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NOETHER PROPERTIES OF LINEAR OPERATORS INDUCED BY ALGEBRAIC ELEMENTS

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Abstract. In this paper we give some algebraic characterizations of an algebraic element with the characteristic polynomial having single roots and then inverstigate the Noether properties of bounded linear operators of the form

$$K=\sum_{(i)\in\Gamma} A_{(i)}T^{(i)}, \tag{1}$$

where

$$\Gamma=\Big\{(i)=(i_1,i_2,\dots,i_m)\mid 0\le i_k\le n_k-1,\ k=1,\dots,m\Big\},$$
$$A_{(i)}=A_{i_1 i_2\dots i_m},\quad T^{(i)}=T_1^{i_1}T_2^{i_2}\dots T_m^{i_m},$$

T_k are the commutative algebraic elements of order n_k, respectively.

Introduction

The Noether theory of singular integral operators with a Carleman shift was considered by several authors (see, e.g. [1], [3]). In 1973, the Noether properties of bounded linear operators with involution in the Banach space were considered by Gokhberg and Krupnic [2].

In this paper we give some algebraic characterizations of an algebraic element with the characteristic polynomial having single roots and then inverstigate the Noether properties of bounded linear operators of the form (1) by means of algebraic methods

1. Characterization of an algebraic element with single roots

Let X be a Banach space over the field $\mathbf{C}$ of complex numbers. Denote by $L_0(X)$ a Banach algebra of bounded linear operators acting in X and by $J(X)$ the two-sided ideal of all compact operators belonging to $L_0(X)$.

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Suppose that there exists a subalgebra $\mathcal{L}(X)$ belonging to $L_0(X)$ such that $AB - BA \in J(X)$ for all $A, B \in \mathcal{L}(X)$.

In the sequel, we assume that $J(X) \subset \mathcal{L}(X)$.

Let $T \in L_0(X)$ be an algebraic operator of order n with characteristic polynomial having single roots, i.e.,

$$P_T(t) = \prod_{i=1}^{n}(t - t_i), \quad t_i \neq t_j \text{ for } i \neq j. \tag{2}$$

Denote by P_j $(j = 1, ..., n)$ the projectors associated with T, we have (see [5])

$$P_i P_j = \delta_{ij} P_j \quad (\delta_{ij} \text{ is the Kronecker symbol}),$$

$$T^k = \sum_{j=1}^{n} t_j^k P_j \quad (\text{where } t_j^0 \overset{\text{def}}{=} 1,\ j = 1, ..., n),$$

$$P_j = \prod_{\substack{\nu=1 \\ \nu \neq j}}^{n} \frac{T - t_\nu I}{t_j - t_\nu}.$$

In particular, if T is an involution of order n, i.e. $t_j = \varepsilon_j$ $(j = 1, \ldots, n)$, where $\varepsilon_1 = \exp(2\pi i/n)$, $\varepsilon_j = \varepsilon_1^j$, then

$$P_j = \prod_{\substack{\nu=1 \\ \nu \neq j}}^{n} \frac{T - \varepsilon_\nu I}{\varepsilon_j - \varepsilon_\nu} = \frac{1}{n} \sum_{\nu=0}^{n-1} \varepsilon_j^{n-\nu} T^\nu.$$

Definition 1.1. We say that an algebraic element $T \in L_0(X)$ with the characteristic polynomial (2) is right $\mathcal{L}(X)$-linearly independent with respect to $J(X)$ if

$$\sum_{k=1}^{n} A_k P_k = 0 \,(\text{mod } J(X)); \quad A_k \in \mathcal{L}(X)$$

implies $A_k = 0 \,(\text{mod } J(X))$, $k = 1, \ldots, n$. (Similarly, if $\sum_{k=1}^{n} P_k A_k = 0$ (mod) $J(X))$, $A_k \in \mathcal{L}(X)$ implies $A_k = 0$ (mod $J(X)$) $(k = 1, \ldots, n)$, then T is said to be left $\mathcal{L}(X)$-linearly independent with respect to $J(X)$).

Definition 1.2. An algebraic element $T \in L_o(X)$ with the characteristic polynomial (2) is said to be $\mathcal{L}(X)$-linearly independent with respect to

$J(X)$ if it is right and left $\mathcal{L}(X)$-linearly independent with respect to $J(X)$.

