ACTA MATHEMATICA VIETNAMICA
Volume 25, Number 3, 2000, pp. 291–299

GROWTH OF MEROMORPHIC FUNCTIONS

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Abstract. In this paper, we prove a general result concerning the growth of meromorphic functions. Our condition is much weaker than that used by Sing-Patil. Our result contains and improves the results of Nevanlinna, Clunie and Goldberg-Ostrovoskii. As an application, we solve a problem proposed by C. C. Yang.

1. Introduction

This paper concerns the following results.

Theorem A (Nevanlinna [3]). *Let $f(z)$ be a meromorphic function of order ρ ($\rho < \infty$) and lower order λ, if $\rho - \lambda < 1$, then*

$$T(r+1, f) \sim T(r, f) \qquad (r \to \infty).$$

Theorem B (Clunie [1]). *Let $f(z)$ be an entire function of order ρ ($\rho < \infty$). If $k > \rho - 1$, then*

$$\log M(r, f) \sim \log M(r - r^{-k}, f) \qquad (r \to \infty).$$

Theorem C (Goldberg and Ostrovoskii [2]). *Let $f(z)$ be a meromorphic function of order ρ ($\rho < \infty$) and lower order λ, if $\rho - \lambda < 1$, then for all constants k, we have*

$$T(r + k \log r, f) \sim T(r, f) \qquad (r \to \infty).$$

Theorem D (Singh and Patil [4]). *Let $f(z)$ be a meromorphic function of order ρ ($\rho < \infty$) and lower order λ. Let E be the set of those r for which*

Received May 19, 1999.
1991 Mathematics Subject Classification. 30D35.
Key words and phrases. Meromorphic function, Order, Lower order, Growth.

$T'(r,f)$ fails to exist and let E be bounded. Let $\phi(r)$ and $\psi(r)$ be two continuous positive increasing functions of r such that

1. $\lim_{r\to\infty} \psi(r)/\phi(r) = 0$,
2. $\lim_{r\to\infty} \log\psi(r)/\log\phi(r) = \alpha \ (\alpha \geq 0)$.

If $\rho - \lambda < 1 - \alpha$, then

$$T(\phi(r) \pm \psi(r), f) \sim T(\phi(r), f) \qquad (r \to \infty).$$

We will omit the differentiability of $T(r,f)$ and $\alpha \geq 0$ in Theorem D and obtain the following result:

Theorem 1. *Let $f(z)$ be a meromorphic function of order ρ ($\rho < \infty$) and lower order λ. Let $\phi(r)$ and $\psi(r)$ be two positive functions, and $\phi(r)$ be continuous and tend to ∞, and $\phi(r)$,$\psi(r)$ satisfy*

1. $\lim_{r\to\infty} \psi(r)/\phi(r) = 0$;
2. $\lim_{r\to\infty} \log\psi(r)/\log\phi(r) = \alpha \ (\alpha < 1)$.

If $\rho - \lambda < 1 - \alpha$, then

1. $T(\phi(r) \pm \psi(r), f) \sim T(\phi(r), f) \qquad (r \to \infty)$;
2. *when $f(z)$ is an entire function, we have*

$$\log M(\phi(r) \pm \psi(r), f) \sim \log M(\phi(r), f) \qquad (r \to \infty).$$

Theorem 1 not only generalizes Theorem D but also contains and improves Theorem A, Theorem B and Theorem C. The fact is:

(1) Take $\phi(r) = r$ and $\psi(r) = 1$, we get

$$\lim_{r\to\infty} \psi(r)/\phi(r) = 0,$$

and

$$\lim_{r\to\infty} \log\psi(r)/\log\phi(r) = 0 = \alpha.$$

Hence by Theorem 1, if $\rho - \lambda < 1 - \alpha$, that is $\rho - \lambda < 1$, we obtain

$$T(r+1, f) \sim T(r, f) \qquad (r \to \infty).$$

This is the result of Theorem A.

