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SOME CHARACTERIZATION OF BEST APPROXIMANTS IN NORMED LINEAR SPACES

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Abstract. Some new characterization of best approximants in normed linear spaces are given.

1. Introduction

Let $(X, \|\cdot\|)$ be a normed linear space and G a nondense linear subspace in X. Suppose $x_0 \in X \setminus Cl(G)$ and $g_0 \in G$.

Definition 1. The element g_0 will be called the best approximation element of x_0 in G if

$$\|x_0 - g_0\| = \inf_{g\in G} \|x_0 - g\| \tag{1.1}$$

and we shall denote by $\mathcal{P}_G(x_0)$ the set of all elements which satisfy (1.1).

The following classic result is due to I. Singer (see for example [4, p. 16])

Theorem 1. *Let X, G, x_0 and g_0 be as above. Then $g_0 \in \mathcal{P}_G(x_0)$ if and only if there exists a functional $f \in X^*$ such that*

$$\|f\| = 1,\ f(g) = 0 \text{ for all } g \in G \text{ and } f(x_0 - g_0) = \|x_0 - g_0\|.$$

For some different consequences as well as for the geometrical interpretation of this fact see [4, p. 16-26].

Another characterization of the best approximation element in terms of the *tangent functional* τ, i.e.,

$$\tau(x, y) := \lim_{t\to 0^+} \frac{(\|x + ty\| - \|x\|)}{t};$$

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$x, y \in X$, $x \neq 0$ is embodied in the following theorem [4, p. 82]:

Theorem 2. *With the above assumptions, the following statements are equivalent:*

(i) $g_0 \in \mathcal{P}_G(x_0)$;

(ii) $\tau(x_0 - g_0, g) \geq 0$ *for all* $g \in G$;

(iii) $-\tau(x_0 - g_0, -g) \leq 0 \leq \tau(x_0 - g_0, g)$ *for all* $g \in G$;

(iv) *For any* $g \in G$, *there exists* $f^g \in X^*$ *such that*

$$\|f^g\| = 1, \ \operatorname{Re} f^g(g) = 0 \ \text{and} \ f^g(x_0 - g_0) = \|x_0 - g_0\|.$$

In 1935, G. Birkhoff (see [4, p. 84]) introduced the following concept of orthogonality in normed spaces:

$$x \perp y \ (B) \text{ iff } \|x + \alpha y\| \geq \|x\| \quad \text{for all } \alpha \in \mathbb{R},$$

which, in the case of real prehilbertian spaces, coincides with the usual orthogonality associated to the inner product, $(\cdot,\cdot)$, which generates the norm.

By the use of Birkhoff's orthogonality, the following characterization of best approximation elements holds.

Lemma 1. *Let* X *be a normed space,* G *its nondense linear subspace,* $x_0 \in X \setminus Cl(G)$ *and* $g_0 \in G$. *Then* $g_0 \in \mathcal{P}_G(x_0)$ *iff* $x_0 - g_0 \perp G\,(B)$, *i.e.,* $x_0 - g_0 \perp g$ *for all* $g \in G$.

For other results in connection with the best approximation element see the monograph [4] as well as the recent papers [1]-[3].

2. The results

Let $(X, \|\cdot\|)$ be a normed space. The mapping $f : X \to \mathbb{R}$ given by $f(x) = \frac{1}{2}\|x\|^2$ is convex on X and thus there exist the following limits (see also [1] and [3]):

$$(x, y)_{i(s)} := \lim_{t \to 0^{-(+)}} \frac{\left(\|y + txt\|^2 - \|y\|^2\right)}{2t}; \quad x, y \in X.$$

The mapping $(\cdot,\cdot)_{i(s)}$ will be called *the inferior (superior) semi-inner product* associated with the norm $\|\cdot\|$.

The following characterization of best approximants holds:

Theorem 3. *Let* $(X, \|\cdot\|)$ *be a real normed space,* G *its closed linear subspace with* $G \neq X$*, and* $x_0 \in X \setminus G$*,* $g_0 \in G$*. The following statements are equivalent:*

(i) $g_0 \in \mathcal{P}_G(x_0)$*;*

(ii) *For every* $f \in (G \oplus Sp(x_0))^*$ *with* $\operatorname{Ker}(f) = G$*, we have the estimation*

$$\|f\|_{G_{x_0}} \left(x, \frac{\lambda_0 (x_0 - g_0)}{\|x_0 - g_0\|}\right)_i \leq f(x) \leq \|f\|_{G_{x_0}} \left(x, \frac{\lambda_0 (x_0 - g_0)}{\|x_0 - g_0\|}\right)_s \tag{2.1}$$

for all $x \in G_{x_0} := G \oplus Sp(x_o)$*, where*

$$\|f\|_{G_{x_0}} := \sup \left\{ \frac{|f(x)|}{\|x\|}, x \in G_{x_0} \right\} \text{ and } \lambda_0 := \operatorname{sgn} f(x_0).$$

