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ON SOME CLASSES OF HYPERBOLIC COMPLEX SPACES

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ABSTRACT. We characterize the hyperbolicity of the following complex spaces in the sense of Kobayashi: complex spaces satisfying the Landau property, compact complex spaces and Zariski open subsets of Moishezon spaces.

INTRODUCTION

Characterizing hyperbolicity in the sense of Kobayashi is one of the most important problems of hyperbolic complex analysis. Much attention has been given to this problem, and the results on this problem can be applied to finiteness theorems for meromorphic and holomorphic mappings. For details we refer the readers to [La] and [Ko2] and [ZL].

In this paper we will characterize the hyperbolicity of some concrete classes of complex spaces.

We now describe more precisely the content of the paper.

In Section 1 we show that a complex space X is hyperbolic if and only if X satisfies the Landau property. This assertion was proved firstly in the case of (nonsingular) complex manifolds by Hahn and Kim [HK] and in the case of complex spaces with discrete singularities by Thai [Th1].

In Section 2 we study the hyperbolicity of compact complex spaces from the viewpoint of extending holomorphic maps. The ideas are due to Kwack [Kw] and Thai [Th2]. They stated that a compact complex space X is hyperbolic if and only if every holomorphic map from the punctured disc Δ^* into X extends holomorphically over Δ. Furthermore, we will prove that the hyperbolicity of a compact complex space is equivalent to the holomorphicity of any Gateaux holomorphic map from an open subset of a Banach space into this space.

Starting from the general philosophy that analytic objects can be approximated by algebraic objects under suitable restrictions, in Section 3 we construct the Kobayashi algebraic hyperbolic pseudodistance on an

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open subset of a Moishezon space and prove that this pseudodistance coincides with the Kobayashi pseudodistance. An analogous assertion was proved in the case of quasi-projective varieties by Demailly, Lempert and Shiffman [DLS].

The main results are the following theorems.

Theorem 1.2. *A complex space X is hyperbolic if and only if X has the Landau property for any Finsler metric H on X.*

Theorem 2.1. *Let X be a compact complex space. Then X is hyperbolic if and only if every Gateaux holomorphic map from any open subset Ω of any Banach space B into X is holomorphic.*

Theorem 3.1. *Let V be a Zariski open subset of a Moishezon space M. Then $d_V(P,Q) = D_V(P,Q)$ for all $P,Q \in V$.*

Now we will recall some basic notions and facts which we shall need in this paper.

We denote the Kobayashi pseudodistance on a complex space X by d_X. X is called to be (complete) hyperbolic if d_X is a (complete) distance.

A complex space X is called to be taut if whenever Y is a complex space and $f_i : Y \to X$ is a sequence of holomorphic maps, then either there exists a subsequence which is compactly divergent or a subsequence which converges uniformly on compact subsets to a holomorphic $f : Y \to X$. It suffices that this condition should hold when $Y = \Delta$, where Δ is the unit disc in $\mathbb{C}$ [Ba]. A complete hyperbolic space is taut, and a taut complex space is hyperbolic [Ba].

Let X be a complex space and TX the Zariski tangent space of X. Put

$$\begin{aligned} &\Delta_r = \{|z| < r\} \subset \mathbb{C}, \quad \Delta_1 = \Delta, \\ &e_0 = \frac{d}{dz}\Big|_{z=0} \in T_0\Delta_r \quad \text{(the unit tangent vector).} \end{aligned}$$

The Royden-Kobayashi cone of X is defined by

$$\operatorname{Con} X := \{v \in TX; \exists \varphi \in \operatorname{Hol}(\Delta_r, X), \exists u \in T_0\Delta_r \text{ such that } \varphi'(u) = v\}.$$

Moreover, we define the Royden-Kobayashi (differential) pseudometric on TX by

$$F_X(v) = \begin{cases} \inf\big\{\dfrac{1}{r}; \exists \varphi \in \operatorname{Hol}(\Delta_r, X), \text{ such that } v = \varphi'(e_0)\big\} \text{ if } v \in \operatorname{Con}X, \\ \infty \text{ if } v \notin \operatorname{Con}X. \end{cases}$$