(2) Take $\phi(r) = r$ and $\psi(r) = k \log r$, we have

$$\lim_{r\to\infty} \psi(r)/\phi(r) = 0,$$

and

$$\lim_{r\to\infty} \log \psi(r)/\log \phi(r) = \lim_{r\to\infty} \frac{\log k + \log\log r}{\log r} = 0 = \alpha.$$

Thus by Theorem 1, if $\rho - \lambda < 1 - \alpha$, that is $\rho - \lambda < 1$, we see that

$$T(r + k \log r, f) \sim T(r, f) \qquad (r \to \infty).$$

This is the result of Theorem C.

(3) Take $\phi(r) = r$ and $\psi(r) = r^{-k}(k > 0)$, then we get

$$\lim_{r\to\infty} \psi(r)/\phi(r) = 0,$$

and

$$\lim_{r\to\infty} \log \psi(r)/\log \phi(r) = \lim_{r\to\infty} \frac{-k\log r}{\log r} = -k = \alpha \ (< 0).$$

So, if $k > \rho - 1$, that is $\rho - \lambda < 1 + k = 1 - \alpha$, by Theorem 1 we have for entire function f

$$\log M(r - r^{-k}, f) \sim \log M(r, f) \qquad (r \to \infty).$$

This is Theorem B.

Applying Theorem 1 we obtain following result:

Theorem 2. *Let $f(z)$ be a meromorphic function of order ρ ($\rho < \infty$) and lower order λ, let $P(z)$ and $Q(z)$ be two polynomials with* $\deg P = m > \deg Q$. *If $\rho - \lambda < 1/m$, then*

$$T(r, f(P(z) + Q(z))) \sim T(r, f(P(z))) \qquad (r \to \infty).$$

Furthermore, C. C. Yang posed the following problem in [5]:

Let f be a meromorphic function , if

$$\lim_{r\to\infty} \frac{T(r, f(z+1))}{T(r, f(z))} = \infty$$

can one prove that the order $\rho_f = \infty$ or furthermore, lower order $\lambda_f = \infty$?

From Theorem 2, we have the following result

Corollary 1. *Let $f(z)$ be a meromorphic function of order ρ and lower order λ, let $P(z)$ and $Q(z)$ be two polynomials with* $\deg P = m > \deg Q$, *and $\rho - \lambda < 1/m$. If*

$$\lim_{r\to\infty} \frac{T(r, f(P+Q))}{T(r, f(P))} = \infty.$$

Then

$$\rho_f = \lambda_f = \infty.$$

Remark. If we take $P(z) = z$, $Q(z) = 1$, then we get the answer to problem of C. C. Yang as follows.

Corollary 2. *Let f be a meromorphic function of order ρ and lower order λ with $\rho - \lambda < 1$. If*

$$\lim_{r\to\infty} \frac{T(r, f(z+1))}{T(r, f)} = \infty.$$

Then

$$\rho_f = \lambda_f = \infty.$$

2. Proof of Theorem 1

2.1. We first prove that $T(\phi(r) + \psi(r), f) \sim T(\phi(r), f)(r \to \infty)$ holds.

Since $T(r, f)$ is a nondecreasing continuous function of r, we can get

$$\varliminf_{r\to\infty} \frac{T(\phi(r) + \psi(r), f)}{T(\phi(r), f)} \geq 1. \tag{1}$$

Thus, we need only to show that

$$\varlimsup_{r\to\infty} \frac{T(\phi(r) + \psi(r), f)}{T(\phi(r), f)} \leq 1.$$

By our assumptions, there exists $r_0 > 0$, such that for $r > r_0$ we have

$$\psi(r) < \phi(r) < \phi(r) + \psi(r) < 2\phi(r).$$

Thus

$$\begin{aligned}(2)\qquad &\log\psi(r)<\log\phi(r)<\log(\phi(r)+\psi(r))\\ &\qquad=\log\phi(r)+\log(1+\psi(r)/\phi(r))<\log 2\phi(r).\end{aligned}$$