Proof. It follows from the following lemma applied for the normed linear space G_{x_0} in which G is a hyperplane. □

Lemma 2. *Let* $(X, \|\cdot\|)$ *be as above* $f \in X^* \setminus \{0\}$*,* $x_0 \in X \setminus \operatorname{Ker}(f)$ *and* $g_0 \in \operatorname{Ker}(f)$*. The following conditions are equivalent:*

(i) $g_0 \in \mathcal{P}_{\operatorname{Ker}(f)}(x_0)$*;*

(ii) *There is the estimation*

$$\|f\| \left(x, \frac{\lambda_0 (x_0 - g_0)}{\|x_0 - g_0\|}\right)_i \leq f(x) \leq \|f\| \left(x, \frac{\lambda_0 (x_0 - g_0)}{\|x_0 - g_0\|}\right)_s \tag{2.2}$$

for all $x \in X$ *and* $\lambda_0 := \operatorname{sgn} f(x_0)$*.*

Proof. (i) $\Longrightarrow$ (ii). Let us assume that $g_0 \in \mathcal{P}_{\operatorname{Ker}(f)}(x_0)$ and denote $w_0 := x_0 - g_0$. Then $w_0 \neq 0$ and, by Lemma 1, we deduce that $w_0 \perp \operatorname{Ker}(f)(B)$. Using the properties of the s.i.p. $(\cdot,\cdot)_{i(s)}$ we have $(y, w_0)_i \leq 0 \leq (y, w_0)_s$ for every $y \in \operatorname{Ker}(f)$.

Now, let x be arbitrary in X. Then the element $y := f(x)w_0 - f(w_0)x$ belongs to $\operatorname{Ker}(f)$, and by the above inequality, we deduce that

$$(f(x)w_0 - f(w_0)x, w_0)_i \leq 0 \leq (f(x)w_0 - f(w_0)x, w_0)_s \tag{2.3}$$

for all $x \in X$.

Using the properties of the mappings $(\cdot,\cdot)_i$ and $(\cdot,\cdot)_s$ one has

$$(f(x)w_0 - f(w_0)x, w_0)_p = f(x)\|w_0\|^2 + (-f(w_0)x, w_0)_p, \quad x \in X,$$

where $p = s$ or $p = i$.

On the other hand, since $w_0 \perp \operatorname{Ker}(f)(B)$ and $w_0 \neq 0$, hence $f(w_0) \neq 0$. Thus, we derive two cases: a) $f(w_0) > 0$ and b) $f(w_0) < 0$.

a) If $f(w_0) > 0$, then by (2.3) we have successively that

$$\begin{aligned} 0 &\leq f(x)\|w_0\|^2 + (-f(w_0)x, w_0)_s \\ &= f(x)\|w_0\|^2 + f(w_0)(-x, w_0)_s \\ &= f(x)\|w_0\|^2 + (-x, f(w_0)w_0)_s \\ &= f(x)\|w_0\|^2 - (x, f(w_0)w_0)_i . \end{aligned}$$

From this we get

$$f(x) \geq \left(x, \frac{f(w_0)\,w_0}{\|w_0\|^2}\right)_i \text{ for all } x \in X. \tag{2.4}$$

Similarly, by (2.3) we deduce that

$$\begin{aligned} 0 &\geq f(x)\|w_0\|^2 + (-f(w_0)x, w_0)_i \\ &= f(x)\|w_0\|^2 - (x, f(w_0)w_0)_s , \end{aligned}$$

which implies

$$f(x) \leq \left(x, \frac{f(w_0)\,w_0}{\|w_0\|^2}\right)_s \text{ for all } x \in X. \tag{2.5}$$

b) The proof goes likewise and we omit the details.

Consequently, in both cases we can state:

$$\left(x, \frac{f(w_0)\,w_0}{\|w_0\|^2}\right)_i \leq f(x) \leq \left(x, \frac{f(w_0)\,w_0}{\|w_0\|^2}\right)_s \text{ for all } x \in X. \tag{2.6}$$

Now, let $u := \dfrac{f(w_0)w_0}{\|w_0\|^2}$. Then by (2.6) we have

$$f(x) \leq (x,u)_s \leq \|x\|\|u\| \text{ for all } x \in X$$

and

$$f(x) \geq (x,u)_i = -(x,u)_s \geq -\|x\|\,\|u\| \text{ for all } x \in X.$$

From this we get

$$-\|u\| \leq \frac{f(x}{\|x\|} \leq \|u\| \text{ for all } x \in X$$

i.e., $\|f\| \leq \|u\|$.

