \dots, \theta_n] \quad \text{or} \quad R^M(1|n) = \mathbb{C}[t^{-1}, t, \theta_1, \dots, \theta_{n-1}, \sqrt{t}\xi].$$

$R^L(1|n)$ can be considered as the superalgebra of complex-valued functions expandable into finite Fourier series or, as superscript indicates, Laurent series. These functions are considered on the real supermanifold $S^{1|n}$ associated with the rank n trivial bundle over the circle. We can forget about φ and think in terms of t considered as the even coordinate on $(\mathbb{C}^*)^{1|n}$.

$R^M(1|n)$ can be considered as the superalgebra of complex-valued functions (expandable into finite Fourier series) on the supermanifold $S^{1|n-1,M}$ associated with the Whitney sum of the Möbius bundle and the trivial bundle of rank $n-1$. Since the Whitney sum of two Möbius bundles is isomorphic to the trivial bundle of rank 2, it suffices to consider one Möbius summand.

Let us introduce the main stringy Lie superalgebras associated with the trivial bundle. These are analogues of $\mathfrak{vect}$, $\mathfrak{svect}$ and $\mathfrak{k}$ obtained by replacing $R(1|n) =$

$\mathbb{C}[t,\theta_1,\dots,\theta_n]$ with $R^L(1|n)$:

$$\begin{aligned}\mathfrak{vect}^L(1|n) &= \mathfrak{der}\ R^L(1|n);\\ \mathfrak{svect}^L_\lambda(1|n) &= \{D\in\mathfrak{vect}^L(1|n)\ :\mathrm{div}(t^\lambda D)=0\}\\ &= \{D\in\mathfrak{vect}^L(1|n)\ :L_D(t^\lambda vol(t,\theta))=0\};\\ \mathfrak{k}^L(1|n) &= \{D\in\mathfrak{vect}^L(n): D(\alpha_1)=f_D\alpha_1 \text{ for}\\ &\qquad \tilde\alpha_1 = dt+\sum\theta_i d\theta_i \text{ and } f_D\in R^L(1|n)\}.\end{aligned}$$

We abbreviate $\mathfrak{svect}^L_0(1|n)$ to $\mathfrak{svect}^L(1|n)$.

The routine arguments prove that the functions $f\in R^L(n)$ generate $\mathfrak{k}^L(n)$, and the formulas for K_f and the contact bracket are the same as for $\mathfrak{k}(1|n)$ and the polynomial f.

Remark. 1) The algebras $\widetilde{\mathfrak{vect}}(1|n)$ and $\widetilde{\mathfrak{svect}}_\lambda(1|n)$ obtained by replacing $R(n)$ with $R^M(1|n)$ are isomorphic to $\mathfrak{vect}^L(1|n)$ and $\mathfrak{svect}^L_{\lambda-\frac12}(1|n)$, respectively.

2) Clearly,

$$\mathfrak{svect}^L_\lambda(1|n)\cong\mathfrak{svect}^L_\mu(1|n) \text{ if } \lambda-\mu\in\mathbb{Z}.$$

In sec. 1.7 we will show a more subtle isomorphism.

3) The following formula is convenient:

$$D=f\partial_t+\sum f_i\partial_i\in\mathfrak{svect}^L_\lambda(1|n)\quad\text{if and only if}\quad \lambda f=-t\mathrm{div}D. \tag{1.1.1}$$

If $\lambda\in\mathbb{Z}$, the Lie superalgebra $\mathfrak{svect}^L_\lambda(1|n)$ has the simple ideal $\mathfrak{svect}^{L\circ}_\lambda(1|n)$ of codimension ε^n:

$$0\longrightarrow\mathfrak{svect}^{L\circ}_\lambda(1|n)\longrightarrow\mathfrak{svect}^L_\lambda(1|n)\longrightarrow\theta_1\cdot\dots\cdot\theta_n\partial_t\longrightarrow 0.$$

• The lift of the contact structure from $S^{1|n}$ to its two-sheeted covering, $S^{1|n,M}$, brings a new structure. Indeed, this lift means replacing θ_n with $\sqrt{t}\theta$; this replacement sends the form $\tilde\alpha_1$ into the Möbius form

$$\tilde{\tilde\alpha}=dt+\sum_{i=1}^{n-1}\theta_i d\theta_i+t\theta d\theta. \tag{$\widetilde{1.1.2}$}$$

It is often convenient to pass to another canonical expression of the *Möbius form*:

$$\hat\alpha=\begin{cases} dt+\sum\limits_{i\le k}(\xi_i d\eta_i+\eta_i d\xi_i)+t\theta d\theta & \text{if } n=2k+1,\\ dt+\sum\limits_{i\le k}(\xi_i d\eta_i+\eta_i d\xi_i+\zeta d\zeta)+t\theta d\theta & \text{if } n=2k+2.\end{cases} \tag{1.1.2}$$

Now, we have two ways for describing the vector fields that preserve $\tilde{\tilde\alpha}$ or $\hat\alpha$:

1) We can set: $(\mathfrak{aut}_{R^M}(\alpha_1))$

$$\mathfrak{k}^M(1|n)=\{D\in\mathfrak{der}\,R^M(1|n)\ :\ L_D(\hat\alpha)=f_D\cdot\hat\alpha,\ \text{ where }\ f_D\in R^M(1|n)\}.$$

In this case the fields K_f are given by the same formulas as for $\mathfrak{k}(1|n)$ but the generating functions belong to $R^M(n)$. The contact bracket between the generating functions from $R^M(n)$ is also given by the same formulas as for the generating functions of $\mathfrak{k}(1|n)$.

2) We can set: $(\mathfrak{aut}_{R^L}(\alpha_1))$

$$\mathfrak{k}^M(1|n) = \{D \in \mathfrak{vect}^L(1|n) : L_D(\alpha_1) = f_D \cdot \alpha_1, \text{ where } f_D \in R^L(1|n)\}.$$

It is not difficult to verify that

$$\mathfrak{k}^M(1|n) = Span(\tilde{K}_f : f \in R^L(1|n)),$$

where the *Möbius contact field* is given by the formula

$$\hat{K}_f = (2-E)(f)\mathcal{D} + \mathcal{D}(f)E + \hat{H}_f, \tag{1.1.3}$$

in which, as in the case of a cylinder $S^{1,n}$, we set

$$E = \sum_{i\leq n-1} \theta_i \frac{\partial}{\partial \theta_i} + \theta \frac{\partial}{\partial \theta},$$

but where

$$\mathcal{D} = \frac{\partial}{\partial t} - \frac{\theta}{2t}\frac{\partial}{\partial \theta} = \frac{1}{2}\hat{K}_1$$

and where

$$\hat{H}_f = (-1)^{p(f)}\left(\sum \frac{\partial f}{\partial \theta_i}\frac{\partial}{\partial \theta_i} + \frac{1}{t}\frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta}\right)$$

in the realization with form $\tilde{\hat{\alpha}}$. In the realization with form $\hat{\alpha}$ we have for $n = 2k$ and $n = 2k+1$, respectively:

$$\hat{H}_f = (-1)^{p(f)}\left(\sum (\frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i}\frac{\partial}{\partial \xi_i}) + \frac{1}{t}\frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta}\right);$$

$$\hat{H}_f = (-1)^{p(f)}\left(\sum (\frac{\partial f}{\partial \xi_i}\frac{\partial}{\partial \eta_i} + \frac{\partial f}{\partial \eta_i}\frac{\partial}{\partial \xi_i}) + \frac{\partial f}{\partial \zeta}\frac{\partial}{\partial \zeta} + \frac{1}{t}\frac{\partial f}{\partial \theta}\frac{\partial}{\partial \theta}\right).$$

The corresponding contact bracket of generating functions will be called the *Ramond bracket*; its explicit form is

$$\{f,g\}_{R.b.} = (2-E)(f)\mathcal{D}(g) - \mathcal{D}(f)(2-E)(g) - \{f,g\}_{MP.b.}, \tag{1.1.4}$$

where the *Möbius-Poisson bracket* $\{\cdot,\cdot\}_{MP.b}$ is

$$\{f,g\}_{MP.b} = (-1)^{p(f)}\left(\sum \frac{\partial f}{\partial \theta_i}\frac{\partial g}{\partial \theta_i} + \frac{1}{t}\frac{\partial f}{\partial \theta}\frac{\partial g}{\partial \theta}\right) \tag{1.1.5}$$

in the realization with form $\tilde{\hat{\alpha}}$.

Observe that

$$L_{K_f}(\alpha_1) = K_1(f)\cdot \alpha_1, \quad L_{\hat{K}_f}(\hat{\alpha}) = \hat{K}_1(f)\cdot\hat{\alpha}. \tag{1.1.6}$$

Remark. Let us give a relation of the brackets with the dot product on the space of functions. (More exactly, for $\mathfrak{k}$ it is defined on $\mathcal{F}_{-2}$, etc., see sec. 1.3.)

$$\begin{aligned}\{f,gh\}_{k.b.}&=\{f,g\}_{k.b.}h+(-1)^{p(f)p(g)}g\{f,h\}_{k.b.}+K_1(f)gh;\\ \{f,gh\}_{R.b.}&=\{f,g\}_{R.b.}h+(-1)^{p(f)p(g)}g\{f,h\}_{R.b.}+\hat{K}_1(f)gh;\\ \{f,gh\}_{m.b.}&=\{f,g\}_{m.b.}h+(-1)^{p(f+1)p(g)}g\{f,h\}_{m.b.}+M_1(f)gh.\end{aligned}\tag{1.1.7}$$

This relation (when the third term vanishes) is often listed as part of the definition of the Poisson algebra which is, certainly, a pure nonsense: it follows from the definition.

Explicitly, an embedding $i:\mathfrak{vect}^L(1|n)\longrightarrow\mathfrak{k}^L(1|2n)$ is given by the following formula in which $\Phi=\sum\limits_{i\leq n}\xi_i\eta_i$:

(1.1.8)

$D\in\mathfrak{vect}^L(1\vert n)$	the generator of $i(D)$
$f(\xi)t^m\partial_t$	$(-1)^{p(f)}\dfrac{1}{2^m}f(\xi)(t+\Phi)^m,$
$f(\xi)t^m\partial_i$	$(-1)^{p(f)}\dfrac{1}{2^m}f(\xi)\eta_i(t+\Phi)^m.$

Clearly, $\mathfrak{svect}^L_\lambda(1|n)$ is the subsuperspace of $\mathfrak{vect}^L(1|n)\subset\mathfrak{k}^L(1|2n)$ spanned by the generating functions

$$f(\xi)(t+\Phi)^m+\sum_i f_i(\xi)\eta_i(t+\Phi)^{m-1}\tag{1.1.9}$$

$$\text{such that } (\lambda+n)f(\xi)=-\sum_i(-1)^{p(f_i)}\frac{\partial f_i}{\partial\xi_i}.$$

The four series of classical stringy superalgebras are: $\mathfrak{vect}^L(1|n)$, $\mathfrak{svect}^L_\lambda(1|n)$, $\mathfrak{k}^L(1|n)$ and $\mathfrak{k}^M(1|n)$.