Let Y, Z be quasi-projective (irreducible, reduced) algebraic varieties. A map $f : \Omega \to Z$ defined on an open subset $\Omega \subset Y$ is said to be Nash algebraic if f is holomorphic and the graph

$$\Gamma_f = \{(y, f(y)) \in \Omega \times Z; y \in \Omega\}$$

is contained in an algebraic subvariety G of $Y \times Z$ of dimension equal to $\dim Y$. If f is Nash algebraic, then the image $f(\Omega)$ is contained in an algebraic subvariety A of Z with $\dim A = \dim f(\Omega) \leq \dim Y$.

An open set Ω in a Stein space Y is said to be a Runge domain if the holomorphic hull with respect to $\mathcal{O}(Y)$ of any compact subset $K \subset \Omega$ is contained in Ω. If Y is an affine algebraic variety, a Stein open set $\Omega \subset Y$ is Runge if and only if the polynomial functions on Y are dense in $\mathcal{O}(\Omega)$.

The following result of Demailly, Lempert and Shiffman will play an important role in this paper.

Theorem [DLS, Theorem 1.1]. *Let Ω be a Runge domain in an affine algebraic variety S, and let $f : \Omega \to X$ be a holomorphic map into a quasi-projective algebraic manifold X. Then for every relatively compact domain $\Omega_0 \subset\subset \Omega$, there is a sequence of Nash algebraic maps $f_j : \Omega_0 \to X$ such that $\{f_j\} \to f$ uniformly on Ω_0.*

1. Hyperbolicity of complex spaces satisfying the Landau property

First of all we recall the following

Definition 1.1. Let X be a complex space and H a Finsler metric on X. The space X is said to have the Landau property for H if for each $p \in X$ and each relatively compact open set W in a coordinate neighbourhood of p, there exists a positive constant $R = R(W)$ such that

$$\sup\{H(f'(0)) : f \in \mathrm{Hol}(\Delta, X) \text{ with } f(0) \in W\} \leq R.$$

We now have the following relation between the hyperbolicity and the classical theorem of Landau.

Theorem 1.2. *A complex space X is hyperbolic if and only if X has the Landau property for any Finsler metric H on X.*

To prove the above-mentioned theorem we need the following

Lemma 1.3. *Let X be a complex space and H a Finsler metric on X. Then X is hyperbolic if and only if for each $p \in X$, there are a neighbourhood U of p and a constant $C > 0$ such that*

$$F_X(\xi_x) \geq CH(\xi_x) \text{ for all } \xi_x \in T_xX \text{ with } x \in U.$$

Proof. ($\Rightarrow$) Let D be a coordinate polydisc about a point p. Since X is hyperbolic, (X, d_X) is tight (see [Ba]) and hence the family $\mathrm{Hol}(\Delta, X)$ is an even family. Thus, there are a disc Δ_δ about 0 and a neighbourhood U of p such that if $\Phi(0) = x \in U$ then $\Phi(\Delta_\delta) \subset D$. If Φ maps Δ_R into X with $\Phi(0) = x \in U$, then $\Phi(\Delta_{\delta R}) \subset D$. Hence for $x \in U$, we have $\delta F_D(\xi_x) \leq F_X(\xi_x)$.

We may assume that $\overline{U}$ is a compact subset of D. Then for $x \in U$ and $\xi_x \in T_xX$, we have $F_X(\xi_x) \geq \delta F_D(\xi_x) \geq CH(\xi_x)$ for some positive constant C.

($\Leftarrow$) Let d_{CH} be the distance function generated on X by CH (see [La]). By the hypothesis, $f^*(CH) \leq ds^2_\Delta$ for all $f \in \mathrm{Hol}(\Delta, X)$, where ds^2_Δ denotes the Bergman-Poincaré metric of Δ. Thus we have

$$d_{CH}(x, y) \leq d_X(x, y) \text{ for } x, y \in X.$$

Hence X is hyperbolic.