Lemma 1.1. *Every algebraic element of order n with the characteristic polynomial having single roots can be written in the form of the polynomial of the involution of order n.*

Proof. Let T be an algebraic element with the characteristic polynomial of the form (2). Putting

$$(3) \qquad S = Q(T), \quad Q_j = \prod_{\substack{\nu=1 \\ \nu\neq j}}^{n} \frac{S - \varepsilon_\nu I}{\varepsilon_j - \varepsilon_\nu}, \quad j = 1, \dots, n,$$

where

$$Q(t) = \sum_{i=1}^{n} \prod_{\substack{\mu=1 \\ \mu\neq i}}^{n} \varepsilon_i \frac{t - t_\mu}{t_i - t_\mu},$$

we obtain

$$(4) \qquad T = \sum_{j=1}^{n} t_j Q_j = \sum_{\nu=0}^{n-1} \Big(\frac{1}{n} \sum_{j=1}^{n} t_j \varepsilon_j^{n-\nu}\Big) S^\nu.$$

Since $P_T(t) = \prod_{i=1}^{n} (t - t_i)$ and $Q(t_i) = \varepsilon_i \neq Q(t_j) = \varepsilon_j$ if $i \neq j$, we have (see [4], [5])

$$P_S(t) = P_{Q(T)}(t) = \prod_{i=1}^{n} (t - Q(t_i)) = \prod_{i=1}^{n} (t - \varepsilon_i) = t^n - 1.$$

Hence S is an involution of order n. The lemma is proved.

Lemma 1.2. *Let $T \in L_0(X)$ be an algebraic element with the characteristic polynomial* (2) *and $S = Q(T)$ be defined by* (3). *The following statements are equivalent*

a) *T is right $\mathcal{L}(X)$-linearly independent with respect to $J(X)$;*

b) $\sum_{k=0}^{n-1} A_k S^k = 0$ *(mod $J(X)$), $A_k \in \mathcal{L}(X)$ implies*

$$(5) \qquad A_k = 0 \pmod{J(X)} \quad (k = 0, \dots, n-1);$$

c) *S is right $\mathcal{L}(X)$ - linearly independent with respect to $J(X)$.*

Proof. a) $\Rightarrow$ b): Suppose that T is right $\mathcal{L}(X)$-linearly independent with respect to $J(X)$ and

$$\sum_{k=0}^{n-1} A_k S^k = 0 \,(\mathrm{mod}\ J(X)),\ \ A_k \in \mathcal{L}(X),\ \ k = 0, ..., n-1,$$

i.e.

$$\sum_{k=0}^{n-1} A_k \Big(\sum_{i=1}^{n} \prod_{\substack{\nu=1 \\ \nu\neq i}}^{n} \varepsilon_i \frac{T - t_\nu I}{t_i - t_\nu} \Big)^k = \sum_{k=0}^{n-1} A_k \Big(\sum_{i=1}^{n} \varepsilon_i^k P_i \Big)$$
$$= \sum_{i=1}^{n} \Big(\sum_{k=0}^{n-1} \varepsilon_i^k A_k \Big) P_i = 0 \,(\mathrm{mod}\ J(X)).$$

Since $\sum\limits_{k=0}^{n-1} \varepsilon_i^k A_k \in \mathcal{L}(X)$ for all $i = 1, \dots, n$, we get

$$\sum_{k=0}^{n-1} \varepsilon_i^k A_k = 0 \,(\mathrm{mod}\ J(X)), \quad i = 1, \dots, n.$$

Hence

$$\sum_{k=0}^{n-1} \varepsilon_i^k [A_k] = 0, \quad i = 1, \dots, n,$$

where $[A]$ is the coset defined by an element $A \in L_0(X)$ in the quotient algebra $L_0(X)/J(X)$. It is easy to see that the determinant of this system is the Vandermonde determinant of the numbers $1, \varepsilon_1, \varepsilon_2, \dots, \varepsilon_{n-1}$. This implies

$$[A_k] = 0 \ \text{ i.e.,} \ \ A_k = 0 \,(\mathrm{mod}\ J(X)), \quad k = 0, ..., n-1.$$

b) $\Rightarrow$ a): Suppose that (5) is satisfied and

$$\sum_{i=1}^{n} A_i P_i = 0 \,(\mathrm{mod}\ J(X)),\ \ A_i \in \mathcal{L}(X),\ \ i = 1, ..., n.$$