1.2. Nonstandard gradings of stringy superalgebras. The Weisfeiler filtrations of vectorial superalgebras with polynomial or formal coefficients are determined by the maximal subalgebra of finite codimension not containing ideals of the whole algebra, cf. [LSh]. The corresponding gradings are natural. We believe that the filtrations and $\mathbb{Z}$-gradings of stringy superalgebras induced by Weisfeiler filtrations of the corresponding vectorial superalgebras with polynomial coefficients are distinguished but we do not know how to characterize such $\mathbb{Z}$-gradings intrinsically.

Explicitely, the isomorphisms $\mathfrak{vect}^L(1|1)\cong\mathfrak{k}^L(1|2)\cong\mathfrak{m}^L(1)$ as abstract (filtrations irnored) Lie superalgebras are as follows. Let x, ξ be the indeterminates that describe $\mathfrak{vect}^L(1|1)$; let t, ξ, η be the indeterminates that describe $\mathfrak{k}^L(1|2)$ and let τ, u, ξ be the indeterminates that describe $\mathfrak{m}^L(1)$. Then the correspondending elements are as follows:

$\mathfrak{vect}^L(1\|1)$	$\mathfrak{k}^L(1\|2)$	$\mathfrak{m}^L(1)$
$x^n\partial_x$	$\frac{1}{2^n}K_{t^n+nt^{n-1}\xi\eta}$	$M_{u^n\xi}$
$\xi x^n\partial_x$	$-\frac{1}{2^n}K_{\xi t^n}$	$\frac{1}{2}M_{u^n\tau\xi}$
$x^n\partial_\xi$	$\frac{1}{2^n}K_{\eta t^n}$	M_{u^n}
$\xi x^n\partial_\xi$	$-\frac{1}{2^n}K_{\xi\eta t^n}$	$\frac{1}{2}M_{u^n(\tau-u\xi)}$

Explicit pasage $\mathfrak{m}^L(1) \longrightarrow \mathfrak{k}^L(1|2)$ in terms of generating functions:

u^n	$\frac{1}{2^n}t^n\eta$
$u^n\tau\xi$	$-\frac{1}{2^{n-1}}\xi t^n$
$u^n\xi$	$\frac{1}{2^n}(t^n+n\xi\eta t^{n-1})$
$u^n\tau$	$\frac{1}{2^{n+1}}(t^{n+1}+(n-3)\xi\eta t^n)$

Explicit pasage $\mathfrak{k}^L(1|2) \longrightarrow^L \mathfrak{m}(1)$ in terms of generating functions:

t^n	$2^{n-2}(4u^n\xi+nu^{n-1}(\tau-u\xi))$
$t^n\xi\eta$	$-2^{n-1}u^n(\tau-u\xi)$
$t^n\xi$	$-2^{n-1}u^n\tau\xi$
$t^n\eta$	2^nu^n

1.3. Modules of tensor fields over stringy superalgebras. Denote by $T^L(V)=\mathbb{C}[t^{-1},t]\otimes V$ the $\mathfrak{vect}(1|n)$-module that differs from $T(V)$ by allowing the Laurent polynomials as coefficients of its elements instead of just polynomials. Clearly, $T^L(V)$ is a $\mathfrak{vect}^L(1|n)$-module. Define the *tensor fields twisted with weight* μ - a version of $T^L(V)$ - by setting:

$$T^L_\mu(V)=\mathbb{C}[t^{-1},t]t^\mu\otimes V.$$

• **The "simplest" modules — the analogues of the standard or identity representation of the matrix algebras**. The simplest modules over the Lie superalgebras of series $\mathfrak{vect}$ are, clearly, the modules of λ-densities, cf. sec. 0.5. These modules are characterized by the fact that they are of rank 1 over $\mathcal{F}$, the algebra of functions.

Over stringy superalgebras, we can also twist these modules and consider Vol_{μ}^{λ}, irreducible if $\lambda \neq 0, 1$. Observe that for $\mu \notin \mathbb{Z}$ the module has only one irreducible submodule, the image of the exterior differential d, see [BL], whereas for $\mu \in \mathbb{Z}$ there is, instead, the kernel of the residue:

$$\begin{aligned}&\mathrm{Res} : Vol^{L} \longrightarrow \mathbb{C}\,,\\ &f vol_{t,\xi} \mapsto \text{ the coefficient of } \frac{\xi_1 \dots \xi_n}{t} \text{ in the power series expansion of } f.\end{aligned}$$

For $Vol^{0} = \mathcal{F}_0$, the space of functions, there is only one submodule, of constants.

• Over $\mathfrak{svect}^{L}(1|n)$ all the spaces Vol^{λ} are, clearly, isomorphic, since their generator, $vol(t, \theta)$, is preserved. So all rank 1 modules over the module of functions are isomorphic to the module of twisted functions $\mathcal{F}_{0;\mu}$ irreducible if $\mu \neq 0$. The module $\mathcal{F}_0$ has a submodule of constants and a codimension 1 submodule of functions with residue 0.

Over $\mathfrak{svect}^{L}_{\lambda}(1|n)$, the simplest module is generated by $t^{\lambda} vol(t, \theta)$.

• Over contact superalgebras $\mathfrak{k}(2n + 1|m)$, it is more natural to express the simplest modules not in terms of λ-densities but via powers of the form $\alpha = \alpha_1$:

$$\mathcal{F}_{\lambda} = \begin{cases} \mathcal{F}\alpha^{\lambda} & \text{for } n = m = 0 \\ \mathcal{F}\alpha^{\lambda/2} & \text{otherwise }. \end{cases} \tag{1.3.1}$$

The twisted modules are denoted by $\mathcal{F}_{\lambda;\mu}$

Observe that

$$Vol^{\lambda} \cong \begin{cases} \mathcal{F}_{\lambda(2n+2-m)} & \text{as } \mathfrak{k}(2n + 1|m)\text{-modules} \\ \mathcal{F}^{M}_{\lambda(2-m+1)} & \text{as } \mathfrak{k}^{M}(1|m)\text{-modules} \end{cases} \tag{1.3.2}$$

To see this, it suffices to compute the degree of $vol_{t,p,q,\theta}$ (the latter is, roughly speaking, $\frac{dtdpdq}{d\theta}$) with the degrees of the indeterminates given by the standard grading, see 1.2. In particular, $\mathfrak{k}(2n + 1|2n + 2) \subset \mathfrak{svect}(2n + 1|2n + 2)$ and

$$\mathfrak{k}(2n + 1|m) \cap \mathfrak{svect}(2n + 1|m) = \mathfrak{po}(2n|m) \text{ for } m \neq 2n + 2.$$

In particular,

$$\mathfrak{k}^{L}(1|4) \simeq Vol \text{ and } \mathfrak{k}^{M}(1|5) \simeq \Pi(Vol). \tag{1.3.3}$$

The module Vol of volume forms over $\mathfrak{k}^{L}(1|m)$ is isomorphic to $\mathcal{F}_{2-m}$ whereas over $\mathfrak{k}^{M}(1|m)$ it is isomorphic to $\mathcal{F}_{3-m}$: the degree of the odd Möbius coordinate vanishes.

Observe also that the Lie superalgebra of series $\mathfrak{k}$ does not distinguish between $\frac{\partial}{\partial t}$ and α^{-1} whereas the Lie superalgebra of series $\mathfrak{k}^M$ does not distinguish between $\mathcal{D}$ and $\hat{\alpha}^{-1}$: their transformation rules are identical. Hence,

$$\mathfrak{k}^L(2n+1|m) \cong \begin{cases} \mathcal{F}_{-1} & \text{for } n=m=0 \\ \mathcal{F}_{-2} & \text{otherwise ;} \end{cases}$$

$$\mathfrak{k}^M(1|m) \cong \begin{cases} \mathcal{F}_{-1} & \text{for } m=1 \\ \mathcal{F}_{-2} & \text{for } m>1. \end{cases} \tag{1.3.4}$$

• For $n=0, m=2$ (we take $\alpha = dt - \xi d\eta - \eta d\xi$) the $\mathfrak{k}^L(1|2)$-modules of rank 1 over $\mathcal{F} = \mathcal{F}_{0,0}$, the algebra of functions, acquire addiional parameter, ν, namely:

$$T(\lambda, \nu)_\mu = \mathcal{F}_{\lambda;\mu} \cdot \left(\frac{d\xi}{d\eta}\right)^{\nu/2}.$$

• Over $\mathfrak{k}^M$, we should replace the form α with $\hat{\alpha}$ and the definition of the $\mathfrak{k}^L(1|m)$-modules $\mathcal{F}_{\lambda;\mu}$ should be replaced with

$$\mathcal{F}^M_{\lambda;\mu} = \begin{cases} \mathcal{F}_{\lambda;\mu}(\hat{\alpha})^{\lambda} & \text{for } m=1 \\ \mathcal{F}_{\lambda;\mu}(\hat{\alpha})^{\lambda/2} & \text{for } m>1. \end{cases}$$

• For $m=3$ and $\hat{\alpha} = dt - \xi d\eta - \eta d\xi - t\theta d\theta$ the $\mathfrak{k}^M(1|3)$-modules of rank 1 over $\mathcal{F} = \mathcal{F}_{0,0}$, the algebra of functions, acquire addiional parameter, ν, namely:

$$T^M(\lambda, \nu)_\mu = \mathcal{F}^M_{\lambda;\mu} \cdot \left(\frac{d\xi}{d\eta}\right)^{\nu/2}.$$

• The simplest $\mathfrak{m}^L(n)$-modules are $\mathcal{F}_\lambda = \mathcal{F} \cdot \alpha_0^{\lambda/2}$. In particular, $\mathfrak{m}(n) \cong \mathcal{F}_{-1}$. For $\mathfrak{m}^L(1)$ the modules $\mathcal{F}_{\lambda,\mu} = \mathcal{F} \cdot \alpha_0^{\lambda/2} \cdot t^\mu$ are naturally defined.

1.4. The four exceptional stringy superalgebras. The "status" of these exceptions is different: A) is a true exception, D) is an exceptional realization; the other two are "drop outs" from the series (like the $\mathfrak{psl}(n|n)$ that have no analogs among $\mathfrak{sl}(m|n)$ with $m \neq n$).