Let $t=\log R$ and denote $F(t)=T(R,f)$. Then F is a positive nondecreasing convex function of $\log r$, by (2) we obtain

$$\begin{aligned}&\frac{F(\log\phi(r)+\log(1+\psi(r)/\phi(r)))-F(\log\phi(r))}{(\log\phi(r)+\log(1+\psi(r)/\phi(r)))-\log\phi(r)}\\ &\leq\frac{F(\log 2\phi(r))-F(\log\phi(r))}{\log 2\phi(r)-\log\phi(r)}.\end{aligned}$$

Hence

$$\frac{F(\log(\phi(r)+\psi(r)))}{F(\log\phi(r))}\leq 1+\frac{\log(1+\psi(r)/\phi(r))}{\log 2}\frac{F(\log 2\phi(r))-F(\log\phi(r))}{F(\log\phi(r))}.$$

Thus, for arbitrary $\varepsilon>0$, there exists $r_1>r_0$, such that for $r>r_1$ we get

$$\begin{aligned}\frac{F(\log(\phi(r)+\psi(r)))}{F(\log\phi(r))}&\leq 1+\frac{1}{\log 2}\log(1+\psi(r)/\phi(r))\Big[\frac{(e^{\log 2\phi(r)})^{\rho+\varepsilon}}{(e^{\log\phi(r)})^{\lambda-\varepsilon}}-1\Big]\\ (3)\qquad&\leq 1+\frac{1}{\log 2}\frac{\psi(r)}{\phi(r)}\big[2^{\rho+\varepsilon}(\phi(r))^{\rho-\lambda+2\varepsilon}-1\big].\end{aligned}$$

Since $\lim\limits_{r\to\infty}\dfrac{\log\psi(r)}{\log\phi(r)}=\alpha(\alpha<1)$, we have $\psi(r)=(\phi(r))^{\alpha(1+\circ(1))}$.
So, by (3) we obtain

$$\begin{aligned}\frac{F(\log(\phi(r)+\psi(r)))}{F(\log\phi(r))}&\leq 1+\frac{1}{\log 2}\frac{(\phi(r))^{\alpha(1+\circ(1))}}{\phi(r)}\big[2^{\rho+\varepsilon}(\phi(r))^{\rho-\lambda+2\varepsilon}-1\big]\\ &=1+\frac{1}{\log 2}\Big[\frac{2^{\rho+\varepsilon}}{\phi(r)^{\varepsilon}}(\phi(r))^{\rho-\lambda+3\varepsilon+\alpha(1+\circ(1))-1}\\ (4)\qquad&\qquad-\frac{1}{\phi(r)^{1-\alpha(1+\circ(1))}}\Big].\end{aligned}$$

Since $\rho-\lambda<1-\alpha$, there exists $\varepsilon_0>0$ such that $\rho-\lambda+3\varepsilon_0<1-\alpha$.
Take $\varepsilon=\varepsilon_0$ we get

$$\rho-\lambda+3\varepsilon_0+\alpha(1+\circ(1))-1<1-\alpha+\alpha(1+\circ(1))-1\to 0(r\to\infty)$$

So, by (4) we have

$$\overline{\lim_{r\to\infty}}\frac{F(\log(\phi(r)+\psi(r))}{F(\log(\phi(r))}\leq 1$$

and so

$$\overline{\lim_{r\to\infty}}\frac{T(\phi(r)+\psi(r),f)}{T(\phi(r),f)}\leq 1. \tag{5}$$

Therefore, by (1) and (5) we obtain

$$T(\phi(r)+\psi(r),f)\sim T(\phi(r),f) \qquad (r\to\infty).$$

2.2. Since $T(\phi(r),f)=T((\phi(r)-\psi(r))+\psi(r),f)$ and $\dfrac{\psi(r)}{\phi(r)}\to 0$ $(r\to\infty)$ and $\dfrac{\log\psi(r)}{\log\phi(r)}\to\alpha$ $(r\to\infty)$. So we get

$$\frac{\psi(r)}{\phi(r)-\psi(r)}=\frac{\psi(r)}{\phi(r)(1-\psi(r)/\phi(r))}\to 0\ (r\to\infty),$$
$$\frac{\log\psi(r)}{\log(\phi(r)-\psi(r))}=\frac{\log\psi(r)}{\log\phi(r)+\log(1-\psi(r)/\phi(r))}\to\alpha\ (r\to\infty).$$

Thus, by 2.1 we see that

$$T(\phi(r),f)=T((\phi(r)-\psi(r))+\psi(r),f)\sim T(\phi(r)-\psi(r),f)(r\to\infty).$$

2.3. If $f(z)$ is an entire function, then $\log M(r,f)$ is a convex function of $\log r$. Similarly, we have

$$\log M(\phi(r)\pm\psi(r),f)\sim\log M(\phi(r),f) \qquad (r\to\infty).$$

This proves Theorem 1.

