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ON THE LOCALLY UNIFORM OPENNESS OF POLYHEDRAL SETS

HUYNH THE PHUNG

ABSTRACT. The paper is concerned with a geometrical property of polyhedral sets. Specifically, we shall prove that every polyhedral set (denoted by M) is locally uniformly opening. As a consequence, we show that for any set-valued map F defined on a polyhedral set M, F is locally Lipschitz on M iff it is locally Lipschitz on each component of M.

1. INTRODUCTION

This paper deals with a geometrical property of a class of subsets in R^n which we call polyhedral sets. They are unions of finitely many polyhedral convex sets called components. Such subsets are usually encountered in theory of optimization. For example, polyhedral multifunctions studied by Robinson [4], Gowda and Sznajder [1] are set-valued maps whose graphs are polyhedral sets. Also, it is well known [2] that the effective domain of the solution map in linear complementarity problems is always a set of this type. We say a subset $M \subset R^n$ to be locally uniformly opening if there exists $\delta > 0$ such that for every $\bar{x} \in M$ there is a neighbourhood U around $\bar{x}$ for which we can find $u \in M$ satisfying $[u, h] \subset M$, $[u, k] \subset M$ and

$$\|h - k\| \geq \delta(\|h - u\| + \|k - u\|) \quad \text{for all } h,\ k \in M \cap U,$$

where $[x, y]$ denotes the closed segment $\mathrm{co}\{x, y\}$.

We shall prove that every polyhedral set is locally uniformly opening. As a consequence, we show that for any set-valued map F defined on a polyhedral set M, F is locally Lipschitz on M iff it is locally Lipschitz on each component of M. Recall that a set-valued map F is said to be locally Lipschitz on a set M if for every $x \in M$ there exist a positive number L and a neighbourhood U of x such that for all $x_1, x_2 \in M \cap U$ we have

$$\mathcal{H}(F(x_1), F(x_2)) \leq L\|x_1 - x_2\|,$$

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where $\mathcal{H}$ stands for the Hausdorff distance.

The following example show that the conclusions mentioned above may not hold when M is not a polyhedral set.

Example 1.1. Let $M = M_1 \cup M_2 \subset R^2$, where

$$M_1 := \{(x, y) \in R^2; x \geq 0, y \geq x^2\},$$
$$M_2 := \{(x, y) \in R^2; x \geq 0, y \leq 0\}.$$

M is not locally uniformly opening. Indeed, for any $\delta > 0$ and any neighbourhood $B(O, r)$ around $O = (0, 0) \in M$ we can choose $h = \left(\frac{1}{n}, \frac{1}{n^2}\right)$, $k = \left(\frac{1}{n}, 0\right)$, where $n > 2\max\left\{\frac{1}{\delta}, \frac{1}{r}\right\}$. Then $h, k \in B(O, r) \cap M$ and $u = O \in M$ is the unique point such that $[u, h] \subset M$, $[u, k] \subset M$. However, it is easy to verify that

$$\|h - k\| < \delta(\|h - O\| + \|k - O\|.$$

Now we consider the function F defined on M by

$$F(x, y) = \begin{cases} 0 & \text{if } (x, y) \in M_2, \\ x & \text{if } (x, y) \in M_1. \end{cases}$$

Then F is locally Lipschitz on M_1 and M_2 but F is not locally Lipschitz on M.

The next section gives some basic lemmas and the main result will be presented in the last section.