A) $\mathfrak{kas}^L$. Certain polynomial functions described in [Sh1] and sec. 0.7 generate $\mathfrak{kas} \subset \mathfrak{k}(1|6)$. Inserting Laurent polynomials in the formulas for the generators of $\mathfrak{kas}$ we get the exceptional stringy superalgebra $\mathfrak{kas}^L \subset \mathfrak{k}^L(1|6)$.

B, C) $\mathfrak{k}^{L\circ}(1|4)$ and $\mathfrak{k}^{M\circ}(1|5)$. It follows from (1.3.3) that the functions with zero residue on $S^{1|4}$ (resp. $S^{1|4;\,M}$) generate an ideal in $\mathfrak{k}^L(4)$ (resp. $\mathfrak{k}^M(5)$). These ideals are, clearly, simple Lie superalgebras denoted in what follows by $\mathfrak{k}^{L\circ}(1|4)$ and $\mathfrak{k}^{M\circ}(1|5)$, respectively.

D) $\mathfrak{m}^L(1)$. On the complexification of $S^{1|2}$, let q be the even coordinate, τ and ξ the odd ones. Set

$$\mathfrak{m}^L(1)=\{D\in\mathfrak{vect}^L(1|2) : D\alpha_0=f_D\alpha_0,\ f_D\in R^L(1|1),\ \alpha_0=d\tau+qd\xi+\xi dq\}.$$

1.5. Deformations. The superalgebrbas $\mathfrak{svect}(1|n)$ and $\mathfrak{svect}^\circ(n)$ do not have deformations, see [LSh]. The stringy superalgebras $\mathfrak{svect}^L(n)$ do have $\mathbb{Z}$-grading preserving deformations discovered by Schwimmer and Seiberg [SS]. More deformations (none of which preserves $\mathbb{Z}$-grading) are described in [KvL]; the complete description of deformations of $\mathfrak{svect}^L(n)$ is an open problem.

Conjecture. $\mathfrak{vect}^L(1|n)$, $\mathfrak{k}^L(1|n)$ and the four exceptional stringy superalgebras are rigid.

1.6. Distinguished stringy superalgebras. In this subsect. 1.6 and in §2 let A_f be the common notation for both K_f and $\hat{K}_f$, depending on whether we consider $\mathfrak{k}^L$ or $\mathfrak{k}^M$, respectively.

1.6.1. The cocycle operator ∇. The *cocycle operator* ∇ is important in applications to integrable dynamical systems. Though it implicitly appears in the second column of Table 1.6.2 according to the formula

$$c: D_1, D_2\mapsto(-1)^{p(D_2)(p(\nabla)+1)}\mathrm{Res}(F(D_1)\cdot\nabla(F(D_2))) \tag{1.6.1}$$

for appropriate functions $F(D)$, it deserves to be described explicitely. (For $\mathfrak{vect}$ and $\mathfrak{svect}$ series the functions $F(D)$ are vector-valued and the dot product $\cdot$ in eq. (1.6.1) is the scalar product $\sum\limits_i F_i(D_1)F_i(D_2)$.)

The functions $F(D)$ are as follows:

$$F(A_f)=F(M_f)=f;$$
$$F(f\frac{\partial}{\partial t}+g_1\frac{\partial}{\partial\xi_1}+g_2\frac{\partial}{\partial\xi_2})=(f,g_1,g_2).$$

Explicitely, the cocycle operator is of the form

algebra	its element	∇
$\mathfrak{vect}^L(1\vert 2)$	$D=f\frac{\partial}{\partial t}+g_1\frac{\partial}{\partial\xi_1}+g_2\frac{\partial}{\partial\xi_2}$	$\left(0,\frac{\partial g_2}{\partial t},-\frac{\partial g_1}{\partial t}\right)$
$\mathfrak{svect}^L_\lambda(1\vert 2)$	$D=f\frac{\partial}{\partial t}+g_1\frac{\partial}{\partial\xi_1}+g_2\frac{\partial}{\partial\xi_2}$	$\left(0,\frac{\partial g_2}{\partial t},-\frac{\partial g_1}{\partial t}\right)$
$\mathfrak{vect}^L(1\vert 1)$	$D=f\frac{\partial}{\partial t}+g\frac{\partial}{\partial\xi}$	$\frac{\partial^2 g}{\partial t^2}-\left((-1)^{p(D)}\frac{\partial^2 f}{\partial t^2}-2\frac{\partial^2 g}{\partial t^2}\right)$

The following Lie superalgebras of contact vector fields are concidered on a supercircle though we skip the superscript:

algebra	its element	∇
$\mathfrak{k}^L(1\|0)$	K_f	$K_1^3(f)$
$\mathfrak{k}^L(1\|1)$ or $\mathfrak{k}^M(1\|1)$	A_f	$A_\theta A_1^2(f)$
$\mathfrak{k}^L(1\|2)$or $\mathfrak{k}^M(1\|2)$	A_f	$A_{\theta_1}A_{\theta_2}A_1(f)$
$\mathfrak{m}^L(1)$	M_f	$(M_\xi)^2(f)$
$\mathfrak{k}^L(1\|3)$or $\mathfrak{k}^M(1\|3)$	A_f	$A_{\theta_1}A_{\theta_2}A_{\theta_3}(f)$
$\left.\begin{matrix}\mathfrak{k}^{L\circ}(4)\\ \mathfrak{k}^M(1\|4)\end{matrix}\right\}$	A_f	(1) $A_{\theta_1}A_{\theta_2}A_{\theta_3}A_{\theta_4}A_1^{-1}(f)$
$\mathfrak{k}^{L\circ}(4)$	K_f	(2) $tK_{t^{-1}}(f)$ (3) $K_1(f)$

1.6.2. Theorem. *The only nontrivial central extensions of the simple stringy Lie superalgebras are those given in the following table.*

algebra	the cocycle c	The name of the extention
$\mathfrak{k}^L(1\|0)$	$\mathrm{Res} f K_1^3(g)$	Virasoro or $\mathfrak{vir}$
$\left.\begin{matrix}\mathfrak{k}^L(1\|1)\\ \mathfrak{k}^M(1\|1)\end{matrix}\right\}$	$\mathrm{Res} f A_\theta A_1^2(g)$	Neveu-Schwarz or $\mathfrak{ns}$ Ramond or $\mathfrak{r}$
$\left.\begin{matrix}\mathfrak{k}^L(1\|2)\\ \mathfrak{k}^M(1\|2)\end{matrix}\right\}$	$(-1)^{p(f)}\mathrm{Res} f A_{\theta_1}A_{\theta_2}A_1(g)$	2-Neveu-Schwarz or $\mathfrak{ns}(2)$ 2-Ramond or $\mathfrak{r}(2)$
$\mathfrak{m}^L(1)$	$M_f, M_g \mapsto \mathrm{Res} f(M_\xi)^2(g)$	$\widehat{\mathfrak{m}^L(1)}$
$\left.\begin{matrix}\mathfrak{k}^L(1\|3)\\ \mathfrak{k}^M(1\|3)\end{matrix}\right\}$	$\mathrm{Res} f A_{\theta_1}A_{\theta_2}A_{\theta_3}(g)$	3-Neveu-Schwarz or $\mathfrak{ns}(3)$ 3-Ramond or $\mathfrak{r}(3)$
$\left.\begin{matrix}\mathfrak{k}^{L\circ}(4)\\ \mathfrak{k}^M(1\|4)\end{matrix}\right\}$	(1) $(-1)^{p(f)}\mathrm{Res} f A_{\theta_1}A_{\theta_2}A_{\theta_3}A_{\theta_4}A_1^{-1}(g)$	4-Neveu-Schwarz = $\mathfrak{ns}(4)$ 4-Ramond = $\mathfrak{r}(4)$
$\mathfrak{k}^{L\circ}(4)$	(2) $\mathrm{Res} f(tA_{t^{-1}}(g))$ (3) $\mathrm{Res} f A_1(g)$	$4'$-Neveu-Schwarz = $\mathfrak{ns}(4')$ 4^0-Neveu-Schwarz = $\mathfrak{ns}(4^0)$

Observe that K_1^{-1} is only defined on $\mathfrak{k}^{L\circ}(4)$ but *not* on $\mathfrak{k}^L(4)$; observe also that the bilinear functionals (2) and (3) *are* defined on $\mathfrak{k}^L(4)$; but whereas they are (trivial) cocycles on $\mathfrak{k}^M(4)$ (with $\hat{K}_f$ instead of K_f, of course, for the Moebius version), they are not even cocycles on $\mathfrak{k}^L(4)$.

$\mathfrak{vect}^L(1\vert 2)$	the restriction of the cocycle (1) defined on $\mathfrak{k}^{L\circ}(4)$: $D_1 = f\frac{\partial}{\partial t} + g_1\frac{\partial}{\partial \xi_1} + g_2\frac{\partial}{\partial \xi_2}$, $D_2 = \tilde f\frac{\partial}{\partial t} + \tilde g_1\frac{\partial}{\partial \xi_1} + \tilde g_2\frac{\partial}{\partial \xi_2}$ $\mapsto\ \mathrm{Res}(g_1\tilde g_2' - g_2\tilde g_1'(-1)^{p(D_1)p(D_2)})$	$\widehat{\mathfrak{vect}}^L(1\vert 2)$
$\mathfrak{svect}^L_\lambda(1\vert 2)$	the restriction of the above	$\widehat{\mathfrak{svect}}^L_\lambda(1\vert 2)$
$\mathfrak{vect}^L(1\vert 1)$	$D_1 = f\frac{\partial}{\partial t} + g\frac{\partial}{\partial \xi},\ D_2 = \tilde f\frac{\partial}{\partial t} + \tilde g\frac{\partial}{\partial \xi} \mapsto$ $\mathrm{Res}(f\frac{\partial^2 \tilde g}{\partial t^2} - g\Big((-1)^{p(D_2)}\frac{\partial^2 \tilde f}{\partial t^2} - 2\frac{\partial^2 \tilde g}{\partial t^2}\Big)$	$\widehat{\mathfrak{vect}}^L(1\vert 1)$

Observe that the restriction of the only nontrivial cocycle existing on $\mathfrak{vect}^L(1|2)$ onto its subalgebra $\mathfrak{witt} \cong Span(f(t)\frac{\partial}{\partial t} : f \in \mathbb{C}[t^{-1}, t])$ is trivial while the the restriction of the only nontrivial cocycle existing $\mathfrak{svect}^L_\lambda(1|2)$ onto its unique subalgebra $\mathfrak{witt}$ is nontrivial. The riddle is solved by a closer study of the embedding $\mathfrak{vect}(1|m) \longrightarrow \mathfrak{k}(1|2m)$: it involves differentiations, see formulas (1.1.8).