Proof of Theorem 1.2. ($\Rightarrow$) Let X be hyperbolic. Suppose that X fails to satisfy the Landau property. Then there exist a point $p_0 \in X$ and a relatively compact open neighbourhood W of p_0 in a local coordinate neighbourhood U of p_0 and a sequence $\{f_k\} \subset \mathrm{Hol}\,(\Delta, X)$ such that $\{f_k(0)\} \subset W$ and $\{H(f'_k(0))\} \to \infty$. By the compactness of $\overline{W}$ we can assume that the sequence $\{f_k(0)\} \to p \in U$. Take $r > 0$ such that $B_r = \{x \in X; d_X(x, p) < r\} \subset U$. We can assume that $\{f_k(0)\} \subset B_{r/2}$. Then there exists a positive number $\delta > 0$ such that $f_k(\Delta_\delta) \subset B_r \subset U$ for all $k \geq 1$.

Without loss of generality we may assume that U is complete hyperbolic. Then U is taut. It follows that the sequence $\{f_k|_{\Delta_\delta}\}$ contains a subsequence $\{f_{k_i}|_{\Delta_\delta}\}$ which converges uniformly to a holomorphic map $f : \Delta_\delta \to X$. This is a contradiction because

$$H(f'(0)) = \lim_{i \to \infty} H(f'_{k_i}(0)) < \infty.$$

($\Leftarrow$) It suffices to prove that X satisfies the conditions of lemma 1.3. Indeed, take $p \in X$ and a relatively compact open neighbourhood W in a coordinate neighbourhood of p. By the mentioned condition, there is a positive number R such that

$$\sup\{H(f'(0)); f \in \text{ Hol}(\Delta, X) \text{ with } f(0) \in W\} \leq R.$$

Given $\xi_x \in T_xW$ with $x \in W$. Let $f \in$ Hol(Δ, X) satisfy $f'(0) = r\xi_x$ $(r > 0)$. We have $H(f'(0)) = H(r\xi_x) = rH(\xi_x) \leq R$ and hence

$$\frac{1}{r} \geq \frac{1}{R} \cdot H(\xi_x).$$

Thus $F_X(\xi_x) \geq C.H(\xi_x)$ with $C = \dfrac{1}{R} > 0$. □

2. Hyperbolicity of compact complex spaces

We now investigate relations between the hyperbolicity of compact complex spaces and the holomorphicity of Gateaux holomorphic maps from an open subset of a Banach space into these spaces.

Theorem 2.1. *Let X be a compact complex space. Then X is hyperbolic if and only if every Gateaux holomorphic map from any open subset Ω of any Banach space B into X is holomorphic.*

Proof. ($\Rightarrow$) Let X be hyperbolic and $f : \Omega \to X$ a Gateaux holomorphic map. Without loss of generality we may assume that Ω is convex. By [DTV] we have

$$d_\Omega = \inf\{d_{\Omega \cap E}; E \subset B, \ \dim E < \infty\}.$$

This equality implies

$$d_X(f(x), f(y)) \leq d_\Omega(x, y) \text{ for } x, y \in \Omega,$$

and hence f is continuous.
($\Leftarrow$) Assume that X is not hyperbolic. By [Bro] there exists a non-constant holomorphic map $\sigma : \mathbb{C} \to X$. Let B be an infinite dimentional Banach space. Choose a discontinuous linear functional $x^\#$ on B. It is easy to see that $\sigma \circ x^\#$ is Gateaux holomorphic. By the hypothesis, $\sigma \circ x^\#$ is holomorphic on B and hence $\sigma \circ x^\#$ is continuous on B. Thus $(\sigma \circ x^\#)^{-1}(\sigma(0))$ is closed in B.