2. Basic Lemmas

Lemma 2.1. *Let $c, u^1, u^2, \ldots, u^m \in R^n$, $c \neq 0$ and $\lambda_i \geq 0$, $i = 1, \ldots, m$, such that*

$$c = \sum_{i=1}^{m} \lambda_i u^i. \tag{2.1}$$

Then there are linearly independent vectors $\{u^{i_1}, \ldots, u^{i_k}\} \subset \{u^1, \ldots, u^m\}$ and numbers $\mu_j \geq 0$, $j = 1, \ldots, k$ satisfying

$$c = \sum_{j=1}^{k} \mu_j u^{i_j}. \tag{2.2}$$

Proof. It suffices to prove that if (2.1) holds and $u^1, \ldots, u^m$ are linearly dependent then there exist $\mu_i \geq 0$, $i = 1, \ldots, m$, with at least one number equals zero, such that

$$c = \sum_{i=1}^{m} \mu_i u^i. \tag{2.3}$$

The situation is trivial if $\lambda_i = 0$ for some i. So, we may assume that $\lambda_i > 0$, $i = 1, \ldots, m$. Since $u^1, \ldots, u^m$ are linearly dependent, there exist $\alpha_1, \alpha_2, \ldots, \alpha_m$, not all zero, such that

$$\sum_{i=1}^{m} \alpha_i u^i = 0. \tag{2.4}$$

Suppose that

$$\frac{\|\alpha_{i_0}\|}{\lambda_{i_0}} = \max\Big\{\frac{\|\alpha_i\|}{\lambda_i},\ i = \overline{1, m}\Big\} > 0. \tag{2.5}$$

By setting $\mu_i := \lambda_i - \dfrac{\lambda_{i_0}\alpha_i}{\alpha_{i_0}}$, $i = 1, \ldots, m$ we have $\mu_{i_0} = 0$ and

$$\mu_i = \lambda_i\Big(1 - \frac{\alpha_i/\lambda_i}{\alpha_{i_0}/\lambda_{i_0}}\Big) \geq \lambda_i\Big(1 - \frac{\|\alpha_i/\lambda_i\|}{\|\alpha_{i_0}/\lambda_{i_0}\|}\Big) \geq 0, \quad i = 1, \ldots, m.$$

Finally,

$$\sum_{i=1}^{m} \mu_i u^i = \sum_{i=1}^{m} \lambda_i u^i - \frac{\lambda_{i_0}}{\alpha_{i_0}} \sum_{i=1}^{m} \alpha_i u^i = c,$$

and the proof is complete. □

Lemma 2.2. *Let $U = \{u^1, u^2, \ldots, u^m\} \subset R^n$. Then there exists $\gamma > 0$ such that for any class of linearly independent vectors $\{v^1, \ldots, v^k\} \subset U$ and any numbers $\lambda_i \geq 0$, $i = 1, \ldots, k$, with $\sum_{i=1}^{k} \lambda_i = 1$, we have*

$$\Big\| \sum_{i=1}^{k} \lambda_i v^i \Big\| \geq \gamma. \tag{2.6}$$

Proof. For each class of linearly independent vectors $V = \{v^1, \dots, v^k\} \subset U$, co$V$ is compact and disjoints from O. Hence there exists $\gamma(V) > 0$ such that for any $\lambda_i \geq 0$, $i = 1, \dots, k$, with $\sum_{i=1}^{k} \lambda_i = 1$, we have

$$\Big\| \sum_{i=1}^{k} \lambda_i v^i \Big\| \geq \gamma(V). \tag{2.7}$$

To complete the proof we can choose

$$\gamma := \min\{\gamma(V) |\ V \text{ is a class of linearly independent vectors in } U\}.$$

Lemma 2.3. *Let h, k, u be distinct vectors in R^n and denote*

$$p := \min\{\|h - u\|, \|k - u\|\}, \tag{2.8}$$

$$\bar{h} := u + \frac{p}{\|h - u\|}(h - u), \quad \bar{k} := u + \frac{p}{\|k - u\|}(k - u), \tag{2.9}$$

$$\bar{s} := \frac{\bar{h} + \bar{k}}{2}. \tag{2.10}$$

Then for all $d \in R^n$ such that

$$\langle u - \bar{s}, d\rangle < 0 \tag{2.11}$$

there exists $t \in (0, 1)$ satisfying

$$\|u + td - h\| + \|u + td - k\| < \|u - h\| + \|u - k\|. \tag{2.12}$$

Proof. By the definition of $\bar{s}$ we have

$$\bar{s} = u + \mu\left(\|u - k\|(h - u) + \|u - h\|(k - u)\right), \tag{2.13}$$

where $\mu = \dfrac{p}{2} \dfrac{1}{\|u - h\|\|u - k\|} > 0$. If (2.11) holds, then from (2.13) we have

$$\|u - k\|\langle u - h, d\rangle + \|u - h\|\langle u - k, d\rangle < 0. \tag{2.14}$$

