ACTA MATHEMATICA VIETNAMICA
Volume 25, Number 3, 2000, pp. 301–313

THE STRUCTURE OF TYPE (Ω) OF SPACES OF BANACH-VALUED HOLOMORPHIC GERMS

NGUYEN VAN KHUE AND NGUYEN DINH LAN

Abstract. This paper gives a sufficient condition which implies the property of type (Ω) of $[\mathcal{H}(K,X)]'$, where $\mathcal{H}(K,X)$ is the space of germs of X-valued holomorphic functions on K.

Introduction

Let K be a compact subset in a Fréchet space E and X a Banach space. By $\mathcal{H}(K,X)$ we denote the space of germs of X-valued holomorphic functions on K. Write $\mathcal{H}(K)$ for $\mathcal{H}(K,C)$. The space $\mathcal{H}(K,X)$, in particular $\mathcal{H}(K)$, is an important subject of infinite dimensional holomorphy. Its structure was investigated by several authors. In [6], Meise and Vogt have shown that $[\mathcal{H}(K)]'$ has (Ω) for every compact subset K in a nuclear Fréchet space E as long as E has (Ω). Recently, this result has been extended to the general case where E is only Fréchet by Nguyen Van Khue and Phan Thien Danh [10]. Earlier, Mujica in [9] has proved that $\mathcal{H}(K)$ is the dual of a quasinormable Fréchet space. The quasinormability of $[\mathcal{H}(K)]'$ was proved in [12]. Hence, by Meise and Vogt [7], $[\mathcal{H}(K)]'$ has (Ω_φ). Several results on quasinormality of spaces of holomorphic functions have appeared recently in [1], [2], [4], etc. However, the quasinormability of $[\mathcal{H}(K,X)]'$, where X is an arbitrary Banach space, cannot be deduced immediately from that of $[\mathcal{H}(K)]'$, which occures only when K is a compact subset of a quasinormable Fréchet space [10]. The main reason is that there is no commutative relation between the inductive tensor product and the inductive limit of a sequence of Banach spaces.

The main aim of the present paper is to study the properties of type (Ω) for $[\mathcal{H}(K,X)]'$. Namely, in Section 2 and Section 3 we extend the results of [9], [12] to $[\mathcal{H}(K,X)]'$, where X is an arbitrary Banach space (Theorem 2.1 and Theorem 3.1). For the proof of Theorem 2.1, we need a characterization of the quasinormability, which is similar to a case of type ($\bar{\Omega}$) in [13]. The proofs of Theorem 2.1 and 3.1 are presented in Section 2

Received August 10, 1999; in revised form January 26, 2000.

and Section 3, respectively. Finally, an application of the quasinormability is given in Section 4.

1. Basic notions

In this article we shall make use of the properties of holomorphic functions on locally convex spaces as in Dineen [4] and the properties of quasinormable spaces as in Meise and Vogt [8].

1.1. A locally convex space E is called *quasinormable* if for every 0-neighbourhood U there exists a 0-neighbourhood V such that for every $\varepsilon > 0$ there exists a bounded set M in E for which $V \subset M + \varepsilon U$.

In [7] Meise and Vogt have proved that a Fréchet space E is quasinormable if and only if E has (Ω_φ). This means that there exists a strictly increasing function $\varphi : (0; +\infty) \longrightarrow (0; +\infty)$ such that for some balanced convex neighbourhood basis $\{U_q\}$ of $0 \in E$, the following holds

$$(\Omega_\varphi) \qquad \forall p\ \exists q\ \ \forall k\ \exists C > 0\ U_q \subseteq\ C\varphi(r)U_k + \frac{1}{r}U_p\,, \quad \forall r > 0.$$

By polarization it is easy to see that (Ω_φ) is equivalent to

$$(\Omega_\varphi)^\circ \qquad \forall p\ \exists q\ \ \forall k\ \exists C > 0 \quad \|\bullet\|_q^* \leq\ C\varphi(r)\,\|\bullet\|_k^* + \frac{1}{r}\,\|\bullet\|_p^*\,, \quad \forall r > 0$$

where $\|u\|_q^* = \sup\{\,|u(x)|\ :\ x \in U_q\,\}$ for $u \in E'$, the dual space of E.

If (Ω_φ) is replaced by

$$(\Omega) \qquad \forall p\ \exists q\ \ \forall k\ \exists C, d > 0 \quad U_q \subseteq\ Cr^dU_k + \frac{1}{r}U_p\,, \quad \forall r > 0,$$

we say that E has the property (Ω).

