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BOUNDARY VALUE CONJUGATION PROBLEMS FOR ELLIPTIC EQUATIONS IN VARIABLE DOMAINS

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Abstract. We study the behaviour of the solutions of boundary value conjugation problems for high order elliptic equations in variable domains (Ω_t, G'_t) $(0 < t \leq 1)$ which depend smoothly on a parameter t in Krein's sense. Considering the domain (Ω_0, G'_0) as the limit of domains (Ω_t, G'_t) when t tends 0, we prove the existence and the uniqueness of the solution of the boundary value conjugation problem in (Ω_0, G'_0).

1. Introduction

Let G_0 be a bounded domain in the space $\mathbb{R}^n$ with sufficiently smooth boundary Γ_0. Let Ω_0 be a domain contained in G_0 with sufficiently smooth boundary γ_0 such that

$$\overline{\Omega}_0 \subset G_0.$$

We will study elliptic differential operators of order $2m$ with smooth coefficients $L_1(x, D)$ in Ω_0 and $L_2(x, D)$ in G_0 and systems of linear differential expressions with smooth coefficients in G_0 $\{B_i^1(x, D)\}$, $\{B_i^2(x, D)\}$ $(i = 1, 2, \dots, 2m)$ of order $m_i \leq 2m - 1$, and $\{B_j^3(x, D)\}$ $(j = 1, 2, \dots, m)$ of order $m_j^3 \leq 2m - 1$.

Consider in the domain G_0 a family of domains $\{G_t\}$ whose boundaries $\{\Gamma_t\}$ depend on a parameter $t \in [0, 1]$ and in the domain Ω_0 a family of domains $\{\Omega_t\}$ with boundaries $\{\gamma_t\}$ depending on the parameter t. In the sequel we suppose that the families $\{\Gamma_t\}$ and $\{\gamma_t\}$ depend smoothly on the parameter $t \in [0, 1]$ in the Krein's sense (see [2] and [3]). Moreover, as t tends to 0,

$$G_t \to G_0, \quad \Omega_t \to \Omega_0. \tag{1.1}$$

Therefore, when t is sufficiently small i.e. $t \in [0, T]$ for some T, $0 < T < 1$, we have

$$\begin{gathered} \Omega_t \subset \Omega_0 \subset G_t, \\ \Omega_{t+\Delta t} \subset \Omega_t, \quad G_{t+\Delta t} \subset G_t \quad (0 < t < t + \Delta t < T). \end{gathered} \tag{1.2}$$

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Let ω_1, ω_2 be two domains in $\mathbb{R}^n$. Let

$$C^\infty(\omega_1,\omega_2) = \big\{v=(v_1,v_2) : v_1 \in C^\infty(\omega_1), v_2 \in C^\infty(\omega_2)\big\},$$
$$H^s(\omega_1,\omega_2) = \big\{v=(v_1,v_2) : v_1 \in H^s(\omega_1), v_2 \in H^s(\omega_2)\big\}, \quad s \geq 0,$$
$$\|v\|^2_{H^s(\omega_1,\omega_2)} = \|v\|^2_{H^s(\omega_1)} + \|v_2\|^2_{H^s(\omega_2)}.$$

Set

$$G'_t = G_t \setminus \Omega_t, \quad \Delta G'_t = G'_t \setminus G'_{t+\Delta t}, \quad \Delta\Omega_t = \Omega_t \setminus \Omega_{t+\Delta_t},$$

where $t \in (0,T]$, $\Delta t > 0$. Applying the results obtained in [1], [2] and [3] we see that for all $\phi(x) \in C^\infty(\Delta\Omega_t, \Delta G'_t)$ which is equal to zero together with all its partial derivatives on the boundary (γ_t, Γ_t) or on $(\gamma_{t+\Delta t}, \Gamma_{t+\Delta t})$, the following estimate holds

$$\|\phi\|_{H^{s-1}(\Delta\Omega_t,\Delta G'_t)} \leq C|\Delta t|\,\|\phi\|_{H^s(\Delta\Omega_t,\Delta G'_t)}, \tag{1.3}$$

where $s \geq 1$ and C is a constant.

Let $u(x) \in H^s(\Omega_t, G'_t)$, where $s \geq 0$ and $t \in [0,1]$. Then $u(x)$ can be extended to a function $u_t(x) = R_t u(x) \in H^s(\Omega_0, G_0)$ by an operator of extension R_t. The operators of extension R_t can be chosen as linear operators from $H^s(\Omega_t, G'_t)$ to $H^s(\Omega_0, G_0)$, uniformly bounded in the norm for all $t \in [0,1]$ and for $0 \leq s \leq N$, where N is a sufficiently large natural number such that the following estimate holds

$$\|R_t S_t u - R_\tau S_\tau u\|_{H^{s-1}(\Omega_0,G_0)} \leq C|t-\tau|\,\|u\|_{H^s(\Omega_0,G_0)}, \tag{1.4}$$

for $0 \leq s \leq N$ and all $u \in H^s(\Omega_0, G_0)$. Here S_t are operators of restriction on $H^s(\Omega_t, G'_t)$ (see [1] and [2]).

Let $(x_0; n) = \big(x_0^1, x_0^2, \ldots, x_0^{n-1}; n\big)$ be a system of local coordinates in a neighbourhood of the boundary Γ_t (respectively γ_t) for $t \in [0,1]$. We assume that a boundary $\Gamma_{t+\Delta t}$ (respectively $\gamma_{t+\Delta t}$) with $|\Delta t|$ sufficiently small is defined by the equation $n = \chi(x_0; t, \Delta t)$, $|\chi| \leq C\cdot|\Delta t|$, where $x_0 = \big(x_0^1, x_0^2, \ldots, x_0^{n-1}\big)$ is the local coordinate on Γ_t (respectively γ_t) (see [1], [2] and [3]). Let $A_{\Delta t}$ be an operator defined on $C^\infty(G_0)$ (respectively $C^\infty(\Omega_0)$) by the formula

$$(A_{\Delta t}\phi)(x_0; n) = \phi\big(x_0; \chi(x_0; t, \Delta t)\big) - \phi(x_0, 0). \tag{1.5}$$

Then $A_{\Delta t}$ can be extended to a continuous operator from $H^{s-\frac{1}{2}+\alpha+1}(G_0)$ (respectively $H^{s-\frac{1}{2}+\alpha+1}(\Omega_0)$) to $H^{s-1+\alpha}(\Gamma_t)$ (respectively $H^{s-1+\alpha}(\gamma_t)$) for all $s \geq 1$, $0 \leq \alpha < 1$, and the following inequality holds

$$\|A_{\Delta t}\phi\|_{H^{s-1+\alpha}(\Gamma_t)} \leq C\cdot|\Delta t|\,\|\phi\|_{H^{s-\frac{1}{2}+\alpha+1}(G_0)}, \tag{1.6}$$

or, respectively,

$$\|A_{\Delta t}\phi\|_{H^{s-1+\alpha}(\gamma_t)} \leq C|\Delta t|\,\|\phi\|_{H^{s-\frac{1}{2}+\alpha+1}(\Omega_0)}.$$

Our aim in this paper is to study the boundary value conjugation problem in the domain (Ω_0, G'_0). In Section 2 we consider the behaviour of the solutions $u(t,x)$ of boundary value conjugation problems in variable domains (Ω_t, G'_t), $0 <$

$t < 1$, and obtain the asymptotic estimate for the solutions $u(t,x)$ when $t \to 0$. In Section 3, we prove the existence and the uniqueness of the solution $u_0(x)$ of the boundary value conjugation problem in the domain (Ω_0, G_0') as the limit (in some sense) when $t \to 0$ of the solutions $u(t,x)$.