On the other hand,

$$\|u+td-h\|^2 = \|u-h\|^2 + t^2\|d\|^2 + 2t\langle u-h, d\rangle, \tag{2.15}$$

$$\|u+td-k\|^2 = \|u-k\|^2 + t^2\|d\|^2 + 2t\langle u-k, d\rangle. \tag{2.16}$$

Combining (2.15) and (2.16) one gets

$$\begin{aligned} &\|u-k\|\cdot\|u+td-h\|^2 + \|u-h\|\cdot\|u+td-k\|^2 \\ &= \|u-k\|\cdot\|u-h\|\cdot(\|u-h\|+\|u-k\|) \\ &\quad + O(t^2) + 2t\left(\|u-k\|\cdot\langle u-h, d\rangle + \|u-h\|\cdot\langle u-k, d\rangle\right). \end{aligned}$$

Thus, by (2.14) there exists $t \in (0,1)$ so that

$$\begin{aligned} &\|u-k\|\cdot\|u+td-h\|^2 + \|u-h\|\cdot\|u+td-k\|^2 \\ &< \|u-k\|\cdot\|u-h\|\cdot(\|u-h\|+\|u-k\|). \end{aligned}$$

Multiplying both sides of the inequality by $\dfrac{(\|u-h\|+\|u-k\|)}{(\|u-h\|\|u-k\|)}$ we obtain

$$\begin{aligned} &\Big(\frac{\|u+td-h\|^2}{\|u-h\|} + \frac{\|u+td-k\|^2}{\|u-k\|}\Big)\cdot(\|u-h\|+\|u-k\|) \\ &< (\|u-h\|+\|u-k\|)^2. \end{aligned} \tag{2.17}$$

Besides, by Bunhiakovskii inequality we have

$$\begin{aligned} &\Big(\frac{\|u+td-h\|^2}{\|u-h\|} + \frac{\|u+td-k\|^2}{\|u-k\|}\Big)\cdot(\|u-h\|+\|u-k\|) \\ &\geq (\|u+td-h\|+\|u+td-k\|)^2, \end{aligned}$$

which together with (2.17) implies (2.12). The proof is complete. □

3. The main result

We first recall [3] that a subset $D \subset R^n$ is a polyhedral convex set if there exist vectors $a^1, a^2, \dots, a^r \in R^n$ and numbers $c_1, c_2, \dots, c_r$ such that

$$D = \left\{x \in R^n | \langle a^i, x\rangle \leq c_i,\ i = 1, \dots, r\right\}. \tag{3.1}$$

Then, for each $k \in D$, one has

$$N_D(k) = \text{conco}\left\{a^i | i \in J(k)\right\}, \tag{3.2}$$

where $J(k) := \{j | \langle a^j, k\rangle = c_j\}$ and $N_D(k) := \{u \in R^n | \langle u, v - k\rangle \leq 0 \text{ for all } v \in D\}$, the normal cone of D at k.

The following proposition is crucial in the sequel.

Proposition 3.1. *Let H and K be polyhedral convex sets in R^n with $H \cap K \neq \emptyset$. Then there exists $\varepsilon > 0$ such that for all $h \in H$ and $k \in K$ there exists $u \in H \cap K$ satisfying*

$$\langle h - u, k - u\rangle \leq (1 - \varepsilon) \cdot \|h - u\| \cdot \|k - u\|. \tag{3.3}$$

Proof. Assume that

$$H = \left\{x \in R^n | \langle a^i, x\rangle \leq c_i, i \in \alpha := \{1, \ldots, l\}\right\},$$
$$K = \left\{x \in R^n | \langle b^j, x\rangle \leq d_j, j \in \beta := \{1, \ldots, m\}\right\},$$

where $a^i \in R^n$, $i \in \alpha$, $b^j \in R^n$, $j \in \beta$. Without loss of generality we may assume that $\|a^i\| = \|b^j\| = 1$ for all nonzero vectors a^i, b^j.