1.2. Let E and F be locally convex spaces and U an open subset in E. A function $f\ :\ U \longrightarrow F$ is called holomorphic if f is continuous and $u \circ f$ is Gâteaux holomorphic for every $u \in F'$. By $\mathcal{H}(U, F)$ we denote the space of F-valued holomorphic functions on U, equipped with the open-compact topology. Instead of $\mathcal{H}(U, C)$ we write $\mathcal{H}(U)$.

Now assume that K is a compact subset in E. A function $f : K \longrightarrow F$ is said to be holomorphic if there exists a F-valued holomorphic function $\hat{f}$ on a neighbourhood U of K such that $\hat{f}|_K = f$.

On $\bigcup\{\mathcal{H}(U, F) : U \supset K$ and U is open$\}$ we define the canonical equivalence. We denote by $\mathcal{H}(K, F)$ the resulting vector space of equivalent classes and the elements $f \in \mathcal{H}(K, F)$ are called holomorphic germs on K. Write $\mathcal{H}(K)$ for $\mathcal{H}(K, C)$. The space $\mathcal{H}(K, F)$ is equipped with the inductive limit topology

$$\mathcal{H}(K, F) = \liminf_{U \supset K, U\ open} [\mathcal{H}(U, F), \tau_\omega]$$

where τ_ω denotes the Nachbin topology on $\mathcal{H}(U, F)$, i.e. the topology on $\mathcal{H}(U, F)$ generated by seminorms ρ satisfying the following property:
There exist a continuous seminorm α on F and a compact subset $A \subset U$ such that for every neighbourhood V of A in U there exists $C > 0$ for which

$$\rho(f) \leq C \sup_{z \in U} \alpha(f(z))$$

for every $f \in \mathcal{H}(U, F)$.

Note that if F is a Banach space, then

$$\mathcal{H}(K, F) = \liminf_{U \supset K, U\ open} H^\infty(U, F),$$

where $H^\infty(U, F)$ is the Banach space of F-valued bounded holomorphic functions on U.

2. The quasinormability of $[\mathcal{H}(K, X)]'$

In this section we prove the following

Theorem 2.1. *Let K be a compact subset in a quasinormable Fréchet space and X a Banach space. Then $[\mathcal{H}(K, X)]'$ is quasinormable.*

We need the following characterization of quasinormable spaces, which is similar to the one of Fréchet spaces having $(\bar{\Omega})$ in [13].

Proposition 2.2. *Let E be a Fréchet space. Then E is quasinormable if and only if there exist a strictly increasing function $\varphi : (0; +\infty) \longrightarrow (0; +\infty)$ and a bounded subset B in E such that*

$$\forall p\ \exists q,\ C > 0\ \forall r > 0 \quad U_q \subseteq\ C\varphi(r)B + \frac{1}{r}U_p. \tag{1}$$

Proof. The sufficiency is obvious. Conversely, assume that E is quasinormable.

By [7] there exist a Banach space X and a nuclear Fréchet space F such that E is a quotient space of $X\hat{\otimes}_\pi F$. Assume $R : X\hat{\otimes}_\pi F \longrightarrow E$ is the quotient map. Given U a neighbourhood of $0 \in E$. Choose a balanced convex neighbourhood $\tilde{U}$ of $0 \in F$ such that $\overline{conv}(W \otimes \tilde{U}) \subset R^{-1}(U)$, where W denotes the unit ball of X. Since F is nuclear, we can find a balanced convex neighbourhood $\tilde{V}$ of $0 \in F$ such that $\tilde{V}$ is relatively compact for the seminorm generated by $\tilde{U}$.

Let $\varepsilon > 0$ be given. Take a finite subset $\tilde{M}$ in $\tilde{V}$ such that $\tilde{V} \subset \tilde{M} + \frac{\varepsilon}{2}\tilde{U}$. Obviously $R\left(\overline{conv}(W \otimes \tilde{M})\right)$ is bounded in $R\left(\overline{conv}(W \otimes \tilde{V})\right)$, a neighbourhood of $0 \in E$. We have

$$R\left(\overline{conv}(W \otimes \tilde{V})\right) \subseteq R\left(\overline{conv}(W \otimes \tilde{M})\right) + \frac{\varepsilon}{2} R\left(\overline{conv}(W \otimes \tilde{U})\right)$$
$$\subseteq R\left(\overline{conv}(W \otimes \tilde{M})\right) + \varepsilon U.$$