1.6.3. The cocycle c in monomial basis. For $\mathfrak{vect}^L(1|2)$ the nonzero values of c are:

$$c(t^k\xi_1\frac{\partial}{\partial \xi_1}, t^l\xi_2\frac{\partial}{\partial \xi_2}) = -k\delta_{k,-l},\quad c(t^k\xi_1\frac{\partial}{\partial \xi_2}, t^l\xi_2\frac{\partial}{\partial \xi_1}) = k\delta_{k,-l},$$
$$c(t^k\xi_1\xi_2\frac{\partial}{\partial \xi_1}, t^l\frac{\partial}{\partial \xi_2}) = -k\delta_{k,-l},\quad c(t^k\xi_1\xi_2\frac{\partial}{\partial \xi_2}, t^l\frac{\partial}{\partial \xi_1}) = k\delta_{k,-l}.$$

In $\mathfrak{svect}^L_\lambda(1|2)$, set:

$$L_m = t^m\left(t\frac{\partial}{\partial t} + \frac{\lambda+m+1}{2}(\xi_1\frac{\partial}{\partial \xi_1} + \xi_2\frac{\partial}{\partial \xi_2})\right),$$
$$S^j_m = t^m\xi_j\left(t\frac{\partial}{\partial t} + (\lambda+m+1)(\xi_1\frac{\partial}{\partial \xi_1} + \xi_2\frac{\partial}{\partial \xi_2})\right).$$

The nonzero values of the cocycles on $\mathfrak{svect}^L_\lambda(1|2)$ are

$$c(L_m, L_n) = \frac{1}{2}m(m^2 - (\lambda+1)^2)\delta_{m,-n},$$
$$c(t^k\frac{\partial}{\partial \xi_i}, S^j_m) = -m(m-(\lambda+1))\delta_{m,-n}\delta_{i,j},$$
$$c(t^m(\xi_1\frac{\partial}{\partial \xi_1} - \xi_2\frac{\partial}{\partial \xi_2}), t^n(\xi_1\frac{\partial}{\partial \xi_1} - \xi_2\frac{\partial}{\partial \xi_2})) = m\delta_{m,-n},$$
$$c(t^m\xi_1\frac{\partial}{\partial \xi_2}, t^n\xi_2\frac{\partial}{\partial \xi_1}) = m\delta_{m,-n}.$$

For $\mathfrak{m}^L(1)$ the nonzero values of c are:

$$c(u^{m+1}\tau\xi, u^{n+1}) = -(n^2+n)\delta_{m+n+1,0},$$
$$c(u^{m+1}\xi, u^n\tau) = (n^2-n)\delta_{m+n,0},$$
$$c(u^m\tau, u^n\tau) = -6n\delta_{m+n,0}.$$

For $\mathfrak{k}^{L\circ}(1|4)$ the nonzero values of c are:

$$c(t^{m+1}, t^{n+1}) = m(m^2-1)\delta_{m+n,0},$$
$$c(t^{m+1}\xi_i, t^{n+1}\xi_i) = -m(m+1)\delta_{m+n+1,0},$$
$$c(t^{m+1}\xi_i\xi_j, t^{n+1}\xi_i\xi_j) = -(m+1)\delta_{m+n+2,0},$$
$$c(t^{m+1}\xi_i\xi_j\xi_k, t^{n+1}\xi_i\xi_j\xi_k) = \delta_{m+n+3,0},$$
$$c(t^{m+1}\xi_1\xi_2\xi_3\xi_4, t^{n+1}\xi_1\xi_2\xi_3\xi_4) = \frac{1}{m+2}\delta_{m+n+4,0}.$$

The restriction of this cocycle determines the nontrivial central extensions of $\mathfrak{k}^L(1|n)$ for $n \leq 3$.

1.6.4. When the nontrivial central extensions of the stringy superalgebras are possible. We find the following quantitative discussion instructive, though it neither replaces the detailed proof (that can be found in [KvL] for all cases except $\mathfrak{kas}^L$; the arguments in the latter case are similar) nor explains the number of nontrivial cocycles on (1|4)-dimensional supercircle with a contact structure.

When we pass from simple finite dimensional Lie algebras to loop algebras, we enlarge the maximal toral subalgebra of the latter to make the number of generators of weight 0 equal to that of positive or negative generators corresponding to simple roots. In this way we get the nontrivial central extensions $\widehat{\mathfrak{g}^{(1)}}$ of the loop algebras $\mathfrak{g}^{(1)}$, called *Kac–Moody* algebras. (Actually, the latter include one more operator of weight 0: the exterior derivation $t\dfrac{d}{dt}$ of $\widehat{\mathfrak{g}^{(1)}}$.)

Similarly, for the Witt algebra $\mathfrak{witt}$ we get:

$\deg K_f$	$\dots$	-2	-1	0	1	2	$\dots$
f	$\dots$	t^{-1}	1	t	t^2	t^3	$\dots$

The depicted elements generate $\mathfrak{witt}$; more exactly,

(a) the elements of degrees -1, 0, 1 generate $\mathfrak{sl}(2)$;

(b) $\mathfrak{witt}$, as $\mathfrak{sl}(2)$-module, is glued from the three modules: the adjoint module and the Verma modules M^{-2}, and M_2 with highest and lowest weights as indicated: -2 and 2, respectively.

It is natural to expect that a central element can be obtained by pairing of the dual $\mathfrak{sl}(2)$-modules M^{-2} and M_2; this actually happens. One of the methods to find all the cocycle is to compute $H^2(\mathfrak{g})$; a simpler way is to compute $H^1(\mathfrak{g};\, \mathfrak{g}^*)$ and interprete them in ters of $H^2(\mathfrak{g})$ (cf. [P] with [Sc], respectively).

• Further on, consider the subalgebra $\mathfrak{osp}(n|2)$ in $\mathfrak{k}^L(1|n)$ and decompose $\mathfrak{k}^L(1|n)$, as an $\mathfrak{osp}(n|2)$-module, into irreducible modules. Denote by $(\chi_0;\, \chi_1, \dots, \chi_r)$ the weight of the irreducible $\mathfrak{osp}(n|2)$-module with respect to the standard basis of Cartan subalgebra of $\mathfrak{sl}(2) \oplus \mathfrak{o}(n)$; here $r = [n/2]$.

These modules and their generators are as follows. Set

$$\alpha = \begin{cases} dt - \sum(\xi_i d\eta_i + \eta_i d\xi_i) & \text{for } n \text{ even} \\ dt - \sum(\xi_i d\eta_i + \eta_i d\xi_i) - \theta d\theta & \text{for } n \text{ odd} \end{cases}$$

$$\zeta = \begin{cases} (\xi, \eta) & \text{for } n \text{ even} \\ (\xi, \eta, \theta) & \text{for } n \text{ odd.} \end{cases}$$

Let $\langle f \rangle$ be a shorthand for the Verma module M with the highest (lowest) weight as indicated by the sub- or superscript, respectively, generated by K_f; we denote by L with the same indices the quotient of M modulo the maximal submodule. Then the irreducible components of the $\mathfrak{osp}(n|2)$-module $\mathfrak{k}^L(1|n)$ are as follows, where $\supset$ denotes the semidirect sum of modules:

n	irreducible factors	of $\mathfrak{k}^L(1\|n)$	as $\mathfrak{osp}(n\|2)$-module
0	$\langle t^{-1} \rangle = L^{-2}$,	$\mathfrak{sl}(2)$	$L_2 = \langle t^3 \rangle$
1	$\langle t^{-1}\theta \rangle = L^{-3}$	$\mathfrak{osp}(1\|2)$	$L_3 = \langle t^2\theta \rangle$
2	$\langle t^{-1}\xi\eta \rangle = L^{-2;\,0}$	$\mathfrak{osp}(2\|2)$	$L_{2;\,0} = \langle t\xi\eta \rangle$
3	$\langle t^{-1}\xi\eta\theta \rangle = L^{-1;\,0}$	$\mathfrak{osp}(3\|2)$	$L_{1;\,0} = \langle \xi\eta\theta \rangle$
4	$\langle t^{-1}\xi_1\xi_2\eta_2 \rangle = L^{-1;\,\varepsilon_1}$ $\Subset L_{0;0} = \langle t^{-1}\xi_1\xi_2\eta_1\eta_2 \rangle$	$\mathfrak{osp}(4\|2)$	$L_{1;\,-\varepsilon_2} = \langle \xi_1\eta_1\eta_2 \rangle$
5	$\langle t^{-1}\xi_1 \dots \eta_2\theta \rangle = M^{1;\,0}$	$\mathfrak{osp}(5\|2)$	$L_{1;\,-\varepsilon_1-\varepsilon_2} = \langle \eta_1\eta_2\theta \rangle$
6	$\langle t^{-1}\xi_1 \dots \eta_3\theta \rangle = M^{2;\,0}$	$\mathfrak{osp}(6\|2)$	$L_{1;\,-\varepsilon_1-\varepsilon_2-\varepsilon_3} \oplus L_{1;\,-\varepsilon_1-\varepsilon_2+\varepsilon_3}$
> 6	$\langle t^{-1}\theta_1 \dots \theta_n \rangle = M^{n-4;\,0}$	$\mathfrak{osp}(n\|2)$	$L_{1;\,-\varepsilon_1-\varepsilon_2-\varepsilon_3} = \langle \eta_1\eta_2\eta_3 \rangle$

For $n > 6$ the module $M_{-1;-\varepsilon_1-\varepsilon_2-\varepsilon_3}$ is always irreducible whereas $M^{n-4;\,0}$ is always reducible:

$$[M^{n-4;\,0}] = [L^{n-7;\,\varepsilon_1+\varepsilon_2+\varepsilon_3}] \Subset [L^{n-4;\,0}].$$

Exceptional cases:

$n = 4$. In this case the Verma module $M^{0;\,0}$ induced from the Borel subalgebra has an irreducible submodule $M^{-1;\,\varepsilon_1}$ dual to $M_{1;\,-\varepsilon_2}$; the subspace of $\mathfrak{k}^L(1|4)$ spanned by all functions except $t^{-1}\xi_1\xi_2\eta_1\eta_2$ is an ideal. An explanation of this phenomenon is given in 1.3.

$n = 6$. Two miracles happen: (1) $L^{6-4;\,\varepsilon_1+\varepsilon_2+\varepsilon_3} = (L_{1;\,-(\varepsilon_1+\varepsilon_2+\varepsilon_3)})^*$ and $L^{6-4;\,0} \cong \mathfrak{osp}(6|2)$. (2) The bilinear form obtained is supersymmetric, see §2.