Applying Lemma 2.2 for the class $U = \{a^i, b^j;\, i \in \alpha, j \in \beta\}$ we can find $\gamma > 0$ such that for any subclass of linearly independent vectors $\{a^i, i \in \bar{\alpha} \subset \alpha;\, b^j, j \in \bar{\beta} \subset \beta\}$ and any $\lambda_i \geq 0$, $i \in \bar{\alpha}$, $\mu_j \geq 0$, $j \in \bar{\beta}$, with

$$\sum_{i \in \bar{\alpha}} \lambda_i + \sum_{j \in \bar{\beta}} \mu_j = 1 \tag{3.4}$$

we have

$$\Big\| \sum_{i \in \bar{\alpha}} \lambda_i a^i + \sum_{j \in \bar{\beta}} \mu_j b^j \Big\| \geq \gamma. \tag{3.5}$$

Clearly, $\gamma \in (0, 1)$. We shall prove that the assertion of the proposition is true with

$$\varepsilon = \gamma^2. \tag{3.6}$$

Take $k \in K$, $h \in H$. If $h \in K$ (respectively, $k \in H$) then (3.3) holds immediately by choosing $u = h$ (respectively, $u = k$). Now suppose that $h \in H \setminus K$ and $k \in K \setminus H$. Consider a functional ϕ defined on R^n by

$$\phi(v) := \|h - v\| + \|k - v\|, \quad v \in R^n. \tag{3.7}$$

It is easy to verify that ϕ is convex, continuous and

$$\lim_{\|v\|\to+\infty} \phi(v) = +\infty. \tag{3.8}$$

On the other hand, $H \cap K$ is a nonempty closed convex subset in R^n. It follows that there exists $u \in H \cap K$ such that

$$\phi(v) \geq \phi(u) \quad \text{for all} \quad v \in H \cap K. \tag{3.9}$$

If $\langle h-u, k-u\rangle \leq 0$ then (3.3) holds. So, we may assume that $\langle h-u, k-u\rangle > 0$ and we denote by $\nu \in [0,1)$ the number with the property

$$\langle h-u, k-u\rangle = (1-\nu)\cdot\|h-u\|\cdot\|k-u\|. \tag{3.10}$$

Now choose p, $\bar h$, $\bar k$, $\bar s$ as in Lemma 2.3. It is easy to see that

$$\bar h \in [u,h] \subset H, \quad \bar k \in [u,k] \subset K, \tag{3.11}$$

and from (2.9), (3.10) we get

$$\langle \bar h - u, \bar k - u\rangle = (1-\nu)\cdot\|\bar h - u\|\cdot\|\bar k - u\| = (1-\nu)p^2. \tag{3.12}$$

We shall prove that

$$\langle u - \bar s, v - u\rangle \geq 0 \quad \text{for all} \quad v \in H \cap K. \tag{3.13}$$

Indeed, otherwise there is $\bar v \in H \cap K$ such that $\langle u - \bar s, \bar v - u\rangle < 0$. Then, by virtue of Lemma 2.3 there exists $t \in (0,1)$ satisfying

$$\|u + t(\bar v - u) - h\| + \|u + t(\bar v - u) - k\| < \|u-h\| + \|u-k\|. \tag{3.14}$$

But from $u, \bar v \in H \cap K$ we get $u + t(\bar v - u) = (1-t)u + t\bar v \in H \cap K$. Therefore, (3.14) conflicts with (3.9).