By Meise and Vogt [7] there exists $\varphi \in M$, the set of strictly increasing functions $\psi : (0; +\infty) \longrightarrow (0; +\infty)$, such that

$$\forall p\ \exists q_{(p)}\ \forall k\ \exists C_k > 0\ \forall r > 0 \quad U_{q_{(p)}} \subseteq C_k \psi(r) U_k + \frac{1}{r} U_p. \tag{2}$$

On the other hand, without loss of generality we may assume that $q(p)$ satisfies the condition:

$\forall \varepsilon > 0\ \exists$ a bounded set $B \subset U_{q(p)}$ such that $U_{q(p)} \subset B + \varepsilon U_p \subseteq B + \varepsilon V_p$.

Fix $p = 1$. Put $V_1 = U_1$ and $V_2 = U_{q_{(1)}}$. Write (2) in the form

$$V_2 \subseteq\ C_k \varphi(r) U_k + \frac{1}{r} V_1 \quad \forall r > 0 \ \ \forall k \geq 1.$$

For each $k \geq q(1)$, we can find a bounded set $B_k^{(1)}$ in U_k such that

$$U_k \subseteq B_k^{(1)} + \frac{1}{k\varphi_{(k)}} V_1.$$

Hence

$$V_2 \subseteq\ C_k \varphi(k) B_k^{(1)} + \frac{1}{k} V_1 \quad \forall k \geq q(1).$$

Put $B_1 = \bigcup_k B_k^{(1)}$. It follows that B_1 is a bounded set in V_1 and

$$V_2 \subseteq\ C_k \varphi(k) B_1 + \frac{1}{k} V_1 \quad \forall k \geq q(1).$$

Choose $\psi_1 \in M$ such that

$$\psi_1(r) \geq C_{k+1}\varphi(r)$$

for $k \leq r < k+1$. Then

$$\begin{aligned} V_2 &\subseteq C_{k+1}\varphi(k+1)B_1 + \frac{1}{k+1}V_1 \\ &\subseteq \psi_1(r)B_1 + \frac{1}{r}V_1 \end{aligned}$$

for $k \leq r < k+1$.

Applying (2) to $U_k \subset V_2$ as above, we can find a neighbourhood $V_3 \subset V_2$, $\psi_2 \in M$ and a bounded set $B_2 \subset V_2$ such that

$$V_3 \subseteq \psi_2(r)B_2 + \frac{1}{r}V_2$$

for $r \geq 1$.

Continuing this process we get a neighbourhood basis $\{V_k\}$ of $0 \in E$, a sequence $\{\psi_k\} \subset M$ and a sequence of bounded sets $\{B_k \subset V_k\}$ such that

$$V_{k+1} \subseteq \psi_k(r)B_k + \frac{1}{r}V_k \tag{3}$$

for $r \geq 1$.

Let $B = \bigcup_{k\geq 1} B_k$ and $\psi \in M$ such that

$$\mathcal{C}_k := \sup_{r\geq 1} \frac{\psi_k(r)}{\psi(r)} < \infty$$

for $k \geq 1$. Then, by (3) we have

$$V_{k+1} \subseteq C_k\psi(r)B + \frac{1}{r}V_k \tag{4}$$

for $r \geq 1$, where $B = \bigcup_{k\geq 1} B_k$ is a bounded set in E. The proposition is proved.

Now we may prove Theorem 2.1 as follows.

Proof of Theorem 2.1. Let $\{U_q\}$ be a decreasing neighbourhood basis of K in E. Since E is quasinormable, so is $[\mathcal{H}(K)]'$ [9], [12]. By applying Proposition 2.2 in the polarization form we can find a bounded set B in $[\mathcal{H}(K)]'$ and a strictly increasing function $\varphi : (0; +\infty) \longrightarrow (0; +\infty)$ such that

$$\forall p\ \exists q,\ C > 0\ \forall r > 0\ \forall f \in \mathcal{H}(K) \subset [\mathcal{H}(K)]'',$$

$$\|f\|_{W_q^\circ} \leq C\varphi(r)\,\|f\|_B + \frac{1}{r}\,\|f\|_{W_p^\circ}\,,$$