2. Conjugation problems in variable domains

For $t \in [0,T]$ we consider the following boundary value conjugation problem

$$\begin{aligned} & L_j(x,D)u^1(x) = f_1(x) \quad \text{in } \Omega_t, \\ (2.1) \qquad & L_2(x,D)u^2(x) = f_2(x) \quad \text{in } G_t' = G_t \setminus \Omega_t, \\ & [B_i(x,D)u(x)] = B_i^1(x,D)u^1(x) + B_i^2(x,D)u^2(x) \\ (2.2) \qquad & \qquad = g_j(x) \quad \text{on } \gamma_t \quad (i = 1,2,\dots,2m), \\ & B_j^3(x,D)u^2(x) = h_j(x) \quad \text{on } \Gamma_t \quad (j = 1,2,\dots,m), \end{aligned}$$

where $f_1(x)$ is a function defined in Ω_0, $f_2(x)$ is a function defined in G_0 and $g_i(x)$, $h_j(x)$ $(i = 1,2,\dots,2m;\ j = 1,2,\dots,m)$ are functions defined in $\overline{G}_0$. Let

$$u(x) = \big(u^1(x), u^2(x)\big) \quad \text{and} \quad Lu = \big(L_1(x,D)u^1, L_2(x,D)u^2\big).$$

We now state the essential asumptions for our latter proofs.

In the domain (Ω_t, G_t'), $t \in (0,T]$, if the Sapiro-Lopatinsky condition (the coercive condition) for the problem (2.1)-(2.2) is satisfied, then the operator of the problem

$$P_s : u(x) \mapsto P_s u = \big\{Lu, \big[B_i u\big]_{\gamma_t}, B_j^3 u^2|_{\Gamma_t}\big\}, \quad (i = 1,2,\dots,2m,\ j = 1,2,\dots,m)$$

from $H^{2m+s}(\Omega_t, G_t')$ to $H^s(\Omega_t, G_t') \times \prod_{i=1}^{2m} H^{2m+s-m_i-\frac{1}{2}}(\gamma_t) \times \prod_{j=1}^{m} H^{2m+s-m_j^2-\frac{1}{2}}(\Gamma_t)$ is Noether in the appropriate spaces for all $s \geq 0$. In addition, the following a priori estimate holds

$$(2.3) \qquad \big\|u\big\|_{H^{2m+s}(\Omega_t,G_t')} \leq$$
$$C(t)\Big\{\big\|Lu\big|_{H^s(\Omega_t,G_t')} + \sum_{i=1}^{2m} \big\|[B_i u]\big\|_{H^{2m+s-m_i-\frac{1}{2}}(\gamma_t)} + \sum_{j=1}^{m} \big\|B_j^3 u\big\|_{H^{2m+s-m_j^3-\frac{1}{2}}(\Gamma_t)}\Big\},$$

for all $s \geq 0$ and all functions $u(x) \in H^{2m+s}(\Omega_t, G_t')$, where $C(t)$ is a function of $t \in [0,T]$. So it is easily seen that under this assumption the problem (2.1)-(2.2) has a unique solution. In the case where $C(t)$ is a constant the problem (2.1)-(2.2) has been investigated by L. Ivanov [1]. The aim of this article is to study the behaviour of the solutions of boundary value conjugation problems in the variable domains (Ω_t, G_t'), $0 < t \leq T$, under the assumption that the function $C(t)$ in the a priori estimate (2.3) is unbounded as $t \to 0$ so that the asymptotic estimate

$$(2.4) \qquad C(t) = 0(t^{-\alpha}),$$

holds for some $0 \le \alpha < 1$. It is necessary to remark that the behaviour of $C(t)$ as $t \to 0$ depends completely on the behaviour of coefficients of the operators $L_1(x,D)$, $L_2(x,D)$ and the expressions $B_i^1(x,D)$, $B_i^2(x,D)$, $B_j^3(x,D)$ $(i = 1,2,\dots,2m; j = 1,2,\dots,m)$ in a neighbourhood of boundary (γ_0, Γ_0). Moreover, the boundary value conjugation problem in (Ω_0, G_0') (for $t = 0$) is considered as the limit (in some meaning) of the problem (2.1)-(2.2) in (Ω_t, G_t') when $t \to 0$. Therefore the problem in (Ω_0, G_0') can be called the boundary value conjugation problem in the limit domain. Furthermore, the results obtained in this paper may be seen as an extension of the ones obtained in [4], [5] and [6] on boundary value elliptic problems to the boundary value conjugation problem in variable domains.

Let $f_1(x) \in C^\infty(\overline{\Omega}_0)$, $f_2(x) \in C^\infty(\overline{G}_0)$. Then $f(x) \in C^\infty(\overline{\Omega}_0, \overline{G}_0)$ and $g_i(x) \in C^\infty(\overline{\Omega}_0)$, $h_j(x) \in C^\infty(\overline{G}_0)$ $(i = 1,2,\dots,2m;\ j = 1,2,\dots,m)$. We define $u(t,x)$ to be the unique solution of problem (2.1)-(2.2), where $t \in (0,T]$. Then $u(t,x) \in C^\infty(\Omega_t, G_t')$. Now put

$$u_t(t,x) = R_t u(t,x) = \big(R_t u^1(t,x), R_t u^2(t,x)\big),$$

where R_t is the operator of extension defined in the introduction. Then $u_t(t,x) \in H^{2m+s}(\Omega_0, G_0)$, $2m + s \le N$. Let $\Delta u_t = u_{t+\Delta t} - u_t$, $(\Delta t > 0)$. It is easily seen that

$$(2.5)\qquad L_1 \Delta u_t^1 = \begin{cases} 0 & \text{if } x \in \overline{\Omega_t \cap \Omega_{t+\Delta t}} \\ L_1 u_{t+\Delta t}^1 - f_1 & \text{if } x \in \Delta\Omega_t \end{cases}$$

$$L_2 \Delta u_t^2 = \begin{cases} 0 & \text{if } x \in \overline{G_t' \cap G_{t+\Delta t}'} \\ L_2 u_{t+\Delta t}^2 - f_2 & \text{if } x \in \Delta G_t' \end{cases}$$

Then $L\Delta u_t = \big(L_1 \Delta u_t^1, L_2 \Delta u_t^2\big)$.