Since $H \cap K = \big\{x \in R^n | \langle a^i, x\rangle \leq c_i,\, i \in \alpha;\, \langle b^j, x\rangle \leq d_j, j \in \beta\big\}$. It follows from (3.13) that

$$\bar s - u \in N_{H\cap K}(u) = \operatorname{conco}\{a^i,\, b^j;\, i \in \alpha_1, j \in \beta_1\}, \tag{3.15}$$

where

$$\alpha_1 := \{i \in \alpha | \langle a^i, u\rangle = c_i\}, \quad \beta_1 := \{j \in \beta | \langle b^j, u\rangle = d_j\}. \tag{3.16}$$

If $\bar{s} - u = 0$ then $\bar{h} - u = -(\bar{k} - u)$, which contradicts (3.12). So $\bar{s} - u \neq 0$. Now, by Lemma 2.1 there exists $\bar{\alpha} \subset \alpha_1$, $\bar{\beta} \subset \beta_1$ and $\lambda_i \geq 0$, $i \in \bar{\alpha}$, $\mu_j \geq 0$, $j \in \bar{\beta}$, such that the class $\{a^i, b^j; i \in \bar{\alpha}, j \in \bar{\beta}\}$ is linearly independent and

$$\bar{s} - u = \sum_{i \in \bar{\alpha}} \lambda_i a^i + \sum_{j \in \bar{\beta}} \mu_j b^j. \tag{3.17}$$

Since $\bar{s} - u \neq 0$, one gets

$$\eta := \sum_{i \in \bar{\alpha}} \lambda_i + \sum_{j \in \bar{\beta}} \mu_j > 0, \tag{3.18}$$

and hence $(\bar{s} - u)/\eta \in \operatorname{co}\left\{a^i, b^j; i \in \bar{\alpha}, j \in \bar{\beta}\right\}$. By the choice of γ we have $\|(\bar{s} - u)/\eta\| \geq \gamma$ or, equivalently,

$$\|\bar{s} - u\| \geq \eta\gamma. \tag{3.19}$$

Besides, it follows from (3.12) that

$$\begin{aligned} \|\bar{s} - u\|^2 &= \frac{1}{4}\|\bar{h} + \bar{k} - 2u\|^2 = \frac{1}{4}\left(\|\bar{h} - u\|^2 + \|\bar{k} - u\|^2 + 2\langle \bar{h} - u, \bar{k} - u\rangle\right) \\ &= \frac{1}{4}\left(2p^2 + 2(1 - \nu)p^2\right) = p^2\left(1 - \frac{\nu}{2}\right) \end{aligned}$$

and

$$\begin{aligned} \|\bar{h} - \bar{k}\|^2 &= \|\bar{h} - u\|^2 + \|\bar{k} - u\|^2 - 2\langle \bar{h} - u, \bar{k} - u\rangle \\ &= 2p^2 - 2(1 - \nu)p^2 = 2\nu p^2. \end{aligned}$$

These imply

$$\|\bar{s} - u\| = p\sqrt{1 - \nu/2} \tag{3.20}$$

and

$$\|\bar{h} - \bar{k}\| = p\sqrt{2\nu}. \tag{3.21}$$

On the other hand, $\|\bar{s} - u\|^2$ can be rewritten as follows

$$\|\bar{s} - u\|^2 = \left\langle \bar{s} - u, \sum_{i \in \bar{\alpha}} \lambda_i a^i + \sum_{j \in \bar{\beta}} \mu_j b^j \right\rangle.$$