On the other hand, we have

$$\|f\| \geq \frac{f(u)}{\|u\|} \geq \frac{(u,u)_i}{\|u\|} = \|u\|,$$

which shows us that

$$\|f\| = \|u\| = \frac{|f(w_0)|}{\|w_0\|}\,.$$

However, $f(w_0) = f(x_0)$ and then

$$\|f\| = \frac{|\ f(x_0)\ |}{\|x_0 - y_0\|} = \frac{f(x_0)\lambda_0}{\|x_0 - y_0\|},\ \text{i.e.,}\ f(x_0) = \lambda_0 \|f\|\,\|x_0 - y_0\|$$

which implies, by (2.6), that the estimation (2.1) holds.

(ii) $\Longrightarrow$ (i). Suppose that (2.1) holds for all $x \in X$. Then we get

$$\left(x, \frac{\lambda_0\,(x_0 - g_0)}{\|x_0 - g_0\|}\right)_i \leq 0 \leq \left(x, \frac{\lambda_0\,(x_0 - g_0)}{}\|x_0 - g_0\|\right)_s$$

for all $x \in \operatorname{Ker}(f)$, which gives, by (iv), that

$$\frac{\lambda_0\,(x_0 - g_0)}{\|x_0 - g_0\|} \perp \operatorname{Ker}(f)\,(B).$$

If $\lambda_0 > 0$, then, obviously, by the above relation we get that $(x_0 - g_0) \perp \operatorname{Ker}(f)\,(B)$, i.e., by Lemma 1 we get $g_0 \in \mathcal{P}_{\operatorname{Ker}(f)}(x_0)$.

If $\lambda_0 < 0$, then also $-(x_0 - g_0) \perp \operatorname{Ker}(f)(B)$ or $(x_0 - g_0) \perp (-\operatorname{Ker}(f))(B)$ and since $-\operatorname{Ker}(f) = \operatorname{Ker}(f)$, we obtain $g_0 \in \mathcal{P}_{\operatorname{Ker}(f)}(x_0)$, and the proof of the lemma is complete. □

The following corollary is important as it gives a criterion of representation for the continuous linear functionals in terms of semi-inner products $(\cdot,\cdot)_{i(s)}$.

Corollary 1. *With the above assumptions and if $x_0 - g_0$ is a point of smoothness of the normed space X, then $g_0 \in \mathcal{P}_G(x_0)$ if and only if for every $f \in G^*_{x_0}$ with* $\mathrm{Ker}(f) = G$, *one has the representation*

$$f(x) = \|f\|_{G_{x_0}} \left(x, \frac{\lambda_0 (x_0 - g_0)}{\|x_0 - g_0\|} \right)_p$$

for all $x \in G_{x_0}$, $p \in \{s, i\}$.

The following theorem contains a variational characterization of best approximation element.

Theorem 4. *Let $(X, \|\cdot\|)$ be a real normed space and G a closed linear subspace in X with $G \neq X$ and $x_0 \in X \setminus G$, $g_0 \in G$. The following statements are equivalent:*

(i) *$g_0 \in \mathcal{P}_G(x_0)$;*

(ii) *For every $f \in G^*_{x_0}$ with* $\mathrm{Ker}\,(f) = G$, *the element*

$$u_0 := \frac{f\,(x_0)\,(x_0 - g_0)}{\|x_0 - g_0\|^2}$$

minimizes the quadratic functional.

$$F_f : G_{x_0} \longrightarrow \mathbb{R}, \quad F_f\,(x) = \|x\|^2 - 2f\,(x)\,.$$

To prove this theorem we need the following lemma which is also interesting in itself.

Lemma 3. *Let $(X, \|\cdot\|)$ be a real normed space, $f \in X^* \setminus \{0\}$ and $w \in X \setminus \{0\}$. The following statements are equivalent:*

(i) *One has the estimation*

$$(x, w)_i \leq f(x) \leq (x, w)_s \quad \textit{for all } x \in X; \tag{2.7}$$

(ii) *The element w minimizes the quadratic functional*

$$F_f : X \longrightarrow \mathbb{R}, \quad F_f(u) := \|u\|^2 - 2f(u).$$

Proof. (i) $\Longrightarrow$ (ii). If w satisfies the relation (2.7), then for $x = w$, we obtain $f(w) = \|w\|^2$.