For $n>6$ there is no chance to have a nondegenerate bilinear form on $\mathfrak{k}^L(1|n)$. The above qualitative arguments, however, do not exclude a degenerate form on $\mathfrak{k}^L(1|n)$, such as a cocycle. There are no cocycles either, for a proof see [KvL].

1.7. Root systems and simple roots for $\mathfrak{svect}^L_\lambda(1|2)$ Set

$$\partial=\frac{\partial}{\partial t},\quad \delta_1=\frac{\partial}{\partial\xi_1},\quad \delta_2=\frac{\partial}{\partial\xi_2}\,.$$

The generators (coroots) coresponding to the *distinguished* system of simple roots are:

$$\begin{array}{lll} X_1^+=\xi_1\delta_2 & X_2^+=t\delta_1 & X_3^+=\xi_2t\partial-(\lambda+1)\xi_1\xi_2\delta_1\\ X_1^-=\xi_2\delta_1 & X_2^-=\lambda\dfrac{\xi_1\xi_2}{t}\delta_2+\xi_1\partial & X_3^-=\delta_2\\ H_1=\xi_1\delta_1-\xi_2\delta_2 & H_2=t\partial+\xi_1\delta_1+\lambda\xi_2\delta_2 & H_3=t\partial+(\lambda+1)\xi_1\delta_1\end{array}\qquad (G1)$$

The reflection in the 2nd root sends $(G1)$ into the following system which, to simplify the expressions, we consider up to factors in square brackets $[\cdot]$:

$$\begin{array}{lll} X_1^+=t\delta_2 & X_2^+=\lambda\dfrac{\xi_1\xi_2}{t}\delta_2+\xi_1\partial & X_3^+=[-\lambda]t\xi_2\delta_1\\ X_1^-=\lambda\dfrac{\xi_1\xi_2}{t}\delta_1-\xi_2\partial & X_2^-=t\delta_1 & X_3^-=[-\lambda]\dfrac{\xi_1}{t}\delta_2\\ H_1=-(t\partial+\lambda\xi_1\delta_1+\xi_2\delta_2) & H_2=t\partial+\xi_1\delta_1+\lambda\xi_2\delta_2 & H_3=\xi_2\delta_2-\xi_1\delta_1\end{array}\qquad (G2)$$

The reflection in the 3rd root sends $(G1)$ into the following system. To simplify the expressions we consider them up to factors in square brackets $[\cdot]$.

$$\begin{array}{lll} X_1^+=\delta_2 & X_2^+=[\lambda+2]t\xi_2\delta_1 & X_3^+=(\lambda+1)\xi_2\xi_1\delta_2-\xi_1t\partial\\ X_1^-=t\xi_2\partial-(\lambda+1)\xi_1\xi_2\delta_1 & X_2^-=[-\lambda]\dfrac{\xi_1}{t}\delta_2 & X_3^-=\delta_1\\ H_1=t\partial+(\lambda+1)\xi_1\delta_1 & H_2=\xi_2\delta_2-\xi_1\delta_1 & H_3=-t\partial-(\lambda+1)\xi_2\delta_2\end{array}\qquad (G3)$$

The corresponding Cartan matrices are:

$$\begin{pmatrix}2&-1&-1\\1-\lambda&0&\lambda\\1+\lambda&-\lambda&0\end{pmatrix},\quad \begin{pmatrix}0&-\lambda+1&-2+\lambda\\1-\lambda&0&\lambda\\-1&-1&2\end{pmatrix},\quad \begin{pmatrix}0&-\lambda&\lambda+1\\-1&2&-1\\1+\lambda&-\lambda-2&0\end{pmatrix}.$$

To compare these matrices, let us reduce them to the following canonical forms $(C1)-(C3)$, respectively, by renumbering generators and rescaling. (Observe that by definition, $\lambda\neq 0,\pm1$, so the fractions are well-defined.) We obtain

$$\begin{pmatrix}2&-1&-1\\-1+\dfrac{1}{\lambda}&0&1\\1+\dfrac{1}{\lambda}&-1&0\end{pmatrix},\quad \begin{pmatrix}2&-1&-1\\-1+\dfrac{1}{1-\lambda}&0&1\\1+\dfrac{1}{1-\lambda}&-1&0\end{pmatrix},\quad \begin{pmatrix}2&-1&-1\\-1+\dfrac{1}{1+\lambda}&0&1\\1+\dfrac{1}{1+\lambda}&-1&0\end{pmatrix}.$$

We see that the transformations $\lambda \mapsto \lambda+1$ and $\lambda \mapsto 1-\lambda$ establish isomorphisms the first of which is obvious, the second one is misterious.

1.8. Simplicity and occasional isomorphisms

Statement. 1) *The Lie superalgebras* $\mathfrak{vect}^L(1|n)$ *for any* n, $\mathfrak{svect}^L_\lambda(1|n)$ *for* $\lambda \notin \mathbb{Z}$ *and* $n>1$, $\mathfrak{svect}^{L0}(1|n)$ *for* $n>1$, $\mathfrak{k}^M(1|n)$ *for* $n \neq 5$ *and* $\mathfrak{k}^L(1|n)$ *for* $n \neq 4$*; and the four exceptional stringy superalgebras are simple.*

2) *The Lie superalgebras* $\mathfrak{vect}^L(1|1)$, $\mathfrak{k}^L(1|2)$ *and* $\mathfrak{m}^L(1)$ *are isomorphic.*

3) *The Lie superalgebras* $\mathfrak{svect}^L_\lambda(1|2) \cong \mathfrak{svect}^L_\mu(1|2)$ *if* μ *can be obtained from* λ *with the help of transformations* $\lambda \mapsto \lambda+1$ *and* $\lambda \mapsto 1-\lambda$*. The Lie superalgebras* $\mathfrak{svect}^L_\lambda(1|2)$ *from the strip* $\operatorname{Re}\lambda \in \left[0, \frac{1}{2}\right]$ *are nonisomorphic.*

The statement on simplicity follows from a criterion similar to the one Kac applied for Lie (super)algebras with polynomial coefficients ([K]). The isomorphism is determined by the gradings listed in sec. 0.4 and arguments of 1.7.

1.9. A relation with Kac–Moody superalgebras. An unpublished theorem of Serganova (1990) states that *the only simple Kac–Moody superalgebras* $\mathfrak{g}(A)$ *of polynomial growth with nonsymmetrizable Cartan matrix* A *are:* $\mathfrak{psq}(n)^{(2)}$ *and an exceptional parametric family with the matrix*

$$A = \begin{pmatrix} 2 & -1 & -1 \\ 1-\alpha & 0 & \alpha \\ 1+\alpha & -\alpha & 0 \end{pmatrix} \cong \begin{pmatrix} 2 & -1 & -1 \\ -1+\dfrac{1}{\lambda} & 0 & 1 \\ 1+\dfrac{1}{\lambda} & -1 & 0 \end{pmatrix}.$$

The exceptional Lie superalgebra $\mathfrak{g}(A)$ can be realized as the distinguished stringy superalgebra $\widehat{\mathfrak{svect}}^L_\alpha(1|2)$. For the description of the relations between its generators see [GL1].

Remark. 1) The Dynkin–Kac diagram corresponding to $\mathfrak{psq}(n)^{(2)}$ is the same as that of $\mathfrak{sl}(n)^{(1)}$ with odd number of nodes replaced with "grey" nodes corresponding to the odd simple roots of type $\mathfrak{sl}(1|1)$, see [FLS], the one-to-one correspondence between Dynkin–Kac diagrams and Cartan matrices gives a description of the possible values of A.

2) The parametric family $\mathfrak{g}(A)$ was found by J. van de Leur around 1986.

Observe, that unlike the Kac–Moody superalgebras of polynomial growth with symmetrizable Cartan matrix, $\hat{\mathfrak{g}} = \widehat{\mathfrak{svect}}^L_\lambda(1|2)$ can not be interpreted as a central extension of any twisted loop algebra. Indeed, the root vectors of the latter are locally nilpotent, whereas the former contains the operator ∂_t with nonzero image of every $\hat{\mathfrak{g}}_i$.

1.10. How conformal are stringy superalgebras. Recall that a Lie algebra is called *conformal* if it preserves up to a factor a metric or, more generally, a (not necessarily symmetric) bilinear form B. It is known (as a theorem of Liouville) that given a metric B on the *real* space V of dimension $\neq 2$, the algebras conformally preserving B are isomorphic to $\mathfrak{c}(\mathfrak{aut}(B)) \cong \mathfrak{co}(V, B)$. If $\dim_{\mathbb{R}} V = 2$, we can consider V as the complex line $\mathbb{C}^1$ with complex coordinate t and identify B with the metric $dt \cdot d\bar{t}$ (the symmetric product of the differentials) on $\mathbb{C}$. The element $f\dfrac{d}{dt}$ from $\mathfrak{witt}$ multiplies dt by f' and, therefore, it multiplies $dt \cdot d\bar{t}$ by $f'\bar{f}'$, so $\mathfrak{witt}$ is *conformal.*

On superspaces V, metrics B can be even and odd, the Lie superalgebras $\mathfrak{aut}(V, B)$ that preserve them are $\mathfrak{osp}^{sy}(Par)$ and $\mathfrak{pe}^{sy}(Par)$ and the corresponding conformal superalgebras are just trivial central extensions of $\mathfrak{aut}(V, B)$ for any dimension.

Let the contact superalgebras $\mathfrak{k}^L$, $\mathfrak{k}^M$ and $\mathfrak{m}^L$ preserve the Pfaff equation $\alpha = 0$ for the correponding form $\alpha = \alpha_1$ or $\alpha = \alpha_0$. Suppose now that we consider a real form of each of the stringy superalgebras considered above and an extension of the complex conjugation (for possibilities see [M]) is defined in the superspace of the generating functions. From formulas (0.3) and (1.1.6) we deduce that the Lie derivative along the elements of these superalgebras multiplies the symmetric product of forms $\alpha \cdot \bar{\alpha}$ by a factor of the form $F\bar{F}$, i.e.,

$$L_{K_f}(\alpha \cdot \bar{\alpha}) = F\bar{F}(\alpha \cdot \bar{\alpha}), \tag{1.10.1}$$

where F is the function determined in (0.3) and (1.1.6). Every element D of the general and divergence-free superalgebra $\mathfrak{svect}^L_\lambda$ multiplies the symmetric product of volume forms $vol(t, \theta) \cdot vol(\bar{t}, \bar{\theta})$ by $\operatorname{div} D \cdot \overline{\operatorname{div} D}$, i.e.,

$$L_D(vol(t, \theta) \cdot vol(\bar{t}, \bar{\theta})) = \operatorname{div} D \cdot \overline{\operatorname{div} D}(vol(t, \theta) \cdot vol(\bar{t}, \bar{\theta})). \tag{1.10.2}$$

None of the tensors considered, i.e., neither $\alpha \cdot \bar{\alpha}$ nor $vol(t, \theta) \cdot vol(\bar{t}, \bar{\theta})$, can be viewed as a metric in the presence of odd parameters except for $\mathfrak{g} = \mathfrak{k}^L(1|1)$ and $\mathfrak{k}^M(1|1)$.