Since $\bar{h} \in H$, $\bar{k} \in K$ so that $\langle a^i, \bar{h} - u\rangle \le 0$, $\langle b^j, \bar{k} - u\rangle \le 0$ for all $i \in \bar{\alpha}$, $j \in \bar{\beta}$, we have

$$\begin{aligned}
\|\bar{s} - u\|^2 &= \frac{1}{2}\Big\langle \bar{h} + \bar{k} - 2u, \sum_{i\in\bar{\alpha}} \lambda_i a^i + \sum_{j\in\bar{\beta}} \mu_j b^j \Big\rangle \\
&= \frac{1}{2}\Big(\sum_{i\in\bar{\alpha}} \lambda_i \langle a^i, \bar{k} - \bar{h}\rangle + 2\sum_{i\in\bar{\alpha}} \lambda_i \langle a^i, \bar{h} - u\rangle \\
&\quad + \sum_{j\in\bar{\beta}} \mu_j \langle b^j, \bar{h} - \bar{k}\rangle + 2\sum_{j\in\bar{\beta}} \mu_j \langle b^j, \bar{k} - u\rangle \Big) \\
&\le \frac{1}{2}\Big(\sum_{i\in\bar{\alpha}} \lambda_i \langle a^i, \bar{k} - \bar{h}\rangle + \sum_{j\in\bar{\beta}} \mu_j \langle b^j, \bar{h} - \bar{k}\rangle \Big) \\
&\le \frac{1}{2}\|\bar{h} - \bar{k}\| \Big(\sum_{i\in\bar{\alpha}} \lambda_i \|a^i\| + \sum_{j\in\bar{\beta}} \mu_j \|b^j\| \Big) \\
&= \frac{\eta}{2}\|\bar{h} - \bar{k}\|. \qquad (3.22)
\end{aligned}$$

Combining (3.19)-(3.22) we obtain

$$\frac{\eta}{2} p\sqrt{2\nu} = \frac{\eta}{2}\|\bar{h} - \bar{k}\| \ge \|\bar{s} - u\|^2 \ge \eta\gamma p\sqrt{1 - \nu/2}.$$

It follows that $\frac{\nu}{2} \ge \gamma^2\big(1 - \frac{\nu}{2}\big)$, and hence $\varepsilon = \gamma^2 \le \frac{\nu}{2-\nu} \le \nu$. This together with (3.10) gives

$$\langle h - u, k - u\rangle \le (1 - \varepsilon)\|h - u\| \cdot \|k - u\|.$$

The proof is complete. □

Corollary 3.2. *Let H and K be polyhedral convex sets in R^n with $H \cap K \neq \emptyset$. Then there exixts $\delta > 0$ such that for all $h \in H$, $k \in K$ there exists $u \in H \cap K$ satisfying*

$$\|h - k\| \ge \delta(\|h - u\| + \|k - u\|). \qquad (3.23)$$

Proof. Indeed, from (3.3) we have

$$\begin{aligned}
\|h - k\|^2 &= \|h - u\|^2 + \|k - u\|^2 - 2\langle h - u, k - u\rangle \\
&\ge \|h - u\|^2 + \|k - u\|^2 - 2(1 - \varepsilon)\|h - u\|\,\|k - u\| \\
&= (1 - \varepsilon)(\|h - u\| - \|k - u\|)^2 + \varepsilon(\|h - u\|^2 + \|k - u\|^2) \\
&\ge \frac{\varepsilon}{2}(\|h - u\| + \|k - u\|)^2.
\end{aligned}$$

This implies that (3.23) holds with $\delta = \sqrt{\varepsilon/2}$, and the proof is complete. $\square$

The following theorem is a generalization of Corollary 3.2.

Theorem 3.3. *All polyhedral sets in R^n are locally uniformly opening.*

Proof. Let M be an arbitrary polyhedral set, whose components are K_i, $i = 1, \dots, s$. By virtue of Corollary 3.2 for each pair K_i, K_j with $K_i \cap K_j \neq \emptyset$ there exists $\delta_{ij} > 0$ such that for all $k_i \in K_i$, $k_j \in K_j$ there exists $u \in K_i \cap K_j$ satisfying

$$\|k_i - k_j\| \geq \delta_{ij}(\|k_i - u\| + \|k_j - u\|).$$

We set

$$\delta = \min\{\delta_{ij} : 1 \leq i \leq j \leq s,\ K_i \cap K_j \neq \emptyset\}.$$