On the other hand, since $\sigma \neq$ const, it follows that

$$(\sigma \circ x^{\#})^{-1}(\sigma(0)) \neq B.$$

Therefore $\ker x^{\#} \subset \overline{\ker x^{\#}} \subset (\sigma \circ x^{\#})^{-1}(\sigma(0)) \neq B$. This implies that $\ker x^{\#} = \overline{\ker x^{\#}}$ and hence $x^{\#}$ is continuous. This is a contradiction. $\square$

Remark 2.2. The assumption on the compactness of a complex space X in Theorem 2.1 cannot be omitted. Indeed, consider

$$X = \{(z_1, z_2) \in \mathbb{C}^2; |z_1| < 1, |z_1 z_2| < 1, |z_2| > 1\} \setminus \{(0, z_2)\}.$$

Then X is not hyperbolic (see [Ko1, p. 130]).

Assume that $f : \Omega \to X$ is a Gateaux holomorphic map, where Ω is an open subset of a Banach space B.

Put $f = (f_1, f_2)$, where $f_1 = \pi_1 \circ f$ and $f_2 = \pi_2 \circ f$.

It is easy to see that f_1 maps Ω into Δ. By a theorem of Mujica [Mu, Prop. 8.6 and Thm. 8.7, p.61], f_1 is holomorphic. On the other hand, f_2 maps Ω into $\{z_2 \in \mathbb{C}; |z_2| > 1\}$. Consider the biholomorphic map $\varphi : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$ given by $\varphi(z) = \dfrac{1}{z}$ for every $z \in \mathbb{C} \setminus \{0\}$. Then, as above, the map $\varphi \circ f_2$ is holomorphic. Therefore f_2 is holomorphic and hence f is holomorphic.

3. Algebraic approximation of Kobayashi pseudodistance

Let V be a Zariski open subset of a Moishezon space M. Let $P, Q \in V$ be arbitrary points. An algebraic chain $\{(C_j, \pi_j, z_j, w_j)\}_{j=0}^{n}$ from P to Q is a family of algebraic curves C_j in V with the normalizations $\pi_j : \widetilde{C}_j \to C_j$ and points $z_j, w_j \in \widetilde{C}_j$ such that

$$P = \pi_0(z_0), \pi_{j-1}(w_{j-1}) = \pi_j(z_j), 1 \leq j \leq n, \pi_n(w_n) = Q.$$

Let $\rho_{\widetilde{C}_j}$ denote the Poincaré distance on $\widetilde{C}_j$, and set

$$D_V(P, Q) = \inf \Big\{ \sum_{j=0}^{n} \rho_{\widetilde{C}_j}(z_j, w_j) \Big\},$$

where the infimum is taken over all possible such chains. We call D_V the algebraic hyperbolic pseudodistance of V. The definition of D_V implies immediately the following properties

(i) $D_V \geq d_V$,

(ii) $D_V(P,Q)$ is a continuous function defining a pseudodistance on V,

(iii) (Distance decreasing principle) For an algebraic morphism $f : V \to V'$, we have $D_V(P,Q) \geq D_{V'}(f(P), f(Q))$ for all $P, Q \in V$. Therefore, D_V is a pseudodistance on V which is biholomorphically invariant.

We now prove the main theorem of this section.

Theorem 3.1. *Let V be a Zariski open subset of a Moishezon space M. Then $d_V(P,Q) = D_V(P,Q)$ for all $P, Q \in V$.*

Proof. It suffices to prove the following inequality

$$d_V(P,Q) \geq D_V(P,Q) \text{ for all } P, Q \in V.$$

Take an arbitrary $\varepsilon > 0$. By definition of d_V, there exists a holomorphic chain $\{(f_j, \lambda_j)\}_{j=0}^n$ such that

$$\sum_{j=0}^{n} \rho_\Delta(0, \lambda_j) \leq d_V(P,Q) + \varepsilon.$$

Fix any $j \in \{0, 1, \ldots, n\}$. Let V_j be the Zariski closure of $f_j(\Delta)$ in V. Since V_j is a Moishezon space, there exists a modification $\pi_j : \widetilde{V}_j \to V_j$ with a projective algebraic manifold.