3. Proof of Theorem 2

(i) Let

$$P(z)=a_m z^m+a_{m-1}z^{m-1}+...+a_1z+a_0 \quad (a_m\neq 0),$$
$$Q(z)=b_n z^n+b_{n-1}z^{n-1}+...+b_1z+b_0 \quad (b_n\neq 0, n<m).$$

Then

$$P(z) + Q(z) = a_m z^m + c_{m-1} z^{m-1} + ... + c_1 z + c_0. \tag{6}$$

Write $\lambda = |a_m| > 0$ and choose μ_1 such that $\mu_1 > |a_{m-1}|$. Then there exists $r_0 > 0$, such that, for any value a and $r \geq r_0$

$$\begin{aligned} m \cdot n\Big(\lambda r^m - \mu_1 r^{m-1}, \frac{1}{f-a}\Big) &\leq n\Big(r, \frac{1}{f(P)-a}\Big) \\ &\leq m \cdot n\Big(\lambda r^m + \mu_1 r^{m-1}, \frac{1}{f-a}\Big). \end{aligned} \tag{7}$$

This implies that

$$\begin{aligned} N\Big(r, \frac{1}{f(P)-a}\Big) - N\Big(r_0, \frac{1}{f(P)-a}\Big) \geq \int_{r_0}^{r} & \frac{m \cdot n\Big(\lambda t^m - \mu_1 t^{m-1}, \frac{1}{f-a}\Big)}{t} dt \\ & + O(\log r). \end{aligned} \tag{8}$$

Put $s = \lambda t^m - \mu_1 t^{m-1}$, then $\dfrac{ds}{s} = \Big(1 + \dfrac{\mu_1}{m(\lambda t - \mu_1)}\Big) m \dfrac{dt}{t}\cdot$

Since $1 + \dfrac{\mu_1}{m(\lambda t - \mu_1)} \to 1$ $(t \to \infty)$, for arbitrarily small $\varepsilon_1 > 0$, we may assume that

$$1 + \frac{\mu_1}{m(\lambda t - \mu_1)} < \frac{1}{1 - \varepsilon_1/2}$$

for $t \geq r_0$. Thus

$$m \frac{dt}{t} \geq \Big(1 - \frac{\varepsilon_1}{2}\Big) \frac{ds}{s}, t \geq r_0. \tag{9}$$

Now, we take a such that $f(z) - a$ has infinitely many zeros. Then for sufficiently large $r \geq r_0$, from (8) and (9) we deduce that

$$N\Big(r, \frac{1}{f(P)-a}\Big) \geq (1 - \varepsilon_1) N\Big(\lambda r^m - \mu_1 r^{m-1}, \frac{1}{f-a}\Big) + O(\log r). \tag{10}$$

Similarly,

$$N\Big(r, \frac{1}{f(P)-a}\Big) \leq (1 + \varepsilon_1) N\Big(\lambda r^m + \mu_1 r^{m-1}, \frac{1}{f-a}\Big) + O(\log r). \tag{11}$$