Since K_i, $i = 1, \dots, s$ are closed, for every $x \in M$ we can choose $r > 0$ small enough such that

$$B(x, r) \cap K_i = \emptyset \quad \text{whenever} \quad K_i \not\ni x. \tag{3.24}$$

Now take arbitrary $h, k \in M \cap B(x, r)$. It follows that $h \in K_i \cap B(x, r)$ and $k \in K_j \cap B(x, r)$ for some i, j. From (3.24) we have $x \in K_i$, $x \in K_j$ and then $K_i \cap K_j \neq \emptyset$. Hence, there exists $u \in K_i \cap K_j \subset M$ such that

$$\|h - k\| \geq \delta_{ij}(\|h - u\| + \|k - u\|) \geq \delta(\|h - u\| + \|k - u\|). \tag{3.25}$$

Besides, it is obvious that $[h, u] \subset K_i \subset M$ and $[k, u] \subset K_j \subset M$. The proof is complete. $\square$

Corollary 3.4. *Let F be a set-valued map from a polyhedral set $M \subset R^n$ into R^m, locally Lipschitz on each component of M. Then F is locally Lipschitz on M.*

Proof. Assume that $M = \bigcup_{i=1}^{s} K_i$ with K_i being polyhedral convex sets. For each $x \in M$ we set $I(x) := \{i | x \in K_i\}$. Since F is local Lipschitz on each K_i, there exists $L_i > 0$, $r_i > 0$, $i \in I(x)$ such that $\forall\, x_1, x_2 \in K_i \cap B(x, r_i)$ we have

$$\mathcal{H}(F(x_1), F(x_2)) \leq L_i \|x_1 - x_2\|. \tag{3.26}$$

Besides, from the proof of Theorem 3.3 there exixts $\delta > 0$, $r > 0$ such that for all h, $k \in M \cap B(x,r)$, there exists $u \in M$ satisfying

$$[u,h] \subset K_i,\ [u,k] \subset K_j \quad \text{for some } i,j \in I(x) \tag{3.27}$$

and

$$\|h-k\| \geq \delta(\|u-h\| + \|u-k\|). \tag{3.28}$$

One sets

$$\bar{r} := \frac{\delta}{\delta+1} \min\{r, \min\{r_i, i \in I(x)\}\} \tag{3.29}$$

and

$$L := \frac{1}{\delta} \max\{L_i,\ i \in I(x)\}. \tag{3.30}$$

Now for all $h,k \in B(x,\bar{r}) \cap M$ there exists $u \in M$ satisfying (3.27) and (3.28). Hence

$$\begin{aligned} 2\|u-x\| &\leq \|u-h\| + \|h-x\| + \|u-k\| + \|k-x\| \\ &\leq \frac{1}{\delta}\|h-k\| + 2\bar{r} \leq 2\bar{r}\Big(\frac{1+\delta}{\delta}\Big). \end{aligned}$$

This together with (3.27) and (3.29) imply that $u \in B(x,r_i) \cap K_i$. On the other hand, $h \in B(x,\bar{r}) \cap K_i \subset B(x,r_i) \cap K_i$. From (3.26) we get

$$\mathcal{H}(F(u),F(h)) \leq L_i\|u-h\|,$$

and, analogously,

$$\mathcal{H}(F(u),F(k)) \leq L_j\|u-k\|.$$

Finally, one gets

$$\begin{aligned} \mathcal{H}(F(h),F(k)) &\leq \mathcal{H}(F(h),F(u)) + \mathcal{H}(F(u),F(k)) \\ &\leq L_i\|u-h\| + L_j\|u-k\| \leq \delta L(\|u-h| + \|u-k\|) \\ &\leq L\|h-k\|. \end{aligned}$$

Since h,k are taken arbitrarily in $M \cap B(x,\bar{r})$ it implies that F is locally Lipschitz on M and the corollary is proved. □

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Department of Mathematics
College of Sciences, University of Hue
Hue, Vietnam