Using a similar approach as for Proposition 1 in [6], we get the following proposition.

Proposition 2.1. *For $s \ge 1$ and $2m + s \le N$ the following estimate holds*

$$(2.6)\qquad \Big\| \frac{\Delta u_t}{\Delta t} \Big\|_{H^{2m+s-1}(\Omega_t, G_t')} \le$$

$$C(t)C(t+\Delta t)\Big\{ \|f\|_{H^s(\Omega_0,G_0)} + \sum_{i=1}^{2m} \|g_i\|_{H^{2m+s-m_i}(\Omega_0)} + \sum_{j=1}^{m} \|h_j\|_{H^{2m+s-m_j^3}(G_0)} \Big\},$$

where $C(t) = 0(t^{-\alpha})$ and $C(t+\Delta t) = 0\big((t+\Delta t)^{-\alpha}\big)$.

Proof. Applying the a priori estimate (2.3) for $t \in (0,T]$ we have

$$\left\|\frac{\Delta u_t}{\Delta t}\right\|_{H^{2m+s-1}(\Omega_t,G'_t)} \le C(t)\Big\{\left\|L\frac{\Delta u_t}{\Delta t}\right\|_{H^{s-1}(\Omega_t,G'_t)} + \sum_{i=1}^{2m}\left\|\left[B_j\frac{\Delta u_t}{\Delta t}\right]\right\|_{H^{2m+s-1-m_i-\frac12}(\gamma_t)} + \sum_{j=1}^{m}\left\|B_j^3\frac{\Delta u_t}{\Delta t}\right\|_{H^{2m+s-1-m_j^3-\frac12}(\Gamma_t)}\Big\}. \tag{2.7}$$

We observe that the expression $L\dfrac{\Delta u_t}{\Delta t} = \dfrac{1}{\Delta t}L\Delta u_t$ defined by (2.5) is equal to zero together with all its partial derivatives on the boundary $(\gamma_{t+\Delta t}, \Gamma_{t+\Delta t})$. Applying the estimate (1.3), we have

$$\begin{aligned}\left\|L\frac{\Delta u_t}{\Delta t}\right\|_{H^{s-1}(\Omega_t,G'_t)} &= \frac{1}{\Delta t}\left\|L\Delta u_t\right\|_{H^{s-1}(\Delta\Omega_t,\Delta G'_t)}\\ &= \frac{1}{|\Delta t|}\left\|Lu_{t+\Delta t}-f\right\|_{H^{s-1}(\Delta\Omega_t,\Delta G'_t)}\\ &\le C\left\|Lu_{t+\Delta t}-f\right\|_{H^{s}(\Delta\Omega_t,\Delta G'_t)}\\ &\le C\Big\{\left\|u_{t+\Delta t}\right\|_{H^{2m+s}(\Omega_t,G'_t)} + \left\|f\right\|_{H^{s}(\Omega_t,G'_t)}\Big\}.\end{aligned} \tag{2.8}$$

Moreover, for the expressions $\left[B_j\dfrac{\Delta u_t}{\Delta t}\right]$ and $B_j^3\dfrac{\Delta u_t}{\Delta t}$ we have

$$\begin{aligned}\left[B_i\frac{\Delta u_t}{\Delta t}\right] &= \frac{1}{\Delta t}\big([B_iu_{t+\Delta t}]-g_i\big)\big|_{\gamma_t} = \frac{A_{\Delta t}g_i - A_{\Delta t}[B_iu_{t+\Delta t}]}{\Delta t},\\ B_j^3\frac{\Delta u_t^2}{\Delta t} &= \frac{1}{\Delta t}\big(B_j^3u_{t+\Delta t}^2 - h_j\big)\big|_{\Gamma_t} = \frac{A_{\Delta t}h_j - A_{\Delta t}[B_j^3u_{t+\Delta t}^2]}{\Delta t}.\end{aligned}$$

Then for $s\ge 1$, $2m+s\le N$, $2m+s-m_i-\dfrac12-1>0$, $2m+s-m_j^3-\dfrac12-1>0$, applying inequality (1.6) we get that

$$\begin{aligned}&\left\|\left[B_t\frac{\Delta u_t}{\Delta t}\right]\right\|_{H^{2m+s-m_i-1-\frac12}(\gamma_t)}\\ &\le \frac{1}{|\Delta t|}\Big\{\left\|A_{\Delta t}[B_ju_{t+\Delta t}]\right\|_{H^{2m+s-1-m_i-\frac12}(\gamma_t)} + \left\|A_{\Delta t}g_i\right\|_{H^{2m+s-1-m_i-\frac12}(\gamma_t)}\Big\}\\ &\le C\Big\{\left\|[B_iu_{t+\Delta t}]\right\|_{H^{2m+s-m_i}(\Omega_0,G_0)} + \left\|g_i\right\|_{H^{2m+s-m_i}(\Omega_0)}\Big\}\\ &\le C\Big\{\left\|u_{t+\Delta t}\right\|_{H^{2m+s}(\Omega_0,G_0)} + \left\|g_i\right\|_{H^{2m+s-m_i}(\Omega_0)}\Big\}.\end{aligned} \tag{2.9}$$

Similarly we have

$$\begin{aligned}\left\|B_j^3\frac{\Delta u_t^2}{\Delta t}\right\|_{H^{2m+s-m_j^3-1-\frac12}(\Gamma_t)} &\le C\Big\{\left\|u_{t+\Delta t}^2\right\|_{H^{2m+s}(G_0)} + \left\|h_j\right\|_{H^{2m+s-m_j^3}(G_0)}\Big\}\\ &\le \Big\{\left\|u_{t+\Delta t}\right\|_{H^{2m+s}(\Omega_0,G_0)} + \left\|h_j\right\|_{H^{2m+s-m_j^3}(G_0)}\Big\}.\end{aligned} \tag{2.10}$$

After obtaining (2.8), (2.9) and (2.10) we return to (2.7) and get

$$\Big\|\frac{\Delta u_t}{\Delta t}\Big\|_{H^{2m+s-1}(\Omega_t,G'_t)} \le C(t)\Big\{\big\|u_{t+\Delta t}\big\|_{H^{2m+s}(\Omega_0,G_0)} + \sum_{i=1}^{2m}\big\|g_i\big\|_{H^{2m+s-m_i}(\Omega_0)} + \sum_{j=1}^{m}\big\|h_j\big\|_{H^{2m+s-m_j^3}(G_0)} + \big\|f\big\|_{H^s(\Omega_t,G'_t)}\Big\}.$$

Using the uniform boundeness of operators R_t and applying the a priori estimate (2.3) to $\big\|u_{t+\Delta t}\big\|_{H^{2m+s}(\Omega_0,G_0)}$ we obtain (2.6). □

The following corollary allows us to estimate $\dfrac{\Delta u_t}{\Delta t}$ in $H^{2m+s-1}(\Omega_0, G_0)$.