This implies

$$A_k P_k = 0 \,(\mathrm{mod}\, J(X))), \quad k = 1, \ldots, n,$$

i.e.

$$A_k \prod_{\substack{i=1 \\ i \neq k}}^{n} \frac{T - t_i I}{t_k - t_i} = 0 \,(\mathrm{mod}\, J(X)). \tag{6}$$

From (4) and (6), we obtain

$$A_k \prod_{\substack{i=1 \\ i \neq k}}^{n} \frac{\sum\limits_{j=1}^{n} t_j Q_j - t_i I}{t_k - t_i} = A_k \sum_{j=1}^{n} \Big(\prod_{\substack{i=1 \\ i \neq k}}^{n} \frac{t_j - t_i}{t_k - t_i} \Big) Q_j$$

$$= A_k Q_k = \frac{1}{n} \sum_{j=0}^{n-1} A_k \varepsilon_k^{n-j} S^j = 0 \,(\mathrm{mod}\, J(X)), \;\; k = 1, \ldots, n.$$

Since $\varepsilon_k^{n-j} A_k \in \mathcal{L}(X)$, we get

$$\varepsilon_k^{n-j} A_k = 0 \,(\mathrm{mod}\, J(X)), \;\; k, j = 1, \ldots, n.$$

Thus

$$A_k = 0 \,(\mathrm{mod}\, J(X)), \quad k = 1, \ldots, n.$$

Hence T is right $\mathcal{L}(X)$-linearly independent with respect to $J(X)$. In the same way, we can prove b) is equivalent to c). □

By similar argument, we obtain the following result

Lemma 1.3. *Let $T \in L_0(X)$ be an algebraic element with the characteristic polynomial* (2) *and $S = Q(T)$ be defined by* (3). *The following statements are equivalent:*

a) *T is left $\mathcal{L}(X)$-linearly independent with respect to $J(X)$;*

b) $\sum\limits_{k=0}^{n-1} S^k A_k = 0 \,(\mathrm{mod}\, J(X))$, *$A_k \in \mathcal{L}(X)$, implies*

$$A_k = 0 \,(\mathrm{mod}\, J(X)), \quad (k = 0, \ldots, n-1);$$

c) *S is left $\mathcal{L}(X)$-linearly independent with respect to $J(X)$.*

Lemma 1.4. *Let $T \in L_0(X)$ be an algebraic element with the characteristic polynomial* (2) *and $S = Q(T)$ be defined by* (3). *If*

$$S^j A S^{n-j} \in \mathcal{L}(X) \quad \text{for all} \quad A \in \mathcal{L}(X) \ \ (j = 1, \dots, n), \tag{7}$$

then the following statements are equivalent:

a) $\sum\limits_{k=0}^{n-1} A_k S^k = 0 \,(\mathrm{mod}\ J(X)),\ \ A_k \in \mathcal{L}(X)$, *implies*

$$A_k = 0 \ \ (\mathrm{mod}\ J(X)) \quad (k = 0, \dots, n-1); \tag{8}$$

b) $\sum\limits_{k=0}^{n-1} S^k A_k = 0 \,(\mathrm{mod}\, J(X)),\ A_k \in \mathcal{L}(X)$, implies

$$A_k = 0 \ \ (\mathrm{mod}\, J(X)), \quad (k = 0, \dots, n-1). \tag{9}$$

Proof. Suppose that (8) is satisfied and

$$\sum_{k=0}^{n-1} S^k A_k = 0 \,(\mathrm{mod}\, J(X)), \quad A_k \in \mathcal{L}(X),$$

i.e.

$$\sum_{k=0}^{n-1} (S^k A_k S^{n-k}) S^k = 0 \,(\mathrm{mod}\, J(X)).$$

Since $S^k A_k S^{n-k} \in \mathcal{L}(X),\ k = 0, \dots, n-1,$ we get

$$S^k A_k S^{n-k} = 0 \,(\mathrm{mod}\, J(X)), \quad k = 0, \dots, n-1.$$

Hence $A_k = 0 \,(\mathrm{mod}\, J(X)),\ k = 0, \dots, n-1$.