Indeed, it is possible to consider $\mathfrak{g} = \mathfrak{k}^L(1|1)$ and $\mathfrak{k}^M(1|1)$ as conformal superalgebras, since the volume form "$dt\dfrac{\partial}{\partial\theta}$" can be considered as, more or less, $d\theta$: their transformation rules under $\mathfrak{g}$ are identical. In fact, consider the quotient $\Omega^1/\mathcal{F}\alpha$ or $\Omega^1/\mathcal{F}\hat{\alpha}$, respectively, of the superspace of differential 1-forms modulo the subspace spanned over functions by the contact form. On the quotient space, the tensor $dt\dfrac{\partial}{\partial\theta} \cdot d\bar{t}\dfrac{\partial}{\partial\bar{\theta}}$ can be viewed as the bilinear form $d\theta \cdot d\bar{\theta}$.

So only $\mathfrak{witt}$, $\mathfrak{k}^L(1|1)$ and $\mathfrak{k}^M(1|1)$ are conformal Lie superalgebras, or as physicists say, superconformal algebras.

2. Invariant bilinear forms on stringy Lie superalgebras

Statement. *An invariant (with respect to the adjoint action) nondegenerate supersymmetric bilinear form on a simple Lie superalgebra $\mathfrak{g}$, if exists, is unique up to proportionality.*

Observe that the form spoken above can be odd. For the proof see [Sh1].

The invariant nondegenerate bilinear form $(\cdot,\cdot)$ on $\mathfrak{g}$ exists if and only if, as $\mathfrak{g}$-modules,

$$\mathfrak{g} \cong \begin{cases} \mathfrak{g}^* & \text{if } (\cdot,\cdot) \text{ is even} \\ \Pi(\mathfrak{g}^*) & \text{if } (\cdot,\cdot) \text{ is odd.} \end{cases}$$

Therefore, let us compare $\mathfrak{g}$ with $\mathfrak{g}^*$ (recall the definition of the modules $\mathcal{F}_\lambda$ and examples from sec. 1.3).

Table. $\mathfrak{g}$-modules $\mathfrak{g}$, *Vol* and $\mathfrak{g}^*$ over contact Lie superalgebras

	0	1	2	3	4	5	6	7	$n>0$
$\mathfrak{g} = \mathfrak{k}^L(1\|n)$	$\mathcal{F}_{-1}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$
Vol	$\mathcal{F}_1$	$\Pi(\mathcal{F}_1)$	$\mathcal{F}_0$	$\Pi(\mathcal{F}_{-1})$	$\mathcal{F}_{-2}$	$\Pi(\mathcal{F}_{-3})$	$\mathcal{F}_{-4}$	$\Pi(\mathcal{F}_{-5})$	$\Pi^n(\mathcal{F}_{2-n})$
$\mathfrak{g}^*$	$\mathcal{F}_2$	$\Pi(\mathcal{F}_3)$	$\mathcal{F}_2$	$\Pi(\mathcal{F}_1)$	$\mathcal{F}_0$	$\Pi(\mathcal{F}_{-1})$	$\mathcal{F}_{-2}$	$\Pi(\mathcal{F}_{-3})$	$\Pi^n(\mathcal{F}_{4-n})$

	1	2	3	4	5	6	7	·	$n>1$
$\mathfrak{g} = \mathfrak{k}^M(1\|n)$	$\mathcal{F}_{-1}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	$\mathcal{F}_{-2}$	·	$\mathcal{F}_{-2}$
Vol	$\Pi(\mathcal{F}_1)$	$\mathcal{F}_1$	$\Pi(\mathcal{F}_0)$	$\mathcal{F}_{-1}$	$\Pi(\mathcal{F}_{-2})$	$\mathcal{F}_{-3}$	$\Pi(\mathcal{F}_{-4})$	·	$\Pi^n(\mathcal{F}_{2-(n-1)})$
$\mathfrak{g}^*$	$\Pi(\mathcal{F}_2)$	$\mathcal{F}_3$	$\Pi(\mathcal{F}_2)$	$\mathcal{F}_1$	$\Pi(\mathcal{F}_0)$	$\mathcal{F}_{-1}$	$\Pi(\mathcal{F}_{-2})$	·	$\Pi^n(\mathcal{F}_{5-n})$

A comparison of $\mathfrak{g}$ with $\mathfrak{g}^*$ shows that there is a nondegenerate bilinear form on $\mathfrak{g} = \mathfrak{k}^L(1|6)$ and $\mathfrak{g} = \mathfrak{k}^M(1|7)$, even and odd, respectively.

Statement. *These forms are supersymmetric and given by the formula*

$$(A_f, A_g) = \mathrm{Res} fg.$$

The restriction of the bilinear form to $\mathfrak{kas}^L$ is identically zero.

A comparison of $\mathfrak{g}$ with *Vol* shows that there is an invariant linear functional on $\mathfrak{g} = \mathfrak{k}^L(1|4)$ and $\mathfrak{g} = \mathfrak{k}^M(1|5)$, even and odd, respectively.

3. The three cocycles on $\mathfrak{k}^{L\circ}(1|4)$ and primary fields

Let us introduce a shorthand notation for the elements of $\mathfrak{k}^{L\circ}(1|4)$:

the elements	their degree	their parity
$L_n = K_{t^{n+1}};\ T_n^{ij} = K_{t^n\theta_i\theta_j};\ S_n = K_{t^{n-1}\theta_1\theta_2\theta_3\theta_4}$	$2n$	$\bar{0}$;
$E_n^i = K_{t^{n+1}\theta_i};\ F_n^i = K_{t^n \frac{\partial \theta_1\theta_2\theta_3\theta_4}{\partial\theta_i}}$	$2n+1$	$\bar{1}$.

In the above "natural" basis the nonzero values of the cocycles are (see [KvL]; here A_n is the group of even permutations):

$$\boxed{\begin{array}{c}
c(L_m,L_n)=\alpha\cdot m(m^2-1)\delta_{m+n,0}\\
c(E^i_m,E^i_n)=\alpha\cdot m(m+1)\delta_{m+n+1,0}\\
c(F^i_m,F^i_n)=\alpha\cdot\delta_{m+n+1,0}\\
c(S_m,S_n)=\alpha\cdot\frac{1}{m}\delta_{m+n,0}\\
\hline
c(L_m,S_n)=(\gamma+\beta\cdot m)\delta_{m+n,0}\qquad(*)\\
c(E^i_m,F^i_n)=\Big(\frac{1}{2}\gamma+\beta\cdot\big(m+\frac{1}{2}\big)\Big)\delta_{m+n+1,0}\\
\hline
c(T^{ij}_m,T^{ij}_n)=\alpha\cdot m\delta_{m+n,0}\\
c(T^{ij}_m,T^{kl}_n)=-\beta\cdot m\delta_{m+n,0},\ \text{where } (i,j,k,l)\in A_4.
\end{array}}$$

To express the cocycle in terms of *primary* fields, we have to eliminate the term $(*)$. To this end, let us embed $\mathfrak{witt}$ differently and, simultaneously, suitably intermix the odd generators:

$$\begin{array}{l}
\tilde L_m=L_m+a_mS_m \text{ for } a_m=-\frac{\beta}{\alpha}\cdot m^2-\frac{\gamma}{\alpha}\cdot m;\\
\tilde E^i_m=E^i_m+b_mF^i_m \text{ for } b_m=\frac{\beta}{2\alpha}\cdot(2m+1)+\frac{\gamma}{2\alpha}.
\end{array}$$

In the new basis the cocycle is of the form:

$$\boxed{\begin{array}{c}
c(\tilde L_m,\tilde L_n)=\Big(\frac{\alpha^2-\beta^2}{\alpha}\cdot m^3-\frac{\alpha^2-\gamma^2}{\alpha}\cdot m\Big)\delta_{m+n,0}\\
c(\tilde E^i_m,\tilde E^i_n)=\Big(\frac{\alpha^2-3\beta^2}{\alpha}\cdot\big(m+\frac{1}{2}\big)^2-\frac{\alpha^2-3\gamma^2}{4\alpha}\Big)\delta_{m+n+1,0}\\
c(F^i_m,F^i_n)=\alpha\cdot\delta_{m+n+1,0}\\
c(S_m,S_n)=\alpha\cdot\frac{1}{m}\delta_{m+n,0}\\
\hline
c(\tilde E^i_m,F^i_n)=(\gamma+\beta\cdot(2m+1))\delta_{m+n+1,0}\\
\hline
c(T^{ij}_m,T^{ij}_n)=\alpha\cdot m\delta_{m+n,0}\\
c(T^{ij}_m,T^{kl}_n)=-\beta\cdot m\delta_{m+n,0},\ \text{where } (i,j,k,l)\in A_4.
\end{array}}$$

It depends on the three parameters and is expressed in terms of primary fields. Observe that the 3-dimensional space of parameters is not $\mathbb{C}^3=\{(\alpha,\beta,\gamma)\}$ but $\mathbb{C}^3$ without a plane, since α can never vanish.

4. The explicit relations between the Chevalley generators of $\mathfrak{kas}^L$

(Other results pertaining here: [GKLP], [GL1].)

4.0. Sergeev's extension. Let ω be a nondegenerate superskew-symmetric odd bilinear form on an (n, n)-dimensional superspace V. In the standard basis of V (all the even vectors come first) the canonical matrix of the form ω is $\begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$ and the elements of $\mathfrak{pe}(n) = \mathfrak{aut}(\omega)$ can be represented by supermatrices of the form $\begin{pmatrix} a & b \\ c & -a^t \end{pmatrix}$, where $b = b^t$, $c = -c^t$. The Lie superalgebra $\mathfrak{spe}(n)$ is singled out by the requirement that $\mathrm{tr}\, a = 0$. Setting

$$\deg\left(\begin{pmatrix} 0 & 0 \\ c & 0 \end{pmatrix}\right) = -1, \quad \deg\left(\begin{pmatrix} a & 0 \\ 0 & -a^t \end{pmatrix}\right) = 0, \quad \deg\left(\begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}\right) = 1, \tag{1.1}$$

we endow $\mathfrak{pe}(n)$ with a $\mathbb{Z}$-grading. It is known ([K]) that $\mathfrak{spe}(n) = \mathfrak{pe}(n) \cap \mathfrak{sl}(n|n)$ is a simple Lie superalgebra for $n \geq 3$.