Now, let $u \in X$. Then

$$\begin{aligned} F_f(u) - F_f(w) &= \|u\|^2 - 2f(u) - \|w\|^2 + 2f(w) \\ &= \|u\|^2 - 2f(u) + \|w\|^2 \geq \|u\|^2 - 2(u,w)_s + \|w\|^2 \\ &\geq \|u\|^2 - 2\|u\|\,\|w\| + \|w\|^2 \\ &= (\|u\| - \|wt\|)^2 \geq 0, \end{aligned}$$

which shows that w minimizes the functional F_f.

(ii) $\Longrightarrow$ (i). If w minimizes the functional F_f, then for all $u \in X$ and $\lambda \in \mathbb{R}$ we have:

$$F_f(w + \lambda u) - F_f(w) \geq 0.$$

On the other hand, a simple calculation yields

$$F_f(w + \lambda u) - F_f(w) = \|w + \lambda u\|^2 - \|w\|^2 - 2\lambda f(u).$$

Thus we obtain

$$2\lambda f(u) \leq \|w + \lambda u\|^2 - \|w\|^2 \tag{2.8}$$

for all $u \in X$ and $\lambda \in \mathbb{R}$.

Now, let us assume that $\lambda > 0$. Then, by (2.8), we have

$$f(u) \leq \frac{\|w + \lambda u\|^2 - \|w\|^2}{2\lambda}, \quad u \in X,$$

which gives us for $\lambda \to 0+$ that $f(u) \leq (u,w)_s$ for all $u \in X$.

Putting $(-u)$ instead of u we get $f(u) \geq -(-u,w)_s = (u,w)_i$ for all $u \in X$, and the lemma is proved. $\square$

The above lemma gives us the following criterion of representation for the continuous linear functionals in normed linear spaces.

Corollary 2. *Let $(X, \|\cdot\|)$ be a real normed space and $f \in X^* \setminus \{0\}$, $w \in X \setminus \{0\}$. Then w is a point of smoothness of X and it minimizes the functional F_f if and only if one has the representation*

$$f(x) = (x,w)_p \qquad \text{for all } x \in X,$$

where $p = s$ or $p = i$.

Proof of Theorem 4. (i) $\Longrightarrow$ (ii). If $g_0 \in \mathcal{P}_G(x_0)$, then by Theorem 3 we deduce that for every $f \in G_{x_0}^*$ with $\operatorname{Ker}(f) = G$ one has the estimation (2.1). If in this relation we put $x = \dfrac{\lambda_0(x_0 - g_0}{\|x_0 - g_0\|}$, a simple calculation gives us

$$\|f\|_{G_{x_0}} = \frac{|f(x_0)|}{\|x_0 - g_0\|},$$

and then (2.1) becomes

$$\left(x, \frac{f(x_0)(x_0 - g_0)}{\|x_0 - g_0\|^2}\right)_i \le f(x) \le \left(x, \frac{f(x_0)(x_0 - g_0)}{\|x_0 - g_0\|^2}\right)_s \tag{2.9}$$

for all $x \in G_{x_0}$.

Now if we apply Lemma 3 for $u_0 = \dfrac{f(x_0)(x_0 - g_0)}{\|x_0 - g_0\|^2}$ on the space G_{x_0} we may conclude that u_0 minimizes the quadratic functional F_f on the space G_{x_0}.

(ii) $\Longrightarrow$ (i). If u_0 given above minimizes the functional F_f on G_{x_0}, then, by Lemma 3, we derive that the estimation (2.7) holds. Furthermore, the interpolation (2.1) is valid, i.e., by Theorem 3, we get that $g_0 \in \mathcal{P}_G$, and the proof is completed. □

References

1. S. S. Dragomir, *A characterization of best approximation elements in real normed spaces*, Studia Univ. "Babes-Bolyai"-Mathematica (Cluj-Napoca) **33** (3) (1988), 74-80.
2. S. S. Dragomir, *On best approximation in sense of Lumer and application*, Riv. Mat. Parma **15** (1989), 253-263.
3. S. S. Dragomir, *Characterizations of proximal, semichebychevian and chebychevian subspaces in real normed spaces*, Num. Funct. Anal. and Optim. **12** (506) (1991), 487-492.
4. I. Singer, *Best Approximation in Normed Linear Spaces by Elements of Linear Subspaces* (Romanian), Ed. Acad. Bucharest, 1967.

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