Conversely, suppose that (9) is satisfied and

$$\sum_{k=0}^{n-1} A_k S^k = 0 \,(\mathrm{mod}\, J(X)),$$

i.e.

$$\sum_{k=0}^{n-1} S^k (S^{n-k} A_k S^k) = 0 \ \ (\mathrm{mod}\, J(X)).$$

Since $S^{n-k}A_kS^k \in \mathcal{L}(X)$, $k = 0, \dots, n-1$, we get

$$S^{n-k}A_kS^k = 0 \,(\operatorname{mod} J(X)),\ \ k = 0, \dots, n-1.$$

Hence $A_k = 0 \,(\operatorname{mod} J(X))$, $k = 0, \dots, n-1$. The Lemma is proved. □

Lemmas 1.2, 1.3 and 1.4 together imply the following result.

Theorem 1.1. *Let $T \in L_o(X)$ be an algebraic element with the characteristic polynomial (2) and $S = Q(T)$ be defined by (3). Suppose that the condition (7) is satisfied. Then T is $\mathcal{L}(X)$-linearly independent with respect to $J(X)$ if and only if S is $\mathcal{L}(X)$-linearly independent with respect to $J(X)$.*

2. Noether properties and the index formula of linear operators induced by an algebraic element

Consider the following operators

$$K_0 = \sum_{k=0}^{n-1} A_kT^k, \tag{10}$$

where $A_k \in \mathcal{L}(X)$ $(k = 0, \dots, n-1)$ and $T \in L_0(X)$ is an algebraic operator with the characteristic polynomial of the form (2).

We assume that $S^jAS^{n-j} \in \mathcal{L}(X)$ for all $A \in \mathcal{L}(X)$ $(j = 1, \dots, n)$, where $S = Q(T)$ is defined by (3).

Then K_0 can be written in the form

$$K_0 = \sum_{k=0}^{n-1} B_kS^k, \tag{11}$$

where

$$B_k = \frac{1}{n}\sum_{i=1}^{n} \varepsilon_i^{n-k}\Big(\sum_{j=0}^{n-1} t_i^j A_j\Big), \quad k = 0, \dots, n-1. \tag{12}$$

Indeed, from Lemma 1.1 we have

$$\begin{aligned} K_0 &= \sum_{k=0}^{n-1} A_kT^k = \sum_{k=0}^{n-1} A_k\Big(\sum_{j=1}^{n}\prod_{\substack{\nu=1\\ \nu\neq j}}^{n} t_j\frac{S-\varepsilon_\nu I}{\varepsilon_j-\varepsilon_\nu}\Big)^k \\ &= \sum_{k=0}^{n-1} A_k\Big(\sum_{j=1}^{n} t_j^kQ_j\Big) = \sum_{k=0}^{n-1}\Big[\frac{1}{n}\sum_{j=1}^{n}\varepsilon_j^{n-\nu}\Big(\sum_{k=0}^{n-1} t_j^kA_k\Big)\Big]S^\nu, \end{aligned}$$

where Q_j $(j = 1, \dots, n)$ are the projectors associated with S.

Denote

$$K_\mu = \sum_{k=0}^{n-1} \varepsilon_\mu^k B_k S^k, \quad \mu = 1, \dots, n-1,$$

where B_k $(k = 0, \dots, n-1)$ are defined by (12).

Consider the operator $E(K_0)$ belonging to $L_0(X^n)$

$$E(K_0) = \Big[C_{ij}\Big]_{i,j=0}^{n-1} = \begin{bmatrix} B_{00} & B_{10} & \dots & B_{n-10} \\ B_{n-11} & B_{01} & \dots & B_{n-21} \\ \vdots & \vdots & \ddots & \vdots \\ B_{1n-1} & B_{2n-1} & \dots & B_{0n-1} \end{bmatrix}, \tag{13}$$

where for $i, j = 0, ..., n-1$,

$$B_{ij} = S^j B_i S^{n-j} \in \mathcal{L}(X),$$

$$C_{ij} = \begin{cases} B_{j-i\ i} & \text{if } j \geq i, \\ B_{n+j-i\ i} & \text{if } j < i. \end{cases}$$

The matrix $E(K_0)$ is said to be the symbol over $\mathcal{L}(X)$ of the operator K_0 of the form (10).