Corollary 2.1. *For $s \ge 1$ the following estimate holds*

$$\Big\|\frac{\Delta u_t}{\Delta t}\Big\|_{H^{2m+s-1}(\Omega_0,G_0)} \le C(t)\cdot C(t+\Delta t)\Big\{\big\|f\big\|_{H^s(\Omega_0,G_0)} + \sum_{i=1}^{2m}\big\|g_i\big\|_{H^{2m+s-m_i}(\Omega_0)} + \sum_{j=1}^{m}\big\|h_j\big\|_{H^{2m+s-m_j^3}(G_0)}\Big\}, \tag{2.11}$$

where $C(t) = 0(t^{-\alpha})$ and $C(t+\Delta t) = 0\big((t+\Delta t)^{-\alpha}\big)$.

Proof. Putting $P_t = R_tS_t$, where R_t is the operator of extension and S_t is the operator of restriction introduced in the introduction, we have

$$\begin{aligned}
&\Big\|\frac{\Delta u_t}{\Delta t}\Big\|_{H^{2m+s-1}(\Omega_0,G_0)} \\
&\le \Big\|P_t\frac{\Delta u_t}{\Delta t}\Big\|_{H^{2m+s-1}(\Omega_0,G_0)} + \Big\|(I-P_t)\frac{\Delta u_t}{\Delta t}\Big\|_{H^{2m+s-1}(\Omega_0.G_0)} \\
&= \Big\|P_t\frac{\Delta u_t}{\Delta t}\Big\|_{H^{2m+s-1}(\Omega_0,G_0)} + \Big\|\frac{1}{\Delta t}(P_{t+\Delta t}-P_t)u_{t+\Delta t}\Big\|_{H^{2m+s-1}(\Omega_0,G_0)} \\
&\le C\Big\|\frac{\Delta u_t}{\Delta t}\Big\|_{H^{2m+s-1}(\Omega_t,G'_t)} + \frac{1}{|\Delta t|}\big\|(P_{t+\Delta t}-P_t)u_{t+\Delta t}\big\|_{H^{2m+s-1}(\Omega_0,G_0)}.
\end{aligned}$$

Applying the inequalities (1.4), (2.6) and (2.3) successively we obtain

$$\begin{aligned}
\Big\|\frac{\Delta u_t}{\Delta t}\Big\|_{H^{2m+s-1}(\Omega_0,G_0)} &\le C\Big\|\frac{\Delta u_t}{\Delta t}\Big\|_{H^{2m+s-1}(\Omega_t,G'_t)} + \big\|u_{t+\Delta t}\big\|_{H^{2m+s}(\Omega_0,G_0)} \\
&\le C\big\{C(t)\cdot C(t+\Delta t) + C(t+\Delta t)\big\}\cdot\Big\{\big\|f\big\|_{H^{2m+s}(\Omega_0,G_0)} \\
&\quad + \sum_{i=1}^{2m}\big\|g_i\big\|_{H^{2m+s-m_i}(\Omega_0)} + \sum_{j=1}^{m}\big\|h_j\big\|_{H^{2m+s-m_j^3}(G_0)}\Big\}.
\end{aligned}$$

This implies (2.11). □

Proposition 2.2. *For $s \geq 1$ the solution $u(t,x)$ of the problem* (2.1)-(2.2) *satisfies the following estimate*

$$\|u_t\|_{H^{2m+s-1}(\Omega_t,G'_t)} \leq C_1(t)\Big\{\|f\|_{H^s(\Omega_0,G_0)} + \sum_{i=1}^{2m}\|g_i\|_{H^{2m+s-m_i}(\Omega_0)} + \sum_{j=1}^{m}\|h_j\|_{H^{2m+s-m_j^3}(G_0)}\Big\}, \tag{2.12}$$

where

$$C_1(t) = \begin{cases} 0(t^{1-2\alpha}) & \text{if } 1-2\alpha < 0 \\ 0\Big(1+\beta(\alpha)ln\dfrac{1}{t}\Big) & \text{if } 1-2\alpha \geq 0, \end{cases} \tag{2.13}$$

and $\beta(\alpha) = 0$ if $\alpha \neq \frac{1}{2}$ and $\beta\Big(\frac{1}{2}\Big) = 1$.

Proof. For $t \in (0,T]$ we divide the interval $[t,T]$ into k equal parts by the points $t = t_0, t_1, t_2, \ldots, t_k = T$, and put $\Delta t = t_\ell - t_{\ell-1}$, $\ell = 1,2,\ldots,k$. Then $u_t = \sum\limits_{\ell=1}^{k}\big(u_{t_{\ell-1}} - u_{t_\ell}\big) + u_T$. Applying the estimate (2.11) to $\big(u_{t_\ell} - u_{t_{\ell-1}}\big)/\Delta t$ and the estimate (2.3) to u_T we obtain

$$\begin{aligned}
\|u_t\|_{H^{2m+s-1}(\Omega_t,G'_t)} &\leq \sum_{\ell=1}^{k}\big\|u_{t_{\ell-1}} - u_{t_\ell}\big\|_{H^{2m+s-1}(\Omega_t,G'_t)} + \|u_T\|_{H^{2m+s-1}(\Omega_t,G'_t)} \\
&\leq \sum_{\ell=1}^{k}\|u_{t_{\ell-1}} - u_{t_\ell}\|_{H^{2m+s-1}(\Omega_0,G_0)} + \|u_T\|_{H^{2m+s-1}(\Omega_0,G_0)} \\
&\leq \Big(\sum_{\ell=1}^{k}C(t_{\ell-1})C(t_\ell)\Delta t + C(T)\Big)\Big\{\|f\|_{H^s(\Omega_0,G_0)} \\
&\quad + \sum_{i=1}^{2m}\|g_i\|_{H^{2m+s-m_i}(\Omega_0)} + \sum_{j=1}^{m}\|h_j\|_{H^{2m+s-m_j^3}(G_0)}\Big\} \\
&\leq \Big\{\sum_{\ell=1}^{k}\frac{\Delta t}{t_{\ell-1}^\alpha \cdot t_\ell^\alpha} + 1\Big\}\Big\{\|f\|_{H^s(\Omega_0,G_0)} \\
&\quad + \sum_{i=1}^{2m}\|g_i\|_{H^{2m+s-m_i}(\Omega_0)} + \sum_{j=1}^{m}\|h_j\|_{H^{2m+s-m_j^3}(G_0)}\Big\} \\
&\leq \Big\{\sum_{\ell=1}^{k}\frac{\Delta t}{t_{\ell-1}^{2\alpha}} + 1\Big\}\Big\{\|f\|_{H^s(\Omega_0,G_0)} + \sum_{i=1}^{2m}\|g_i\|_{H^{2m+s-m_i}(\Omega_0)} \\
&\quad + \sum_{j=1}^{m}\|h_j\|_{H^{2m+s-m_j^3}(G_0)}\Big\},
\end{aligned}$$