where W_q denotes the unit ball in $H^\infty(U_q)$ and

$$\|f\|_{W_q^\circ} = \sup\left\{|\mu(f)|\ \ :\ \ \mu \in W_q^\circ\right\}.$$

Note that by the Hahn Banach theorem we have

$$\|f\|_{W_q^0} = \|f\|_{U_q} := \sup\left\{|f(z)| : z \in U_q\right\}$$

Let $\widetilde{B}$ be the subset of $[\mathcal{H}(K,X)]'$ consisting of elements of the forms μ_{x^*} with $x^* \in X'$, $\mu \in B$ such that

$$\mu_{x^*}(f) = \mu(x^* f)$$

for every $f \in \mathcal{H}(K,X)$. Then, if $\widetilde{W_q}$ is the unit ball of $H^\infty(U_q,X)$, we have

$$\begin{aligned}
\|f\|_{\widetilde{W}_q^\circ} &= \sup_{z\in U_q}\|f(z)\| = \sup_{||x^*||\leq 1}\|x^* f\|_{W_q^\circ}\\
&\leq C\varphi(r)\sup_{||x^*||\leq 1}\|x^* f\|_B + \frac{1}{r}\sup_{||x^*||\leq 1}\|x^* f\|_{W_p^\circ}\\
&= C\varphi(r)\,\|f\|_{\widetilde{B}} + \frac{1}{r}\,\|f\|_{\widetilde{W}_p^\circ}
\end{aligned}$$

for every $r > 0$ and every $f \in \mathcal{H}(K,X)$. By the $\sigma\,[[\mathcal{H}(K,X)]', \mathcal{H}(K,X)]$-compactness of $\widetilde{B}^{\circ\circ}$, this yields that

$$\widetilde{W}_q^\circ \subseteq C\varphi(r)\widetilde{B}^{\circ\circ} + \frac{1}{r}\widetilde{W}_p^\circ \quad \forall r > 0.$$

Consequently, $[\mathcal{H}(K,X)]'$ is quasinormable. The proof of Theorem 2.1 is now complete.

3. The property (Ω) of $[\mathcal{H}(K, X)]'$

In this section we prove the following

Theorem 3.1. *Let E be a Fréchet space having (Ω) and K a compact subset in E. Then, $[\mathcal{H}(K, X)]'$ has (Ω) for every Banach space X.*

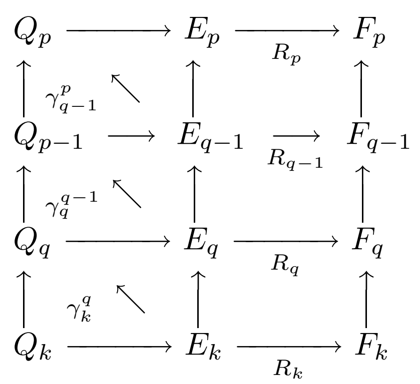
For the proof of Theorem 3.1 we need the following two lemmas

Lemma 3.2. *Let $E = \lim proj\, E_k$ and $F = \lim proj\, F_k$ be Fréchet spaces and let $R : E \longrightarrow F$ be the continuous linear map induced by continuous linear surjections $R_k : E_k \longrightarrow F_k$. Assume that E can be written in an other form $E = \lim proj\, Q_k$ such that E is dense in Q_k for $k \geq 1$ and the projective spectrum $\{E_k\}$ is equivalent to $\{Q_k\}$. Then, F has (Ω) if E has (Ω).*

Proof. Choose a balanced convex neighbourhood basis $\{W_k\}$ of $0 \in E$ such that $Q_k = E(W_k)$, the Banach space associated to W_k, for $k \geq 1$. Given $p \geq 1$. Since E has (Ω), we have

$$\exists q\ \forall k\ \exists C, d > 0 : \|\cdot\|^{*1+d}_{W_{q-1}} \leq C \|\cdot\|^{*}_{W_k} \|\cdot\|^{*d}_{W_p}$$

Since the projective spectrums $\{Q_k\}$ and $\{E_k\}$ are equivalent, without loss of generality we may assume that there exists the following diagram



Moreover, we may assume that

$$R(W_k) \subset V_k, \quad R_k(\hat{U}_k) \subset \hat{V}_k \quad \text{and} \quad \left(\gamma^q_k\right)^{-1}(W_q) \supset \hat{U}_k$$

for $k \geq q \geq p$, where $\hat{U}_k$ and $\hat{V}_k$ are the unit balles of E_k and F_k respectively.