Lemma 2.1. *Suppose that* K_0 *is defined by* (10) *and*

$$K_0' = \sum_{k=0}^{n-1} A_k' T^k, \ \ A_k' \in \mathcal{L}(X) \ \ (k = 0, \dots, n-1).$$

Then

a) $E(\lambda K_0) = \lambda E(K_0), \ \lambda \in \mathbf{C}$;

b) $E(K_0 + K_0') = E(K_0) + E(K_0')$;

c) $E(K_0 K_0') = E(K_0) E(K_0')$.

Proof. Evidently, $E(\lambda K_0) = \lambda E(K_0)$ and $E(K_0 + K_0') = E(K_0) + E(K_0')$. We have

$$\begin{aligned} K_0 K_0' &= \sum_{k=0}^{n-1} A_k T^k \sum_{j=0}^{n-1} A_j' T^j = \sum_{k=0}^{n-1} B_k S^k \sum_{j=0}^{n-1} B_j' S^j \\ &= \sum_{k=0}^{n-1} \sum_{j=0}^{n-1} B_k B_{jk}' S^{j+k} = \sum_{j=0}^{n-1} D_j S^j, \end{aligned}$$

where $D_j = B_0 B'_{j0} + B_1 B'_{j-1\,1} + \ldots + B_{n-1} B'_{j+1\,n-1}$, $\quad j = 0, \ldots, n-1$, $D_j \in \mathcal{L}(X)$. Hence

$$E(K_0 K'_0) = \left[P_{ij}\right]_{i,j=0}^{n-1},$$

where for $i, j = 0, 1, \ldots, n-1$,

$$\begin{aligned} (14) \qquad P_{ij} &= D_{j-i\ i} \\ &= B_{0i} B'_{j-i\ i} + B_{1i} B'_{j-i-1\ i+1} + \ldots + B_{n-1\ i} B'_{j-i+1\ i-1}, \end{aligned}$$

$$D_{j-i\,i} := \begin{cases} D_{j-i\ i} & \text{if } j \geq i, \\ D_{n+j-i\ i} & \text{if } j < i. \end{cases}$$

On the other hand, if

$$E(K_0) E(K'_0) = \left[Q_{ij}\right]_{i,j=0}^{n-1}$$

then

$$Q_{ij} = \sum_{k=0}^{n-1} B_{k-i\ i} B'_{j-k\ k}. \tag{15}$$

From (14) and (15), we get $P_{ij} = Q_{ij}$, $i, j = 0, \ldots, n-1$. The lemma is proved. □

It is easy to prove that the set $\mathcal{K}(X)$ of all operators K_0 of the form (10) and the set $\mathcal{H}(X^n)$ of all operators $E(K_0)$ of the form (13) form algebras.

Lemma 2.2. *Suppose that T is $\mathcal{L}(X)$-linearly independent with respect to $J(X)$. Then K_0 is a compact operator if and only if $E(K_0)$ is a compact operator.*

Proof. Denote by $J(X^n)$ the two-side ideal of all compact operators belonging to $L_0(X^n)$. Evidently, if $E(K_0) = 0 \pmod{J(X^n)}$ then $K_0 = 0 \pmod{J(X)}$.

Conversely, suppose that

$$K_0 = 0 \pmod{J(X)}. \tag{16}$$

The assumption and Theorem 1.1 together imply S is $\mathcal{L}(X)$-linearly indenpendent with respect to $J(X)$. According to Lemma 1.2, from (16) we have

$$B_k = 0 \pmod{J(X)}, \quad k = 0, \ldots, n-1.$$

Hence

$$E(K_0) = 0 \pmod{J(X^n)}.$$

Denote

$$[\mathcal{K}(X)] = \mathcal{K}(X)/J(X), \quad [\mathcal{H}(X^n)] = \mathcal{H}(X^n)/J(X^n).$$

Lemmas 2.1 and 2.2 together imply the following result

Theorem 2.1. *Suppose that T is $\mathcal{L}(X)$-linearly independent with respect to $J(X)$. Then $[\mathcal{K}(X)]$ and $[\mathcal{H}(X^n)]$ are isomorphic.*