It is easily seen that when k tends to infinity, the sum $\sum_{\ell=1}^{k} \frac{\Delta t}{t_{\ell-1}^{2\alpha}}$ tends to the integral

$$\int_t^T \frac{dx}{x^{2\alpha}} = \begin{cases} \frac{1}{1-2\alpha}\left(T^{1-2\alpha} - t^{1-2\alpha}\right) & \text{if } \alpha \neq \frac{1}{2}\,, \\ \ln\frac{T}{t} & \text{if } \alpha = \frac{1}{2}\,\cdot \end{cases}$$

Therefore, for k sufficiently large, the sum $\sum_{\ell=1}^{k} \frac{\Delta t}{2\alpha} + 1$ can not exceed

$$C\Big\{1 + t^{1-2\alpha} + \delta\Big(\frac{1}{2} - \alpha\Big)\ln\Big(\frac{1}{t}\Big)\Big\}\,,$$

where $\delta\Big(\frac{1}{2} - \alpha\Big) = 0$ if $\alpha \neq \frac{1}{2}$ and $\delta\Big(\frac{1}{2} - \alpha\Big) = 1$ if $\alpha = \frac{1}{2}$. Hence

$$C_1(t) = C\Big\{1 + t^{1-2\alpha} + \delta\Big(\frac{1}{2} - \alpha\Big)\ln\frac{1}{t}\Big\}$$

and we obtain (2.12) with $C_1(t)$ satisfying (2.13). The proof of Proposition 2.2 is complete. □

We now consider the case where $1 - 2\alpha \leq 0$. Firstly, we observe that for all $s \geq 2$, by applying the a priori estimate (2.3) to $\frac{\Delta u_t}{\Delta t}$ we have

$$\Big\|\frac{\Delta u_t}{\Delta t}\Big\|_{H^{2m+s-2}(\Omega_t, G'_t)} \leq C(t)\Big\{\Big\|L\frac{\Delta u_t}{\Delta t}\Big\|_{H^{s-2}(\Omega_t, G'_t)} + \sum_{i=1}^{2m}\Big\|\Big[B_i\frac{\Delta u_t}{\Delta t}\Big]\Big\|_{H^{2m+s-2-m_i-\frac{1}{2}}(\gamma_t)} + \sum_{j=1}^{m}\Big\|B_j^3\frac{\Delta u_t^2}{\Delta t}\Big\|_{H^{2m+s-2-m_j^3-\frac{1}{2}}(\Gamma_t)}\Big\}.$$

Using the reasoning in the proof of Proposition 2.1 we obtain

$$\Big\|L\frac{\Delta u_t}{\Delta t}\Big\|_{H^{s-2}(\Omega_t, G'_t)} \leq C\Big\{\big\|u_{t+\Delta t}\big\|_{H^{2m+s-1}(\Omega_t, G'_t)} + \big\|f\big\|_{H^{s-1}(\Omega_t, G'_t)},$$

$$\Big\|\Big[B_i\frac{\Delta u_t}{\Delta t}\Big]\Big\|_{H^{2m+s-2-m_i-\frac{1}{2}}(\gamma_t)} \leq C\Big\{\big\|u_{t+\Delta t}\big\|_{H^{2m+s-1}(\Omega_0, G_0)} + \big\|g_i\big\|_{H^{2m+s-1-m_i}(\Omega_0)}\Big\},$$

$$\Big\|B_j^3\frac{\Delta u_t^2}{\Delta t}\Big\|_{H^{2m+s-2-m_j^3-\frac{1}{2}}(\Gamma_t)} \leq C\Big\{\big\|u_{t+\Delta t}\big\|_{H^{2m+s-1}(\Omega_0, G_0)} + \big\|h_j\big\|_{H^{2m+s-m_j^3}(G_0)}\Big\}.$$

Therefore,

$$\Big\|\frac{\Delta u_t}{\Delta t}\Big\|_{H^{2m+s-2}(\Omega_t, G'_t)} \leq C(t)\Big\{\big\|u_{t+\Delta t}\big\|_{H^{2m+s-1}(\Omega_0, G_0)} + \big\|f\big\|_{H^{s-1}(\Omega_t, G_t)} + \sum_{i=1}^{2m}\big\|g_i\big\|_{H^{2m+s-1-m_i}(\Omega_0)} + \sum_{j=1}^{m}\big\|h_j\big\|_{H^{2m+s-1-m_j^3}(G_0)}\Big\}.$$

Hence, under the uniformly bounded condition of operators R_t, we can apply the estimate (2.12) to $\|u_{t+\Delta t}\|_{H^{2m+s-1}(\Omega_0,G_0)}$ to obtain

$$\Big\|\frac{\Delta u_t}{\Delta t}\Big\|_{H^{2m+s-2}(\Omega_t,G'_t)} \le C(t)C_1(t+\Delta t)\Big\{\|f\|_{H^s(\Omega_0,G_0)} + \sum_{i=1}^{2m}\|g_i\|_{H^{2m+s-m_i}(\Omega_0)} + \sum_{j=1}^{m}\|h_j\|_{H^{2m+s-m_j^3}(G_0)}\Big\}. \tag{2.14}$$

Using inequality (2.14) and applying the argument in the proof of Corollary 2.1 we obtain the following estimate in (Ω_0, G_0) for $\dfrac{\Delta u_t}{\Delta t}$ with $s \ge 2$:

$$\Big\|\frac{\Delta u_t}{\Delta t}\Big\|_{H^{2m+s-2}(\Omega_0,G_0)} \le C(t)C_1(t+\Delta t)\Big\{\|f\|_{H^s(\Omega_0,G_0)} + \sum_{i=1}^{2m}\|g_i\|_{H^{2m+s-m_i}(\Omega_0)} + \sum_{j=1}^{m}\|h_j\|_{H^{2m+s-m_j^3}(G_0)}\Big\}. \tag{2.15}$$

Thus, estimate (2.15) yields the following proposition.

Proposition 2.3. *For $s \ge 2$ and $2m+s \le N$, the solution of the problem* (2.1)-(2.2) *satisfies the following estimate*

$$\|u_t\|_{H^{2m+s-2}(\Omega_t,G'_t)} \le C_2(t)\Big\{\|f\|_{H^s(\Omega_0,G_0)} + \sum_{i=1}^{2m}\|g_j\|_{H^{2m+s-m_i}(\Omega_0)} + \sum_{j=1}^{m}\|h_j\|_{H^{2m+s-m_j^3}(G_0)}\Big\} \tag{2.16}$$

where

$$C_2(t) = \begin{cases} o\big(t^{2-3\alpha}\big) & \text{if } 2-3\alpha < 0 \\ o\big(1+\delta\big(\frac{2}{3}-\alpha\big)\, ln\frac{1}{t}\big) & \text{if } 2-3\alpha \ge 0 \end{cases} \tag{2.17}$$

with $\delta\Big(\dfrac{2}{3}-\alpha\Big) = 0$ *if* $\alpha \ne \dfrac{2}{3}$; $\delta\Big(\dfrac{2}{3}-\alpha\Big) = 1$ *if* $\alpha = \dfrac{2}{3}\cdot$