Given $v \in F'$, $\|v\|^*_{V_p} < +\infty$. Choose $\hat{v} \in F'_q$ such that $\hat{v}$ is an extension of v and $\|\hat{v}\|^*_{\hat{V}_q} = \|v\|^*_{V_q}$. We have

$$\begin{aligned}\|v\|^*_{V_q} &= \|\hat{v}\|^*_{\hat{V}_q} = \|\hat{v}R_q\|^*_{\hat{U}_q} \le \|\hat{v}R_{q-1}\|^*_{W_{q-1}} \\ &\le C^{1/1+d}\|\hat{v}R_{q-1}\|^{*\ 1/1+d}_{W_k}\|\hat{v}R_{q-1}\|^{*d/1+d}_{W_p} \\ &\le C^{1/1+d}\|v\|^{*1/1+d}_{V_k}\|v\|^{*d/1+d}_{V_p}.\end{aligned}$$

Hence the lemma is proved.

Lemma 3.3. *Let E be a Frechet space with the property (Ω). Then there exists a fundamental systems of seminorms $\{\|\cdot\|_k\}$ of E such that the image of every canonical map $E'' \to E''_k$ is dense for all $k \ge 1$. Here E_k denotes the Banach space associated to $\|\cdot\|_k$.*

Proof. By Vogt [14] there exists a Banach space B and a continuous linear surjection $R : B\widehat{\otimes}_\pi s \to E$, where s is the space of the rapidly decreasing sequences. Moreover, by [10] R can be chosen such that $R' : E' \to (B\widehat{\otimes}_\pi s)'$ is an embedding. Assume that $W_k = \overline{conv}(U \otimes U_k)$, $k \ge 1$ where U is the unit ball of B and

$$U_k = \Big\{x = (x_j) \in S : \|x\|_k := \sum_{j=1}^{\infty} |x_j| j^k \le 1\Big\}.$$

Then $\{W_k\}$ forms a neighbourhood basis of $0 \in B \otimes_\pi s$ and hence so is $\{V_k = R(W_k)\}$ for $0 \in E$. Note that R induce continuous linear surjections

$$R_k : \big(B\widehat{\otimes}_\pi s)_{W_k} \to E_{V_k},$$

where $\big(B\widehat{\otimes}_\pi s\big)_{W_k}$ and E_{V_k} are Banach spaces associated to W_k and V_k, respectively.

Since s is nuclear, we have

$$\lim_k \mathrm{ind}\big(B\widehat{\otimes}_\pi s\big)'_{W_k} \cong \big(B\widehat{\otimes}_\pi s\big)' \cong B'\widehat{\otimes}_\pi s' \cong B'\widehat{\otimes}_\pi\big(\lim_k \mathrm{ind} s'_k\big)$$

$$\cong B'\widehat{\otimes}_\varepsilon\big(\lim_k \mathrm{ind} s'_k\big) \cong \lim_k \mathrm{ind}\big(B'\widehat{\otimes}_\varepsilon s'_k\big) \cong \lim_k \mathrm{ind}\big(B'\widehat{\otimes}_\pi s'_k\big),$$

where the last isomorphism follows from the nuclearity of the canonical maps $s'_k \to s'_{k+1}$. Thus, by the surjectivity of $R''_k : \big(B\widehat{\otimes}_\pi s\big)''_{W_k} \to E''_k$, it suffices to check that $\big(B'\widehat{\otimes}_\pi s'_k\big)'$ is dense in $\big(B'\widehat{\otimes}_\pi s'_{k-1}\big)'$ for the topology of $\big(B'\widehat{\otimes}_\pi s_{k-2}\big)'$ for $k \geq 3$. But this follows from the relations

$$\big(B'\widehat{\otimes}_\pi s'_k\big)' \cong \mathcal{L}(B', s_k)$$

and

$$\sum_{j=1}^{\infty} \|e_j^*\|_k \|e_j\|_{k-2} = \sum_{j=1}^{\infty} \frac{1}{j^2} < \infty.$$

On the other hand, since $R''_k : B''\hat{\otimes}_\pi s_k \longrightarrow E''_k$ is surjective and $B''\hat{\otimes}_\pi s$ is dense in $B''\hat{\otimes}_\pi s_k$, it follows that E'' is dense in E''_k for $k \geq 1$. This proves Lemma 3.3. □

Now we are able to prove Theorem 3.1.