Theorem 2.2. *Suppose that there exists a Noether operator U belonging to $L_0(X)$ such that*

$$\begin{aligned} UA_k - A_kU \in J(X) \quad (k = 0, \dots, n-1), \\ US = \varepsilon_1 SU, \end{aligned} \tag{17}$$

where $S = Q(T)$ is defined by (3)*. Then*

a) *Either K_μ $(\mu = 0, \dots, n-1)$ are Noether operators or aren't Noether operators, simutaneously;*

b) *If K_μ $(\mu = 0, \dots, n-1)$ are Noether operators, simutaneously, then*

$$\text{Ind } K_\mu = \text{Ind } K_0 \quad (\mu = 1, \dots, n-1);$$

c) *Either K_0 and $E(K_0)$ are Noether operators or aren't Noether operators, simutaneously. In the case K_0 and $E(K_0)$ are Noether operators, the following equality holds*

$$\text{Ind } K_0 = \frac{1}{n} \text{ Ind } E(K_0).$$

Proof. a) From (17), we have

$$\begin{aligned} &UB_k - B_kU \in J(X), \\ &U^jS^k = \varepsilon_j^k S^kU^j, \quad k, j = 1, 2, \dots \end{aligned}$$

Hence

$$\begin{aligned} U^jK_0 &= U^j \sum_{k=0}^{n-1} A_kT^k = U^j \sum_{k=0}^{n-1} B_kS^k \\ &= \Big(\sum_{k=0}^{n-1} \varepsilon_j^k B_kS^k \Big) U^j = K_jU^j \pmod{J(X)}. \end{aligned}$$

Thus

$$U^j K_0 = K_j U^j (\text{mod } J(X)), \text{ i.e } K_0 = R^j K_j U^j (\text{mod } J(X)),$$

where R^j is a regularized of U^j to $J(X)$. This implies K_0 and K_j are Noether operators or aren't Noether operators, simultaneously.

b) If K_j $(j = 0, \dots, n-1)$ are Noether operators, we have

$$\text{Ind } (U^j K_0) = \text{Ind } (K_j U^j).$$

Hence

$$\text{Ind } U^j + \text{Ind } K_0 = \text{Ind } K_j + \text{Ind } U^j,$$

i.e.

$$\text{Ind } K_0 = \text{Ind } K_j, \quad j = 1, \dots, n-1.$$

c) Denote

$$\mathcal{A}_n = \begin{pmatrix} \mathrm{I} & S^{n-1} & S^{n-2} & \dots & S \\ \mathrm{I} & \varepsilon_{n-1} S^{n-1} & \varepsilon_{n-2} S^{n-2} & \dots & \varepsilon_1 S \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \mathrm{I} & \varepsilon_{n-1}^{n-1} S^{n-1} & \varepsilon_{n-2}^{n-1} S^{n-2} & \dots & \varepsilon_1^{n-1} S \end{pmatrix}$$

$$\mathcal{B}_n = \begin{pmatrix} \mathrm{I} & \mathrm{I} & \mathrm{I} & \dots & \mathrm{I} \\ S & \varepsilon_1 S & \varepsilon_2 S & \dots & \varepsilon_{n-1} S \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ S^{n-1} & \varepsilon_1^{n-1} S^{n-1} & \varepsilon_2^{n-1} S^{n-1} & \dots & \varepsilon_{n-1}^{n-1} S^{n-1} \end{pmatrix}$$

$$\mathcal{N}_n = n \ \text{diag}\,(K_0, K_1, \dots, K_{n-1}).$$

A direct check shows that the following equalities are satisfied

$$\mathcal{A}_n \mathcal{B}_n = \mathcal{B}_n \mathcal{A}_n = n \ \text{diag } (I, I, \dots, I), \tag{18}$$

$$\mathcal{A}_n E(K_0) \mathcal{B}_n = \mathcal{N}_n. \tag{19}$$

From (18), we have

$$\text{Ind } \mathcal{A}_n = \text{Ind } \mathcal{B}_n = 0.$$

This and (19) together imply that either $E(K_0)$ and $\mathcal{N}_n$ are Noether operators and Ind $E(K_0)$ = ind $\mathcal{N}_n$ or $E(K_0)$ and $\mathcal{N}_n$ aren't Noether operators.