Proof. The estimate (2.16) can be proved in a similar manner as it was done for Proposition 2.1 except for one thing that the sum $\sum_{\ell=1}^{k} C(t_{\ell-1})\cdot C(t_\ell)\Delta t + C(T)$ can be replaced by $\sum_{\ell=1}^{k} C(t_{\ell-1})\cdot C_1(t_\ell)\Delta t + C(T)$ for which the following estimate holds

$$\sum_{\ell=1}^{k} C(t_{\ell-1})C_1(t_\ell)\Delta t + C(T) \le C\Big\{\sum_{\ell=1}^{k}\frac{\Delta t}{t_{\ell-1}^{\alpha}t_\ell^{2\alpha-1}}+1\Big\}$$
$$\le C\Big\{\sum_{\ell=1}^{k}\frac{\Delta t}{t_{\ell-1}^{3\alpha-1}}+1\Big\}.$$

Note that when t tends to infinity the sum $\sum_{\ell=1}^{k}\frac{\Delta t}{t_{\ell-1}^{3\alpha-1}}$ tends to the integral

$$\int_t^T \frac{dx}{x^{3\alpha-1}} = \begin{cases}\frac{1}{2-3\alpha}\big(T^{2-3\alpha}-t^{2-3\alpha}\big) & \text{if } 2-3\alpha\neq 0\\ \ln\frac{T}{t} & \text{if } 2-3\alpha=0.\end{cases}$$

Therefore, for k sufficiently large, the sum $\sum_{\ell=1}^{k} C(t_{\ell-1})\cdot C_1(t_\ell)\Delta t + C(T)$ cannot exceed $C\big\{1+t^{2-3\alpha}+\delta(2/3-\alpha)\ln(1/t)\big\}$. Now, putting

$$C_2(t) = \big\{1+t^{2-3\alpha}+\delta(2/3-\alpha)\ln(1/t)$$

we obtain the estimate (2.16) with $C_2(t)$ satisfying (2.17). The proof of Proposition 2.3 is complete. □

If $2-3\alpha$ is not positive, using the above reasoning for $s\ge 3$, $s\ge 4,\ldots$, we obtain after k steps the following estimate for $u_t(t,x)$ for $s\ge k$, $2m+s\le N$.

$$\big\|u_t\big\|_{H^{2m+s-k}(\Omega_t,G'_t)} \le C_k(t)\Big\{\big\|f\big\|_{H^s(\Omega_0,G_0)}+\sum_{i=1}^{2m}\big\|g_i\big\|_{H^{2m+s-m_i}(\Omega_0)}$$
$$+\sum_{j=1}^{m}\big\|h_j\big\|_{H^{2m+s-m_j^3}(G_0)}\Big\} \tag{2.18}$$

where

$$C_k(t) = \begin{cases}0\big(t^{k-(k+1)\alpha}\big) & \text{if } \alpha>\frac{k}{k+1},\\ 0\Big(1+\delta\big(\frac{k}{k+1}-\alpha\big)\ln\frac{1}{t}\Big) & \text{if } \alpha\le\frac{k}{k+1},\end{cases} \tag{2.19}$$

and

$$\delta\big(\frac{k}{k+1}-\alpha\big) = \begin{cases}0 & \text{if } \alpha\neq\frac{k}{k+1},\\ 1 & \text{if } \alpha=\frac{k}{k+1}.\end{cases}$$

Observe that the behaviour of the solutions of the boundary value problems (2.1)-(2.2) in the variable domains (Ω_t, G'_t) are presented by the estimate (2.18) which is essentially better than that obtained in [4].

3. Conjugation problem in limit domain

Consider the following boundary value conjugation problem

$$L_1(x,D)u^1(x) = f_1(x) \quad \text{in } \Omega_0,$$

$$L_2(x,D)u^2(x) = f_2(x) \quad \text{in } G_0' = G_0 \setminus \Omega_0, \tag{3.1}$$

$$[B_i(x,D)u(x)] = B_i^1(x,D)u^1(x) + B_i^2(x,D)u^2(x) = g_i(x) \text{ on } \gamma_0,$$

$$B_j^3(x,D)u^2(x) = h_j(x) \text{ on } \Gamma_0 \quad (i = 1,2,\dots 2m,\ j = 1,2,\dots,m). \tag{3.2}$$

In the sequel, the solution $u_0(x)$ of the problem (3.1)-(3.2) will be considered as the limit (in some sense) of the solutions $u_t(t,x)$ of problem (2.1)-(2.2) in (Ω_t, G_t') for $t \in (0,T]$ when $t \to 0$. Firstly, as $0 < \alpha < 1$, we can find the least positive integer k_0 such that.

$$0 < \alpha < \frac{k_0}{k_0+1} \cdot \tag{3.3}$$

Based on the estimates (2.18), (2.19) and (3.3) for $s \geq k_0$ we find that there exists a constant C such that the solution of the problems (2.1)-(2.2) satisfies the following condition

$$\begin{aligned} \|u_t\|_{H^{2m+s-k_0}(\Omega_t,G_t')} \leq C\Big\{\|f\|_{H^s(\Omega_0,G_0)} + \sum_{i=1}^{2m} \|g_i\|_{H^{2m+s-m_i}(\Omega_0)} \\ + \sum_{j=1}^{m} \|h_j\|_{H^{2m+s-m_j^3}(G_0)}\Big\}, \end{aligned} \tag{3.4}$$

where $2m + s \leq N$, $s \geq k_0$. Using the estimate (3.4) and the reasoning used in the proof of Proposition 2.1 and Corollary 2.1 we get the following estimate for $\Delta u_t/\Delta t$ in the domain (Ω_0, G_0):

$$\begin{aligned} \Big\|\frac{\Delta u_t}{\Delta t}\Big\|_{H^{2m+s-(k_0+1)}(\Omega_0,G_0)} \leq C\Big\{\|f\|_{H^s(\Omega_0,G_0)} + \sum_{i=1}^{2m} \|g_i\|_{H^{2m+s-m_i}(\Omega_0)} \\ + \sum_{j=1}^{m} \|h_j\|_{H^{2m+s-m_j^3}(G_0)}\Big\}, \end{aligned}$$

for all $s \geq k_0 + 1$ and $2m + s \leq N$. Therefore, for $s \geq k_0 + 1$ we have

$$\begin{aligned} \|\Delta u_t\|_{H^{2m+s-(k_0+1)}(\Omega_0,G_0)} \leq C \cdot |\Delta t| \cdot \Big\{\|f\|_{H^s(\Omega_0,G_0)} + \sum_{i=1}^{2m} \|g_i\|_{H^{2m+s-m_i}(\Omega_0)} \\ + \sum_{j=1}^{m} \|h_j\|_{H^{2m+s-m_j^3}(G_0)}\Big\}. \end{aligned} \tag{3.5}$$

The inequality (3.5) shows that the function $u_t(t,x)$ can be considered as an abstract function of value in the space $H^{2m+s-(k_0+1)}(\Omega_0, G_0)$ for $s \geq k_0 + 1$ and

$2m+s \leq N$, which is uniformly continuous in $t \in (0,T]$. Hence there exists the limit

$$\lim_{t\to 0} u_t(t,x) = u_0(x), \tag{3.6}$$

in $H^{2m+s-(k_0+1)}(\Omega_0, G_0)$.