Proof of Theorem 3.1. Assume that X is complemented in X''. Then so is $\mathcal{H}(K, X)$ in $\mathcal{H}(K, X'')$. Thus, it remains to show that $[\mathcal{H}(K, X'')]'$ has (Ω). By [3], there exists a Fréchet space $G(K) = \lim proj\, G_n$, where G_n are Banach spaces, such that

$$\mathcal{H}(K, X'') \cong \lim ind\, L(G_n, X'') \cong \big[\lim proj (G_n \hat{\otimes}_\pi X')\big]'_i,$$

the bornological dual of $\lim proj (G_n \hat{\otimes}_\pi X')$.

Let $\{U_n\}$ be a decreasing neighbourhood basis of K in E and let W_q and $\widetilde{W}_q$, for each $q \geq 1$, be the unit balls in $\mathcal{H}^\infty(U_q)$ and $\mathcal{H}^\infty(U_q, X'')$, respectively. Since $[\mathcal{H}(K)]'$ has (Ω) [10], we have

(1) $$\forall p \ \ \exists q \ \ \forall k \ \ \exists C,\, d > 0 \ \ \forall f \in \mathcal{H}(K) \subseteq [\mathcal{H}(K)]'',$$

$$\|f\|_{W_q^\circ}^{1+d} \leq C\, \|f\|_{W_k^\circ}\, \|f\|_{W_p^\circ}^d\ .$$

From the relations

$$\|f\|_{W_q^\circ} = \sup_{U_q} |f(z)|\,,$$

for $f \in \mathcal{H}(K)$, and

$$\|f\|_{\widetilde{W}_q^\circ} = \sup_{U_q} \|f(z)\| = \sup_{\|x^*\| \leq 1} \|x^* f\|_{W_q^\circ}$$

for $f \in \mathcal{H}(K, X'')$, it follows that (1) holds also for $\mathcal{H}(K, X'')$:

$$\forall p \ \ \exists q \ \ \forall k \ \ \exists C,\, d > 0 \quad \|f\|_{\widetilde{W}_q^\circ}^{1+d} \leq C\, \|f\|_{\widetilde{W}_k^\circ}\, \|f\|_{\widetilde{W}_p^\circ}^d$$

for $f \in \mathcal{H}(K, X'') \cong [\lim proj(G_n \hat{\otimes}_\pi X')]_i$. This means that $\lim proj(G_n \hat{\otimes}_\pi X')$ has (Ω) and hence,

$$[\mathcal{H}(K, X'')]' \cong [\lim proj(G_n \hat{\otimes}_\pi X')]''$$

also has (Ω).

Now we consider the general case. Let $\{V_n\}$ be a decreasing neighbourhood basis of K. For each $n \geq 1$, we denote by R_n the canonical map from $H^\infty(V_n, X)$ into $H^\infty(V_n, X'')$.

We have

$$[\mathcal{H}(K, X'')]' \cong \lim proj\, [H^\infty(V_n, X'')]'$$
$$[\mathcal{H}(K, X)]' \cong \lim proj\, [H^\infty(V_n, X)]'$$

If $R : \mathcal{H}(K, X) \longrightarrow \mathcal{H}(K, X'')$ is the canonical map, then $R' : \mathcal{H}(K, X'')]' \to [\mathcal{H}(K, X)]'$ is induced by the continuous linear surjections

$$R'_n : [H^\infty(V_n, X'')]' \longrightarrow [H^\infty(V_n, X)]'.$$

On the other hand, since $Q := \lim proj\, (G_n \hat{\otimes}_\pi X')$ has (Ω), we can write $Q = \lim proj\, Q_n$ such that Q'' is dense in Q''_n for $n \geq 1$. For each $n \geq 1$, consider the canonical map $Q'_n \longrightarrow \lim ind\, H^\infty(V_n, X'')$. Then there exists $\alpha(n) \geq n$ such that Q'_n is continuously mapped into $\mathcal{H}^\infty(V_{\alpha(n)}, X'')$. Similarly, we can find $\beta\alpha(n) \geq \alpha(n)$ such that $\mathcal{H}^\infty(V_{\alpha(n)}, X'')$ is continuously embedded into $Q'_{\beta\alpha(n)}$. This yields that the inductive spectrums $\{Q'_n\}$ and $\{H^\infty(V_n, X'')\}$ are equivalent. Hence the projective spectrums $\{Q''_n\}$ and $\{[H^\infty(V_n, X'')]'\}$ are also equivalent. As we have seen at the beginning of the proof, $[\mathcal{H}(K, X'')]' \in (\Omega)$, and Lemma 3.2 yields that $[\mathcal{H}(K, X)]' \in (\Omega)$.