A. Sergeev proved (1977, unpublished) that there exists just one nontrivial central extension of $\mathfrak{spe}(n)$. It exists for $n = 4$ and is denoted by $\mathfrak{as}$. Let us represent an arbitrary element $A \in \mathfrak{as}$ as a pair $A = x + d \cdot z$, where $x \in \mathfrak{spe}(4)$, $d \in \mathbb{C}$ and z is the central element. In the matrix form the bracket in $\mathfrak{as}$ is

$$\left[\begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} + d \cdot z, \begin{pmatrix} a' & b' \\ c' & -a'^t \end{pmatrix} + d' \cdot z\right] = \left[\begin{pmatrix} a & b \\ c & -a^t \end{pmatrix}, \begin{pmatrix} a' & b' \\ c' & -a'^t \end{pmatrix}\right] + \mathrm{tr}\, cc' \cdot z.$$

Clearly, $\deg z = -2$ with respect to the grading (1.1).

The Lie superalgebra $\mathfrak{as}$ can also be described with the help of the spinor representation. Consider $\mathfrak{po}(0|6)$, the Lie superalgebra whose superspace is the Grassmann superalgebra $\Lambda(\xi, \eta)$ generated by $\xi_1, \xi_2, \xi_3, \eta_1, \eta_2, \eta_3$ and the bracket is the Poisson bracket. Recall that $\mathfrak{h}(0|6) = Span(H_f : f \in \Lambda(\xi, \eta))$.

Now, observe that $\mathfrak{spe}(4)$ can be embedded into $\mathfrak{h}(0|6)$. Indeed, setting $\deg \xi_i = \deg \eta_i = 1$ for all i we introduce a $\mathbb{Z}$-grading on $\Lambda(\xi, \eta)$ which, in turn, induces a $\mathbb{Z}$-grading on $\mathfrak{h}(0|6)$ of the form $\mathfrak{h}(0|6) = \mathop{\oplus}\limits_{i \geq -1} \mathfrak{h}(0|6)_i$. Since $\mathfrak{sl}(4) \cong \mathfrak{o}(6)$, we can identify $\mathfrak{spe}(4)_0$ with $\mathfrak{h}(0|6)_0$.

It is not difficult to see that the elements of degree -1 in $\mathfrak{spe}(4)$ and $\mathfrak{h}(0|6)$ constitute isomorphic $\mathfrak{sl}(4) \cong \mathfrak{o}(6)$-modules. It is subject to a direct verification that it is possible to embed $\mathfrak{spe}(4)_1$ into $\mathfrak{h}(0|6)_1$.

Sergeev's extension $\mathfrak{as}$ is the result of the restriction onto $\mathfrak{spe}(4) \subset \mathfrak{h}(0|6)$ of the cocycle that turns $\mathfrak{h}(0|6)$ into $\mathfrak{po}(0|6)$. The quantization deforms $\mathfrak{po}(0|6)$ into $\mathfrak{gl}(\Lambda(\xi))$; the through maps $T_\lambda : \mathfrak{as} \longrightarrow \mathfrak{po}(0|6) \longrightarrow \mathfrak{gl}(\Lambda(\xi))$ are representations of $\mathfrak{as}$ in the $4|4$-dimensional modules spin_λ *distinct* for distinct values λ of the central element z. (Here $\lambda \in \mathbb{C}$ plays the role of Planck's constant.) The explicit form of T_λ is as follows:

$$T_\lambda : \begin{pmatrix} a & b \\ c & -a^t \end{pmatrix} + d \cdot z \mapsto \begin{pmatrix} a & b - \lambda\tilde{c} \\ c & -a^t \end{pmatrix} + \lambda d \cdot 1_{4|4}, \tag{1.2}$$

where $1_{4|4}$ is the unit matrix and for a skew-symmetric matrix $c_{ij} = E_{ij} - E_{ji}$ we set $\tilde{c}_{ij} = c_{kl}$ for the even permutation $(1234) \mapsto (ijkl)$. Clearly, T_λ is an irreducible representation.

In [Sh1] it is demonstrated that

1) the Cartan prolong $\mathfrak{f}^\lambda = (\mathrm{spin}_\lambda, \mathfrak{as})_*$ is infinite dimensional and simple for $\lambda \neq 0$.

2) $\mathfrak{f}^\lambda \cong \mathfrak{f}^\mu$ if $\lambda \cdot \mu \neq 0$. Observe that though the representations T_λ are distinct for $\lambda \neq 0$, the corresponding Cartan prolongs are isomorphic.

For brevity, we denote the isomorphic superalgebras $\mathfrak{f}^\lambda = (\mathrm{spin}_\lambda, \mathfrak{as})_*$ for any $\lambda \neq 0$ by $\mathfrak{kas}$.

4.1. The Chevalley generators in $\mathfrak{kas}^L$ in terms of $\mathfrak{o}(6)$. Let Λ^k be the subsuperspace of $\mathfrak{k}^L(1|6)$ generated by the k-th degree monomials in the odd indeterminates θ_i. Then the basis elements of $\mathfrak{k}^L(1|6)$ with their degrees with respect to K_t are given by the following table:

...	-2	-1	0	1	2	...
...	1		t		t^2	...
...		Λ		$t\Lambda$		...
...	$\frac{\Lambda^2}{t}$		Λ^2		$t\Lambda^2$	...
...		$\frac{\Lambda^3}{t}$		Λ^3		...
...	$\frac{\Lambda^4}{t^2}$		$\frac{\Lambda^4}{t}$		Λ^4	...
...		$\frac{\Lambda^5}{t^2}$		$\frac{\Lambda^5}{t}$		...
...	$\frac{\Lambda^6}{t^3}$		$\frac{\Lambda^6}{t^2}$		$\frac{\Lambda^6}{t}$	...

Explicitly, in terms of the generating functions, the basis elements of $\mathfrak{kas}^L$ are given by the following formulas, where $\Theta = \xi_1\xi_2\xi_3\eta_1\eta_2\eta_3$, $\hat{\eta}_i = \xi_i$ and $\hat{\xi}_i = \eta_i$. Let $\tilde{T}^{ij}$ $(i = 1, \dots, 6)$ be the matrix skew-symmetric with respect to the side diagonal with only (i,j)-th and (j,i)-th nonzero entries equal to ± 1; let $\tilde{G}^i = \theta_i$, where $\theta = (\xi_1, \xi_2, \xi_3, \eta_3, \eta_2, \eta_1)$. Let the $\tilde{S}$ denote the generators of one of the two irreducible components in the $\mathfrak{o}(6)$-module $\Lambda^3(\mathrm{id})$. We will later identify $\tilde{G}$ with the space of skew-symmetric 4×4 matrices and $\tilde{S}$ with the $\mathfrak{sl}(4)$-module $S^2(\mathrm{id})$ of symmetric 4×4 matrices, namely, $\tilde{S}^{\pm\varepsilon_i}$ for $i = 1, 2, 3$ will be the symmetric off-diagonal matrices; $\tilde{S}^{2,0,0}$, $\tilde{S}^{-2,2,0}$, $\tilde{S}^{0,-2,2}$ and $\tilde{S}^{0,0,-2}$ the diagonal matrix units (the superscripts of $\tilde{S}$ are the weights of the matrix elements of the symmetric bilinear form with respect to $\mathfrak{sl}(4)$, see sec. 4.2). Set

$$\tilde{G} = \begin{pmatrix} 0 & -\xi_1 & -\xi_2 & \eta_3 \\ & 0 & \xi_3 & \eta_2 \\ & & 0 & \eta_1 \\ & & & 0 \end{pmatrix},$$

$$\tilde{S} = \begin{pmatrix} \xi_1\xi_2\xi_3 & \xi_1(\xi_2\eta_2+\xi_3\eta_3) & \xi_2(\xi_1\eta_1+\xi_3\eta_3) & \eta_3(\xi_1\eta_1-\xi_2\eta_2) \\ & \xi_1\eta_2\eta_3 & \xi_3(\xi_1\eta_1+\xi_2\eta_2) & \eta_2(\xi_1\eta_1-\xi_3\eta_3) \\ & & \xi_2\eta_1\eta_3 & \eta_1(\xi_2\eta_2-\xi_3\eta_3) \\ & & & \xi_3\eta_1\eta_2 \end{pmatrix},$$

i.e., to ξ_1 we assign the matrix $\tilde{G}^{12} = E_{21} - E_{12}$, etc., to $\xi_1(\xi_2\eta_2+\xi_3\eta_3)$ we assign the matrix $\tilde{S}^{12} = E_{21} + E_{12}$, etc. These generators, expressed via monomial generators of $\mathfrak{k}^L(1|6)$, are rather complicated. Let us pass to simpler ones using the isomorphism $\mathfrak{sl}(4) \cong \mathfrak{o}(6)$. Explicitly, this isomorphism is defined as follows:

$$\begin{pmatrix} & \xi_2\eta_3 & \xi_1\eta_3 & \xi_1\xi_2 \\ \xi_3\eta_2 & & \xi_1\eta_2 & -\xi_1\xi_3 \\ -\xi_3\eta_1 & \xi_2\eta_1 & & \xi_2\xi_3 \\ \eta_1\eta_2 & \eta_1\eta_3 & \eta_2\eta_3 & \end{pmatrix}, \quad \begin{array}{c} H_1 = -(\xi_2\eta_2 - \xi_3\eta_3), \\ H_2 = -(\xi_1\eta_1 - \xi_2\eta_2), \\ H_3 = (\xi_2\eta_2 + \xi_3\eta_3), \end{array}$$

i.e., to $\xi_2\eta_3$ we assign the matrix E_{12}, to $\xi_2\eta_2 - \xi_3\eta_3$ we assign the matrix $-(E_{11} - E_{22})$, etc.