Theorem 3.1. *Suppose that*

$$\begin{aligned} &f_1(x) \in H^s(\Omega_0), \quad f_2(x) \in H^s(G_0),\\ &g_i(x) \in H^{2m+s-m_i}(\Omega_0) \quad (i = 1,2,\dots,m),\\ &h_j(x) \in H^{2m+s-m_j^3}(G_0) \quad (j = 1,2,\dots m)\end{aligned}$$

for all $s \geq k_0+1$, *where* k_0 *is the least integer that satisfies* (3.3). *Then the function* $u_0(x)$ *defined in* (3.6) *is a solution of the boundary value conjugation problem* (3.1)-(3.2) *and the following estimate holds*

$$\begin{aligned}\|u_0\|_{H^{2m+s-(k_0+1)}(\Omega_0,G_0')} \leq C\Big\{\|f\|_{H^s(\Omega_0,G_0)} + \sum_{i=1}^{2m}\|g_i\|_{H^{2m+s-m_i}(\Omega_0)}\\ + \sum_{j=1}^{m}\|h_j\|_{H^{2m+s-m_j^3}(G_0)}\Big\}\end{aligned} \tag{3.7}$$

for all $s \geq k_0+1$.

Proof. For $t \in (0,T]$ and $s \geq k_0+1$ we have

$$\begin{aligned}&\|L_1u_0^1 - f_1\|_{H^{s-(k_0+1)}(\Omega_0)}\\ &\leq \|L_1u_0^1 - L_1u_t^1\|_{H^{s-(k_0+1)}(\Omega_0)} + \|L_1u_t^1 - f_1\|_{H^{s-(k_0+1)}(\Omega_0)}\\ &\leq C_1\|u_0^1 - u_t^1\|_{H^{2m+s-(k_0+1)}(\Omega_0)} + \|L_1u_t^1 - f_1\|_{H^{s-(k_0+1)}(\Omega_0\backslash\Omega_t)'}\end{aligned}$$

where

$$\begin{aligned}&\|L_1u_t^1 - f_1\|_{H^{s-(k_0+1)}(\Omega_0\backslash\Omega_t)}\\ &\leq \|L_1u_t^1 - L_1u_0^1\|_{H^{s-(k_0+1)}(\Omega_0\backslash\Omega_t)} + \|L_1u_0^1 - f_1\|_{H^{s-(k_0+1)}(\Omega_0\backslash\Omega_t)}\\ &\leq C_2\|u_t^1 - u_0^1\|_{H^{2m+s-(k_0+1)}(\Omega_0)} + \|L_1u_0^1 - f_1\|_{H^{s-(k_0+1)}(\Omega_0\backslash\Omega_t)}.\end{aligned}$$

Then

$$\begin{aligned}\|L_1u_0^1 - f_1\|_{H^{s-(k_0+1)}(\Omega_0)} \leq C\|u_t^1 - u_0^1\|_{H^{2m+s-(k_0+1)}(\Omega_0)}\\ + \|L_1u_0^1 - f_1\|_{H^{s-(k_0+1)}(\Omega_0\backslash\Omega_t)}.\end{aligned}$$

Applying a similar argument to $L_2u_0^2 - f_2$ we have

$$\begin{aligned}\|L_2u_0^2 - f_2\|_{H^{s-(k_0+1)}(G_0')} \leq C\|u_t^2 - u_0^2\|_{H^{2m+s-(k_0+1)}(G_0')}\\ + \|L_2u_0^2 - f_2\|_{H^{s-(k_0+2)}(G_0'\backslash G_t')},\end{aligned}$$

where $s \geq k_0 + 1$. Finally we obtain the estimate

$$\begin{aligned}\|Lu_0 - f\|_{H^{s-(k_0+1)}(\Omega_0, G_0')} \leq C\|u_t - u_0\|_{H^{2m+s-(k_0+1)}(\Omega_0, G_0)} \\ + \|Lu_0 - f\|_{H^{s-(k_0+1)}(\Omega_0 \backslash \Omega_t, G_0' \backslash G_t')}\end{aligned}$$

for all $s \geq k_0 + 1$ and $2m + s \leq N$. If t tends to 0, then the first term in the right hand side of the above expression tends to 0 by (3.6) and the second term also tends to 0 because $Lu_0 - f \in H^{s-(k_0+1)}(\Omega_0, G_0')$. Therefore $\|Lu_0 - f\|_{H^{s-(k_0+1)}(\Omega_0, G_0')} = 0$, i.e. $Lu_0 = f$ in (Ω_0, G_0'). We now verify that $u_0(x)$ satisfies the boundary condition (3.2). First of all, the following estimate holds for $s \geq k_0 + 1$,

$$\begin{aligned}&\|[B_i u_0] - g_i\|_{H^{2m+s-m_i-(k_0+1)-\frac{1}{2}}(\gamma_0)} \\ &\leq \|[B_i u_0] - g_i\|_{H^{2m+s-m_i-(k_0+1)-\frac{1}{2}}(\gamma_t)} + \|A_t([B_i u_0] - g_i)\|_{H^{2m+s-m_i-(k_0+1)-\frac{1}{2}}(\gamma_t)} \\ &\leq \|[B_i u_0] - [B_i \dot{u}_t]\|_{H^{2m+s-m_i-(k_0+1)-\frac{1}{2}}(\gamma_t)} + \|A_t([B_i u_0] - g_i)\|_{H^{2m+s-m_i-(k_0+1)-\frac{1}{2}}(\gamma_0)} \\ &\leq C\|u_t - u_0\|_{H^{2m+s-(k_0+1)}(\Omega_0, G_0)} + \|A_t([B_i u_0] - g_i)\|_{H^{2m+s-m_i-(k_0+1)-\frac{1}{2}}(\gamma_0)},\end{aligned}$$

where A_t is the operator $A_{\Delta t}$ defined in (1.5) for the case $\Delta t = t - 0$. By (1.6) the second term in the last sum tends to 0 as $t \to 0$, while the first term tends to 0 by (3.6). Therefore, when $t \to 0$ we have

$$\|[B_i u_0] - g_i\|_{H^{2m+s-m_i-(k_0+1)-\frac{1}{2}}(\gamma_0)} = 0, \quad (i = 1, 2, \ldots, m).$$

By a similar reasoning we also obtain

$$\|B_j^3 u_0^2 - g_j\|_{H^{2m+s-m_j^3-(k_0+1)-\frac{1}{2}}(\Gamma_0)} = 0, \quad (j = 1, 2, \ldots, m).$$