The proof of Theorem 3.1 is complete.

4. An application of the quasinormability of $[\mathcal{H}(K, X)]'$

In this section we are interested in the problem whether $\mathcal{H}(K, X)$ is a closed subspace of $\mathcal{H}(K, Y)$ for every compact subset K in a Fréchet space E and every closed subspace X of a Banach space Y. Up to now, the completeness of $\mathcal{H}(K, Y)$ is an open problem. However, in [3] the authors have shown that $\mathcal{H}(K, Y)$ is complete in some cases, in particular, when Y is complemented in Y''. The above problem seems to be not easy. By applying Theorem 2.1, we prove the following.

Theorem 4.1. *Let E be a quasinormable Fréchet space with an absolute basis and X a closed subspace of a Banach space Y. Then $\mathcal{H}(O_E, X)$ is a closed subspace of $\mathcal{H}(O_E, Y)$, where O_E denotes the zero-element of E.*

Proof. Let Z be the quotient space Y/X with the quotient map $R : Y \longrightarrow Z$ and the embedding : $X \longrightarrow Y$. By Theorem 2.1, $[\mathcal{H}(O_E, Z)]'$ is quasinormable. Hence, by [8], it suffices to show that the sequence

$$O \longrightarrow [\mathcal{H}(O_E, Z)]' \xrightarrow{\hat{R}'} [\mathcal{H}(O_E, Y)]' \xrightarrow{\hat{S}'} [\mathcal{H}(O_E, X)]' \longrightarrow O,$$

is exact, where $\hat{R}$ and $\hat{S}$ are induced by R and S respectively.

Choose a balanced convex neighbourhood basis $\{U_n\}$ of $O_E \in E$ such that $3U_{n+1} \subseteq U_n$ for $n \geq 1$.

First we will show that $\hat{R} : \mathcal{H}(O_E, Y) \longrightarrow \mathcal{H}(O_E, Z)$ is surjective and hence, it is open [11].

Given $g \in \mathcal{H}(O_E, Z)$ Take $n \geq 1$ such that $g \in H^\infty(U_n, Z)$. Write the Taylor expansion of g at $O_E \in E$

$$g(z) = \sum_{k \geq 0} P_k g(z),$$

where

$$P_k g(z) = \frac{1}{2\pi i} \int\limits_{|\lambda|=1} \frac{g(\lambda z)}{\lambda^{k+1}} d\lambda \text{ and } z \in U_n.$$

We have

$$\begin{aligned} \|P_k g\|_{U_{n+1}} &= \sup_{z \in U_{n+1}} \left| \frac{1}{2\pi i} \int\limits_{|\lambda|=3} \frac{g(\lambda z)}{\lambda^{k+1}} d\lambda \right| \\ &\leq \left(\frac{1}{3}\right)^k \|g\|_{U_{n+1}} \end{aligned}$$

for $k \geq 0$.

For each k, let $\hat{P}_k g$ denote the continuous symmetric k-linear map associated to $P_k g$. Since E_n, the Banach space associated to U_n, is isomorphic to l^1 it is easy to see that for each $k \geq 0$ we can find a continuous symmetric k-linear map

$$\hat{Q}_k \ : \ \underbrace{E \times \cdots \times E}_{k} \longrightarrow Y$$

such that

$$Q_k(Rz) := \hat{Q}_k(\underbrace{Rz, \dots, Rz}_{k}) = P_k g(z)$$

for $z \in E$ and

$$\|Q_k\|_{U_{n+1}} := \sup_{U_{n+1}} \|Q_k(z)\| \leq C^k \left\|\hat{P}_k g\right\|_{\underbrace{U_{n+1} \times \cdots \times U_{n+1}}_{k}},$$

where $C > 1$ is chosen so that

$$\sum_{k \geq 0} \left(\tfrac{C}{3}\right)^k \frac{k^k}{k} < \infty.$$

Thus, the form

$$f(z) = \sum_{n \geq 0} Q_k(z),$$

for $z \in U_{n+1}$, defines $f \in \mathcal{H}(O_E, Y)$ such that $\hat{R}f = g$. Hence $\mathrm{Ker}\hat{R}' = O$ and $\mathrm{Ker}\hat{S}' = \mathrm{Im}\hat{R}'$.