On the other hand, it is easy to prove that $\mathcal{N}_n$ is Noether operator if and only if K_j $(j = 0, \dots, n-1)$ are Noether operators and

$$\operatorname{Ind} \mathcal{N}_n = \sum_{j=0}^{n-1} \operatorname{Ind} K_j.$$

Apply the results of a) and b), we obtain c). The theorem is proved. □

3. Noether properties and the index formula of linear operators induced by several algebraic elements

Let $T_1, T_2, \dots, T_m$ be commutative algebraic elements with the characteristic roots t_{jk_j} $(k_j = 1, \dots, n_j;\ j = 1, \dots, m)$ and let P_{jk_j} $(k_j = 1, \dots, n_j)$ be the projectors associated with T_j, $j = 1, \dots, m$.

Denote

$$\begin{aligned}
\Gamma &= \big\{(i) = (i_1, i_2, \dots, i_m) \mid 0 \le i_k \le n_k - 1,\ k = 1, \dots, m\big\},\\
T &= T_1 T_2 \dots T_m,\\
T^{(i)} &= T_1^{i_1} T_2^{i_2} \dots T_m^{i_m},\\
P_{(j)} &= P_{1j_1} P_{2j_2} \dots P_{mj_m},\\
t_{(j)} &= t_{1j_1} t_{2j_2} \dots t_{mj_m},\\
t_{(j)}^{(i)} &= t_{1j_1}^{i_1} t_{2j_2}^{i_2} \dots t_{mj_m}^{i_m},\ (i), (j) \in \Gamma,\\
\Lambda &= \big\{(j) \in \Gamma \,:\, P_{(j)} \ne 0\big\}.
\end{aligned} \tag{20}$$

In the sequel, we assume that

$$t_{(i)} \ne t_{(j)} \text{ if } (i) \ne (j),\ (i), (j) \in \Lambda \tag{21}$$

where $(i) = (j) \Leftrightarrow i_k = j_k \ \ \forall k = 1, \dots, m$. Consider the following operators

$$K_0 = \sum_{(i) \in \Gamma} A_{(i)} T^{(i)}, \tag{22}$$

where

$$A_{(i)} = A_{i_1 i_2 \dots i_m} \in \mathcal{L}(X), \quad (i) \in \Gamma.$$

Lemma 3.1 [6]. *The following equalities hold:*

a) $\sum_{(i)\in\Lambda} P_{(i)} = I$,

b) $P_{(i)}P_{(j)} = \begin{cases} 0 & \textit{if } (i) \neq (j) \\ P_{(j)} & \textit{if } (i) = (j) \end{cases}.$

Lemma 3.2 *The following equalities hold:*

a) $T^k = \sum_{(j)\in\Lambda} t^k_{(j)} P_{(j)}, \;\; k = 0, 1, \ldots;$

b) $TP_{(j)} = t_{(j)}P_{(j)}, \;\; (j) \in \Lambda.$

Proof. We have

$$T = \prod_{i=1}^{m} T_i = \prod_{i=1}^{m} \Big(\sum_{j_i=1}^{n_i} t_{ij_i} P_{ij_i} \Big)$$
$$= \sum_{\substack{j_i=1,\ldots,n_i \\ i=1,\ldots,m}} t_{1j_1} t_{2j_2} \ldots t_{mj_m} P_{1j_1} P_{2j_2} \ldots P_{mj_m},$$

Hence

$$T = \sum_{(j)\in\Lambda} t_{(j)} P_{(j)}. \tag{23}$$

From Lemma 3.1 and formula (23), we have

$$T^k = \sum_{(j)\in\Lambda} t^k_{(j)} P_{(j)},$$
$$TP_{(j)} = \Big(\sum_{(i)\in\Lambda} t_{(i)} P_{(i)} \Big) P_{(j)}$$
$$= t_{(j)} P_{(j)}, \quad (j) \in \Lambda.$$

Lemma 3.3. *T of the form* (20) *is an algebraic element with the characteristic roots $t_{(j)}, \;\; (j) \in \Lambda$.*

Proof. Let $P(t) = \prod_{(i)\in\Lambda} (t - t_{(i)})$, we have (see [4])

$$P(T) = \prod_{(i)\in\Lambda} (T - t_{(i)} I) = 0.$$

Let

$$P_1(t) = \prod_{(i)\in\Lambda\setminus\{(j)\}} (t - t_{(i)}), \quad (j) \in \Lambda,$$

we shall show that $P_1(T) \neq 0$. Indeed,

$$\begin{aligned} P_1(T) &= \sum_{(k)\in\Lambda} \prod_{(i)\in\Lambda\setminus\{(j)\}} (T - t_{(i)}I)P_{(k)} \\ &= \prod_{(i)\in\Lambda\setminus\{(j)\}} (T - t_{(i)}I)P_{(j)} \\ &= \prod_{(i)\in\Lambda\setminus\{(j)\}} (T - t_{(j)}I + t_{(j)}I - t_{(i)}I)P_{(j)} \\ &= \prod_{(i)\in\Lambda\setminus\{(j)\}} (t_{(j)} - t_{(i)})P_{(j)}. \end{aligned}$$