Hence $u_0(x)$ satisfies the boundary condition (3.2). Moreover, for $s \geq k_0 + 1$, using (3.4) we have

$$\begin{aligned}\|u_0\|_{H^{2m+s-(k_0+1)}(\Omega_0, G_0')} &\leq \|u_0 - u_t\|_{H^{2m+s-(k_0+1)}(\Omega_0, G_0')} + \|u_t\|_{H^{2m+s-(k_0+1)}(\Omega_0, G_0')} \\ &\leq \|u_0 - u_t\|_{H^{2m+s-(k_0+1)}(\Omega_0, G_0')} + C\Big\{\|f\|_{H^s(\Omega_0, G_0')} \\ &\quad + \sum_{i=1}^{2m} \|g_i\|_{H^{2m+s-m_i}(\Omega_0)} + \sum_{j=1}^{m} \|h_j\|_{H^{2m+s-m_j^3}(G_0)}\Big\}.\end{aligned}$$

Let t to 0. Then we obtain the estimate (3.7). Theorem 3.1 is proved. □

Remark. By value of interpolation theorem we get from (3.4) and (3.5) the following estimate for $s \geq k_0 + 1$ and $0 \leq \varepsilon \leq 1$

$$\begin{aligned}\|\Delta u_t\|_{H^{2m+s-(k_0+\varepsilon)}(\Omega_0, G_0)} \leq C|\Delta t|^{\varepsilon}\Big\{\|f\|_{H^s(\Omega_0, G_0)} + \sum_{i=1}^{2m} \|g_i\|_{H^{2m+s-m_i}(\Omega_0)} \\ + \sum_{j=1}^{m} \|h_j\|_{H^{2m+s-m_j^3}(G_0)}\Big\}.\end{aligned} \tag{3.8}$$

Therefore the solution $u_0(x)$ of boundary value conjugation problem (3.1)-(3.2) in the domain (Ω_0, G_0') belongs to $H^{2m+s-(k_0+\varepsilon)}(\Omega_0, G_0')$.

Theorem 3.2. *Under the assumptions of Theorem 3.1, the solution of problem* (3.1)-(3.2) *is unique.*

Proof. Let $u(x) \in H^{2m+s-(k_0+1)}(\Omega_0, G_0')$ be a solution of the boundary value conjugation problem

$$\begin{aligned} &L_1(x,D)u^1(x) = 0 \quad \text{in } \Omega_0, \\ (3.9) \qquad &L_2(x,D)u^2(x) = 0 \quad \text{in } G_0' = G_0 \setminus \Omega_0, \\ &[B_i(x,D)u(x)] = 0 \quad \text{on } \gamma_0 \quad (i = 1, 2, \dots, 2m), \\ (3.10) \qquad &B_j^3(x,D)u^2(x) = 0 \quad \text{on } \Gamma_0 \quad (j = 1, 2, \dots, m). \end{aligned}$$

Applying the a prior estimate (2.3) to $R_0u(x)$, where R_0 is the operator of extension to (Ω_0, G_0), we have

$$\begin{aligned} &\|R_0u\|_{H^{2m+s-(k_0+1)}(\Omega_t, G_t')} \le C(t)\Big\{\|L(R_0u)\|_{H^{s-(k_0+1)}(\Omega_t, G_t')} \\ &+ \sum_{i=1}^{2m} \|[B_iR_0u]\|_{H^{2m+s-(k_0+1)-m_i-\frac{1}{2}}(\gamma_t)} + \sum_{j=1}^{m} \|B_j^3R_0u^2\|_{H^{2m+s-(k_0+1)-m_j^3-\frac{1}{2}}(\Gamma_t)}\Big\}, \end{aligned}$$

for $s \ge k_0 + 1$ and $t \in (0, T]$. Observe that for $t \in (0, T]$ and $\Omega_1 \subset \Omega_0$, under condition (3.9) we have $L_1R_0u^1 = 0$ in Ω_t. Moreover, $\Omega_0 \subset G_t \subset G_0$. Hence $G_t' = G_t \setminus \Omega_t = (G_t \setminus \Omega_0) \cup (\Omega_0 \setminus \Omega_t)$. Under condition (3.9) we have

$$L_2R_0u^2 = \begin{cases} L_2R_0u^2(x) & \text{if } x \in \Omega_0 \setminus \Omega_t, \\ 0 & \text{if } x \in G_t \setminus \Omega_0. \end{cases}$$

Therefore, from (1.3) we deduce

$$\begin{aligned} &\|LR_0u\|_{H^{s-(k_0+1)}(\Omega_t, G_t')} = \|L_2R_0u^2\|_{H^{s-(k_0+1)}(\Omega_0\setminus\Omega_t)} \\ &\le C \cdot t\|L_2R_0u^2\|_{H^{s-k_0}(\Omega_0\setminus\Omega_t)} \le C \cdot t\|R_0u^2\|_{H^{2m+s-k_0}(\Omega_0)}. \end{aligned}$$

Applying the inequality (1.6) we get the estimates

$$\begin{aligned} \|[B_iR_0u]\|_{H^{2m+s-(k_0+1)-m_i-\frac{1}{2}}(\gamma_t)} &= \|[B_iR_0u]_{\gamma_t} - [B_iR_0u]_{\gamma_0}\|_{H^{2m+s-(k_0+1)-m_i-\frac{1}{2}}(\gamma_t)} \\ &= \|A_t[B_iR_0u]\|_{H^{2m+s-(k_0+1)-m_i-\frac{1}{2}}(\gamma_t)} \\ &\le C \cdot t\|[B_iR_0u]\|_{H^{2m+s-k_0-m_i}(\Omega_0, G_0)} \\ &\le C \cdot t\|R_0u\|_{H^{2m+s-k_0}(\Omega_0, G_0)}, \end{aligned}$$

$$\begin{aligned} \|B_j^3R_0u^2\|_{H^{2m+s-(k_0+1)-m_j^3-\frac{1}{2}}(\Gamma_t)} &= \|A_tB_j^3R_0u^2\|_{H^{2m+s-(k_0+1)-m_j^3-\frac{1}{2}}(\Gamma_t)} \\ &\le C \cdot t\|B_j^3R_0u^2\|_{H^{2m+s-k_0-m_j^3}(G_0)} \\ &\le C \cdot t\|R_0u^2\|_{H^{2m+s-k_0}(G_0)}. \end{aligned}$$

Then we have

$$\|R_0u\|_{H^{2m+s-(k_0+1)}(\Omega_t,G'_t)} \leq C(t)\cdot t\cdot \|R_0u\|_{H^{2m+s-k_0}(\Omega_0,G_0)}.$$

Therefore, under the condition (2.4) we have

$$\lim_{t\to 0}\|R_0u\|_{H^{2m+s-(k_0+1)}(\Omega_t,G'_t)} = 0.$$

Hence $R_0u \equiv 0$ in $H^{2m+s-(k_0+1)}(\Omega_0, G'_0)$, or in other words, $u(x) \equiv 0$ in (Ω_0, G'_0). Theorem (3.2) is proved. □

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