By [8], to prove $\mathrm{Im}\hat{S}' = [\mathcal{H}(O_E, X)]'$ it remains to check that

$$\mathrm{Im}\left([H^\infty(U_{n+1}, Z)]' \longrightarrow [H^\infty(U_{n-1}, Z)]'\right)$$

is dense in

$$\mathrm{Im}\left([H^\infty(U_n, Z)]' \longrightarrow [H^\infty(U_{n-1}, Z)]'\right)$$

for $n \geq 2$.

Given $\mu \in \mathrm{Im}([H^\infty(U_n, Z)]' \longrightarrow [H^\infty(U_{n-1}, Z)]')$ and $\varepsilon > 0$. Take $k_\circ$ such that

$$\sum_{k > k_\circ} \left(\frac{1}{3}\right)^k < \varepsilon.$$

Put $\mu_\varepsilon(f) = \sum_{0 \leq k \leq k_\circ} \mu(P_k f)$ for $f \in H^\infty(U_{n+1}, Z)$. Then $\mu_\varepsilon \in [H^\infty(U_{n+1}, Z)]'$ and

$$\begin{aligned}
|\mu(g) - \mu_\varepsilon(g)| &= \left| \sum_{k \geq 0} \mu(P_k g) - \sum_{0 \leq k \leq k_\circ} \mu(P_k g) \right| \\
&\leq \sum_{k > k_\circ} |\mu(P_k g)| \\
&\leq \|\mu\| \, \|g\|_{U_{n-1}} \sum_{k > k_\circ} \left(\tfrac{1}{3}\right)^k \\
&\leq \varepsilon \, \|\mu\| \, \|g\|_{U_{n-1}}
\end{aligned}$$

for $g \in H^\infty(U_{n-1}, Z)$. Theorem 4.1 is now proved.

References

1. J. M. Ansemil, *On the quasi-normability of* $H_b(U)$, Extracta Math. **9**, (1) (1994), 71-74.
2. J. M. Ansemil, F. Blasco, S. Ponte, *Quasi-normability and topologies on spaces of polynomials*, J. Math. Anal. Appl. **213** (1997), 534-539.
3. J. Bonet, P. Domanski and J. Mujica, *Complete spaces of vector-valued holomorphic germs*, Math. Scand. **75** (1994), 150-160.
4. S. Dineen, *Quasinormable spaces of holomorphic functions*, Note Math. **12** (1) (1993), 155-195.
5. S. Dineen, *Complex Analysis on Infinite Dimensional Spaces*, Springer-Verlag London Berlin Heidelberg, 1999.
6. R. Meise and D. Vogt, *Structure of spaces of holomorphic functions on infinite dimensional polydiscs,* Studia Math. **75** (1983), 235-252.
7. R. Meise and D. Vogt, *A characterization of quasinormable Fréchet spaces,* Math. Nachr. **122** (1985), 141-150.
8. R. Meise and D. Vogt, *Introduction to Functional Analysis,* Clarendon Press-Oxford 1997.
9. J. Mujica, *A completeness criterion for inductive limits of Banach spaces, Functional analysis, Holomorphy and Approximation Theory II*, North-Holland Math. Stud. **86** (1984), 319-329.
10. Nguyen Van Khue and Phan Thien Danh, *Structure of spaces of germs of holomorphic functions*, Publ. Mat. **41** (1997), 467-480.
11. A. P. Robertson and W. Robertson, *Topological Vector Spaces,* Cambridge University Press 1964.
12. Nguyen Ha Thanh, *On spaces of germs of holomorphic functions on quasinormable spaces,* Publ. of Center of Functional and Complex Analysis **1** (1997), 143-150.
13. D. Vogt, *Eine Charakterisierung der Poterzreihenenräume von endlichen Typ und ihre Folgerungen,* Manuscript Math. **37** (1982), 269-301.
14. D. Vogt, *On two classes of* (F)*-spaces*, Arch. Math. **45** (1985), 255-266.

Department of Mathematics,
Pedagogical Institute Hanoi
Tu Liem, Hanoi, Vietnam

Department of Mathematics,
HoChiMinh city University of Education
280 An Duong Vuong , District 5,
HoChiMinh City, Vietnam
E-mail: ngdlanesed@hcm.vnn.vn