In terms of $\tilde{T}$, $\tilde{G}$ (the sign $+$ corresponds to $\mathfrak{kas}^\xi$ and $-$ to $\mathfrak{kas}^\eta$) and $\tilde{S}$ the generators of $\mathfrak{kas}$ $(n \geq 0)$ and $\mathfrak{kas}^L$ $(n \in \mathbb{Z})$ are as folows:

the element	its generating function
$L(2n-2)$	$t^n \pm n(n-1)(n-2)t^{n-3}\Theta$
$\tilde{G}^i(2n-1)$	$t^n\theta_i \pm n(n-1)t^{n-2}\dfrac{\partial\Theta}{\partial\hat{\theta}_i}$
$\tilde{T}^{ij}(2n)$	$t^n\theta_i\theta_j \pm nt^{n-1}\dfrac{\partial}{\partial\hat{\theta}_i}\dfrac{\partial}{\partial\hat{\theta}_j}\Theta,$
$\tilde{S}^{\varepsilon_i}(2n+1)$	$t^n\xi_i(\xi_j\eta_j + \xi_k\eta_k)$
$\tilde{S}^{-\varepsilon_i}(2n+1)$	$t^n\eta_i(\xi_j\eta_j - \xi_k\eta_k)$
$\tilde{S}^{2,0,0}(2n+1)$	$t^n\xi_1\xi_2\xi_3$
$\tilde{S}^{-2,2,0}(2n+1)$	$t^n\xi_1\eta_2\eta_3$
$\tilde{S}^{0,-2,2}(2n+1)$	$t^n\xi_2\eta_1\eta_3$
$\tilde{S}^{0,0,-2}(2n+1)$	$t^n\xi_3\eta_1\eta_2$

where $\tilde{S}$ are the above symmetric matrices and where the skew-symmetric matrices $\tilde{G}^i$ are defined as $\tilde{G}^{\varepsilon_i}$, where ε_i and $-\varepsilon_i$ is the weight of ξ_i and η_i, respectively, with respect to $(H_1, H_2, H_3) \in \mathfrak{sl}(4)$.

4.2. The multiplication table in $\mathfrak{kas}^L$. In terms of $\mathfrak{sl}(4)$-modules we get a more compact expression of the elements of $\mathfrak{kas}^L$. Let ad be the adjoint module, S the symmetric square of the identity 4-dimensional module id and $G = \Lambda^2(\mathrm{id}^*)$; let $\mathbb{C} \cdot 1$ denote the trivial module. Then the basis elements of $\mathfrak{kas}^L$ with their degrees with respect to to K_t is given by the following table in which u is a new indeterminate of degree 2:

degree	-2	-1	0	1	2	...
space	$\mathbb{C}\cdot u^{-1}$, $\mathrm{ad}\cdot u^{-1}$	$S\cdot u^{-1}$, $G\cdot u^{-1}$	$\mathbb{C}\cdot 1$, ad	S, G	$\mathbb{C}\cdot u$, $\mathrm{ad}\cdot u$	...

Though it is impossible to embed $\mathfrak{witt}\supsetplus\mathfrak{gl}(4)^{(1)}$ into $\mathfrak{kas}^L$ (only $\mathfrak{witt}\supsetplus\mathfrak{sl}(4)^{(1)}$ can be embedded into $\mathfrak{kas}^L$), it is convenient to express the brackets in $\mathfrak{kas}^L$ in terms of the matrix units of $\mathfrak{gl}(4)^{(1)}=\mathfrak{gl}(4)\otimes\mathbb{C}[u^{-1},u]$; we will denote these units by $T^i_j(a)$; we further set $H_1(a)=T^1_1(a)-T^2_2(a)$, $H_2(a)=T^2_2(a)-T^2_2(a)$ and $H_3(a)=T^2_2(a)-T^3_3(a)$. Clearly, the right hand side in the last line of the following multiplication table can be expressed via the $H_i(a)$. We denote the basis elements of the trivial $\mathfrak{sl}(4)$-module of degree a by $L(a)$ and norm them so that they commute as the usual basis elements of $\mathfrak{witt}$.

The multiplication table in $\mathfrak{kas}^L$ is given by the following table:

$$\begin{array}{rcl}
[L(a),\ L(b)] &=& (b-a)L(a+b),\\
[L(a),\ T^i_j(b)] &=& bT^i_j(a+b),\\
[L(a),\ S^{ij}(b)] &=& (b+\frac{1}{2}a)S^{ij}(a+b),\\
[L(a),\ G_{ij}(b)] &=& (b-\frac{1}{2}a)G_{ij}(a+b),\\
[T^i_j(a),\ T^k_l(b)] &=& \delta^k_jT^i_l(a+b)-\delta^i_lT^k_j(a+b),\\
[T^i_j(a),\ S^{kl}(b)] &=& \delta^k_jS^{il}(a+b)+\delta^l_jS^{ik}(a+b),\\
[T^i_j(a),\ G_{kl}(b)] &=& \delta^i_kG_{lj}(a+b)+\delta^i_lG_{jk}(a+b)+a\sigma(j,k,l,m)S^{im}(a+b),\\
[S^{ij}(a),\ S^{kl}(b)] &=& 0,\\
[S^{ij}(a),\ G_{kl}(b)] &=& 2\left(\delta^i_kT^j_l(a+b)-\delta^i_lT^j_k(a+b)+\delta^j_kT^i_l(a+b)-\delta^j_lT^i_k(a+b)\right),
\end{array}$$

$$\begin{aligned}
[G_{ij}(a),\ G_{kl}(b)] = 2(b-a)\big(&\delta_{j,k}\sigma(i,j,l,m)T^m_j(a+b)+\delta_{i,k}\sigma(i,j,l,m)T^m_i(a+b)\\
&+\delta_{j,l}\sigma(i,k,l,m)T^m_j(a+b)+\delta_{i,l}\sigma(j,i,k,m)T^m_i(a+b)\big)\\
&+\sigma(i,j,k,l)\big(-4L(a+b)+(b-a)(T^i_i(a+b)\\
&+T^j_j(a+b)-T^k_k(a+b)-T^l_l(a+b))\big),
\end{aligned}$$

where $\sigma(j,i,k,m)$ is the sign of the permutation (j,i,k,m).

4.3. The relations between Chevalley generators in $\mathfrak{kas}^L$ in terms of $\mathfrak{sl}(4)$. Denote: $T^a_{ij}=T^j_i(a)$. For the positive Chevalley generators we take same of $\mathfrak{sl}(4)=Span(T^j_i:1\leq i,j\leq 4)$ and the lowest weight vectors S^1_{44} and G^1_{12} of S^1 and G^1, respectively. For the negative Chevalley generators we take same of $\mathfrak{sl}(4)$ and the highest weight vectors S^{-1}_{11} and G^{-1}_{34} of S^{-1} and G^{-1}, respectively. Then the defining relations, stratified by weight, are the following ones united with the usual Serre relations in $\mathfrak{sl}(4)$ (we skip them) and the relations that describe the highest (lowest) weight vectors:

$[T^0_{23},[T^0_{23},\ G^1_{12}]]=0$; $[T^0_{34},\ [T^0_{34},\ [T^0_{34},\ S^1_{44}]]]=0$;
$[[G^1_{12},\ [T^0_{23},\ G^1_{12}]],\ [T^0_{34},\ S^1_{44}]]=0$; $[S^1_{44},\ [T^0_{34},\ S^1_{44}]]=0$;
$[[G^1_{12},\ [T^0_{23},\ T^0_{34}]],\ [[T^0_{23},\ G^1_{12}],\ [T^0_{34},\ S^1_{44}]]]=0$;

$$[[G^{-1}_{34},\ [T^0_{23},\ T^0_{34}]],\ [[T^0_{23},\ G^{-1}_{34}],\ [T^0_{34},\ S^1_{44}]]] = 0;$$
$$[L^0,\ S^{-1}_{11}] - S^{-1}_{11} = 0;\ \ [L^0,\ G^{-1}_{34}] - G^{-1}_{34} = 0;$$
$$[H^0_1,\ S^{-1}_{11}] - 2S^{-1}_{11} = 0;\ \ [H^0_1,\ G^{-1}_{34}] = 0;\ \ [H^0_2,\ S^{-1}_{11}] = 0;$$
$$[H^0_2,\ G^{-1}_{34}] - G^{-1}_{34} = 0;\ \ [H^0_3,\ S^{-1}_{11}] = 0;\ \ [H^0_3,\ G^{-1}_{34}] = 0;$$
$$[L^0,\ S^1_{44}] - S^1_{44} = 0;\ \ [L^0,\ G^1_{12}] - G^1_{12} = 0;$$
$$[H^0_1,\ S^1_{44}] = 0;\ \ [H^0_1,\ G^1_{12}] = 0;\ \ [H^0_2,\ S^1_{44}] = 0;$$
$$[H^0_2,\ G^1_{12}] + G^1_{12} = 0;\ \ [H^0_3,\ S^1_{44}] + 2S^1_{44} = 0;$$
$$[H^0_3,\ G^1_{12}] = 0;\ \ [T^0_{12},\ S^{-1}_{11}] = 0;\ \ [T^0_{12},\ G^{-1}_{34}] = 0;$$
$$[T^0_{23},\ S^{-1}_{11}] = 0;\ \ [T^0_{23},\ G^{-1}_{34}] = 0;\ \ [T^0_{34},\ S^{-1}_{11}] = 0;$$
$$[T^0_{34},\ G^{-1}_{34}] = 0;\ \ [S^1_{44},\ T^0_{21}] = 0;\ \ [S^1_{44},\ T^0_{32}] = 0;$$
$$[S^1_{44},\ T^0_{43}] = 0;\ \ [G^1_{12},\ T^0_{21}] = 0;\ \ [G^1_{12},\ T^0_{32}] = 0;$$
$$[G^1_{12},\ T^0_{43}] = 0;\ \ [T^0_{12},\ S^1_{44}] = 0;\ \ [T^0_{12},\ G^1_{12}] = 0;$$
$$[T^0_{23},\ S^1_{44}] = 0;\ \ [T^0_{34},\ G^1_{12}] = 0;\ \ [T^0_{21},\ G^{-1}_{34}] = 0;$$
$$[T^0_{32},\ S^{-1}_{11}] = 0;\ \ [T^0_{43},\ S^{-1}_{11}] = 0;\ \ [T^0_{43},\ G^{-1}_{34}] = 0;$$
$$[S^1_{44},\ S^1_{44}] = 0;\ \ [S^1_{44},\ G^1_{12}] = 0;\ \ [G^1_{12},\ G^1_{12}] = 0;$$
$$[S^{-1}_{11},\ S^{-1}_{11}] = 0;\ \ [S^{-1}_{11},\ G^{-1}_{34}] = 0;\ \ [G^{-1}_{34},\ G^{-1}_{34}] = 0;$$
$$[S^1_{44},\ S^{-1}_{11}] = 0;\ \ [S^1_{44},\ G^{-1}_{34}] + 4T^0_{43} = 0;$$
$$[G^1_{12},\ S^{-1}_{11}] - 4T^0_{12} = 0;$$
$$[G^1_{12},\ G^{-1}_{34}] + 4L^0 - 2H^0_1 - 4H^0_2 - 2H^0_3 = 0.$$

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