By Nevanlinna Theory, we can take two suitable values b_1 and b_2 such that

$$N\Big(\lambda r^m \pm \mu_1 r^{m-1}, \frac{1}{f-b_1}\Big) \sim T(\lambda r^m \pm \mu_1 r^{m-1}, f) \qquad (r \to \infty)$$

and

$$N\Big(r, \frac{1}{f(P)-b_2}\Big) \sim T(r, f(P)) \qquad (r \to \infty).$$

Thus for arbitrary small ε_2, there exists $r_1 > r_0$, such that, for $r > r_1$, we have

$$\begin{aligned}(1-\varepsilon_2)T(\lambda r^m \pm \mu_1 r^{m-1}, f) &< N\Big(\lambda r^m \pm \mu_1 r^{m-1}, \frac{1}{f-b_1}\Big)\\ &< (1+\varepsilon_2)N(\lambda r^m \pm \mu_1 r^{m-1}, f)\end{aligned} \tag{12}$$

and

$$(1-\varepsilon_2)N\Big(r, \frac{1}{f(P)-b_2}\Big) < T(r, f(P)) < (1+\varepsilon_2)N\Big(r, \frac{1}{f(P)-b_2}\Big). \tag{13}$$

From (10), (11), (12) and (13) we see that if $r > r_2 \geq r_1$, then

$$\begin{aligned}(1-\varepsilon)T(\lambda r^m - \mu_1 r^{m-1}, f) + O(\log r) &< T(r, f(P))\\ &< (1+\varepsilon)T(\lambda r^m + \mu_1 r^{m-1}, f) + O(\log r),\end{aligned} \tag{14}$$

where $\varepsilon = \varepsilon(\varepsilon_1, \varepsilon_2)$ is arbitrary, with $0 < \max\{\varepsilon_1, \varepsilon_2\} < \varepsilon < 1$.

(ii) By (6), we can choose μ_2 such that $\mu_2 > |c_{m-1}|$. By the same reasoning as above, for arbitrary small $\varepsilon' > 0$, there exists $r_0' > 0$, such that for all $r > r_0'$

$$\begin{aligned}(1-\varepsilon')T(\lambda r^m - \mu_2 r^{m-1}, f) + O(\log r) &< T(r, f(P+Q))\\ &< (1+\varepsilon')T(\lambda r^m + \mu_2 r^{m-1}, f) + O(\log r).\end{aligned} \tag{15}$$

Thus it follows from (14) and (15) that, for all sufficiently large values of r,

$$\begin{aligned}\frac{(1-\varepsilon)T(\lambda r^m - \mu_1 r^{m-1}, f) + O(\log r)}{(1+\varepsilon')T(\lambda r^m + \mu_2 r^{m-1}, f) + O(\log r)} &\leq \frac{T(r, f(P))}{T(r, f(P+Q))}\\ &\leq \frac{(1+\varepsilon)T(\lambda r^m + \mu_1 r^{m-1}, f) + O(\log r)}{(1-\varepsilon')T(\lambda r^m - \mu_2 r^{m-1}, f) + O(\log r)}.\end{aligned} \tag{16}$$

Take $\phi(r) = \lambda r^m$, $\psi_i(r) = \mu_i r^{m-1}$ $(i = 1, 2)$, then $\dfrac{\psi_i(r)}{\phi(r)} \to 0$ $(r \to \infty,$ $i = 1, 2)$ and $\log \psi_i(r)/\log \phi(r) \to (m-1)/m$ $(r \to \infty, i = 1, 2)$.

Since $\rho_f - \lambda_f < 1 - (m-1)/m = 1/m$, by Theorem 1 we have

$$T(\lambda r^m, f) \sim T(\lambda r^m \pm \mu_1 r^{m-1}, f) \sim T(\lambda r^m \pm \mu_2 r^{m-1}, f) \qquad (r \to \infty).$$

Since ε, ε' are arbitrary $(0 < \varepsilon < 1, 0 < \varepsilon' < 1)$,

$$T(r, f(P+Q)) \sim T(r, f(P)) \qquad (r \to \infty).$$

This proves Theorem 2.

4. Proof of Corollary 1

Suppose on the contrary $\lambda_f < \infty$, since $\rho_f - \lambda_f < 1/m$, $\rho_f < \infty$. Thus, by Theorem 2 we have

$$\lim_{r\to\infty} \frac{T(r, f(P+Q))}{T(r, f(P))} = 1.$$

This contradicts the known condition $\displaystyle\lim_{r\to\infty} \frac{T(r, f(P+Q))}{T(r, f(P))} = \infty$.

Hence Corollary 1 is proved.

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