Since $t_{(j)} \neq t_{(i)}$, if $(j) \neq (i)$ $((i),(j) \in \Lambda)$, we have $P_1(T) \neq 0$. Thus

$$P_T(t) = P(t) = \prod_{(i)\in\Lambda} (t - t_{(i)}).$$

Remark. The deteminant of the system

$$T^k = \sum_{(i)\in\Lambda} t^k_{(i)} P_{(i)}, \quad k = 0, ..., N-1, \tag{24}$$

(where N is cardinality of Λ) with respect to the unknows $P_{(i)}$ $((i) \in \Lambda)$ is the Vandermonde deteminant of the number $t_{(i)}$ $((i) \in \Lambda)$. Since $t_{(i)} \neq t_{(j)}$ if $(i) \neq (j)$, the system (24) has a unique solution of the form

$$P_{(i)} = \sum_{k=0}^{N-1} c_{(i)k} T^k, \quad (i) \in \Lambda, \tag{25}$$

where $c_{(i)k}$ are defined in terms of $t_{(j)}$ $((j) \in \Lambda)$, $i \in \Lambda$, $k = 0, \ldots, N-1$.

K_0 can be written in the form

$$K_0 = \sum_{k=0}^{N-1} B_k T^k, \tag{26}$$

where $B_k = \sum_{(j)\in\Lambda} \sum_{(i)\in\Gamma} t^{(i)}_{(j)} c_{(j)k} A_{(i)}$ $c_{(i)k}$ is defined by (25). Indeed,

$$\begin{aligned}
K_0 &= \sum_{(i)\in\Gamma} A_{(i)} T^{(i)} \\
&= \sum_{(i)\in\Gamma} A_{(i)} \prod_{k=1}^{m} \Big(\sum_{j_k=1}^{n_k} t_{kj_k} P_{kj_k} \Big)^{i_k} \\
&= \sum_{(i)\in\Gamma} A_{(i)} \prod_{k=1}^{m} \Big(\sum_{j_k=1}^{n_k} t^{i_k}_{kj_k} P_{kj_k} \Big) \\
&= \sum_{(i)\in\Gamma} A_{(i)} \Big(\sum_{\substack{j_k=1,\dots,n_k \\ k=1,\dots,m}} t^{i_1}_{1j_1} t^{i_2}_{2j_2} \dots t^{i_m}_{mj_m} P_{1j_1} P_{2j_2} \dots P_{mj_m} \Big) \\
&= \sum_{(i)\in\Gamma} A_{(i)} \Big(\sum_{(j)\in\Lambda} t^{(i)}_{(j)} P_{(j)} \Big) \\
&= \sum_{(j)\in\Lambda} \Big(\sum_{(i)\in\Gamma} t^{(i)}_{(j)} A_{(i)} \Big) P_{(j)} \\
&= \sum_{(j)\in\Lambda} \Big(\sum_{(i)\in\Gamma} t^{(i)}_{(j)} A_{(i)} \Big) \sum_{k=0}^{N-1} c_{(j)k} T^k \\
&= \sum_{(j)\in\Lambda} \sum_{k=0}^{N-1} \Big(\sum_{(i)\in\Gamma} t^{(i)}_{(j)} c_{(j)k} A_{(i)} \Big) T^k \\
&= \sum_{k=0}^{N-1} \Big(\sum_{(j)\in\Lambda} \sum_{(i)\in\Gamma} t^{(i)}_{(j)} c_{(j)k} A_{(i)} \Big) T^k \\
&= \sum_{k=0}^{N-1} B_k T^k.
\end{aligned}$$

The operator K_0 of the form (26) is induced by an algebraic element with the characteristic polynomial having single roots. Hence, applying the results of §2, we shall obtain the Noether properties and the index formula of K_0.

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