Consider the following commutative diagram

$$W @> p >> \widetilde{V}_j$$

$$@VqVV @VV\pi_j V$$

$$\Delta @> f_j >> V_j$$

where $W = \Delta \times_{V_j} \widetilde{V}_j$ is the fiber product, p and q are the canonical maps. Denote S_j the singular locus of V_j for the modification $\pi_j : \widetilde{V}_j \to V_j$, i.e the restriction

$$\pi_j|_{\widetilde{V}_j \setminus \pi_j^{-1}(S_j)} : \widetilde{V}_j \setminus \pi_j^{-1}(S_j) \to V_j \setminus S_j$$

is a biholomorphism. Then $f_j(\Delta) \not\subset S_j$.

Note that q is proper and $q : q^{-1}(U) \to U$ is biholomorphic, where $U = \Delta \setminus f_j^{-1}(S_j)$. Moreover, $\dim W = 1$.

Take an irreducible branch H of W such that $q^{-1}(U)$ is contained in H. Since H is not compact and $\dim H = 1$, H is a Stein space [GR]. Consider the map $q^{-1} : U \to H$. Since the holomorphic map $q|_H : H \to \Delta$ is proper and $f_j^{-1}(S_j)$ is discrete in Δ, by the Riemann theorem, the holomorphic map $q^{-1} : U \to H$ extends to a holomorphic map $h : \Delta \to H$. Thus f_j lifts to a holomorphic map $\widetilde{f}_j : \Delta \to \widetilde{V}_j$.

By a theorem of Demailly, Lempert and Shiffman [DLS], there is a Nash algebraic map $g_{\delta_j} : \Delta_{1-\delta} \to \widetilde{V}_j$ such that $g_{\delta_j}(0) = \widetilde{f}_j(0)$, $g_{\delta_j}(\lambda_j) = \widetilde{f}_j(\lambda_j)$. Put $C_{\delta_j} = \pi_j(g_{\delta_j}(\Delta_{1-\delta}))$. Let $\widetilde{\pi}_{\delta_j} : \widetilde{C}_{\delta_j} \to C_{\delta_j}$ be the normalization and choose points $z_{\delta_j}, w_{\delta_j} \in \widetilde{C}_{\delta_j}$ with $\widetilde{\pi}_{\delta_j} = \pi_j(g_{\delta_j}(0))$, $\widetilde{\pi}_{\delta_j}(w_{\delta_j}) = \pi_j(g_{\delta_j}(\lambda_j))$.

By the distance decreasing principle, we have

$$d_{\widetilde{C}_{\delta_j}}(z_{\delta_j}, w_{\delta_j}) \leq d_{\Delta_{1-\delta}}(0, \lambda_j).$$

It follows from the definition of D_V that for $0 < \delta < 1$

$$\begin{aligned} D_V(P, Q) &\leq \sum_{j=0}^{n} \rho_{\widetilde{C}_{\delta_j}}(z_{\delta_j}, w_{\delta_j}) \\ &= \sum_{j=0}^{n} d_{\widetilde{C}_{\delta_j}}(z_{\delta_j}, w_{\delta_j}) \\ &\leq \sum_{j=0}^{n} d_{\Delta_{1-\delta}}(0, \lambda_j). \end{aligned}$$

On the other hand, there is a small $\delta_j > 0$ such that

$$d_{\Delta_{1-\delta}}(0, \lambda_j) < d_{\Delta}(0, \lambda_j) + \frac{\varepsilon}{n+1} \text{ if } 0 < \delta < \delta_j.$$

Hence, if $0 < \delta < \min\limits_{0 \leq j \leq n} \delta_j$ we have

$$D_V(P, Q) \leq \sum_{j=0}^{n} \left\{ d_{\Delta}(0, \lambda_j) + \frac{\varepsilon}{n+1} \right\} < d_V(P, Q) + 2\varepsilon.$$

Letting $\varepsilon \to 0$ we finally get $D_V(P, Q) \leq d_V(P, Q)$. The proof of Theorem 3.1 is now complete. $\square$

In [DLS] Demailly, Shiffman and Lempert defined the algebraic pseudodistance $\overline{D}_V$ on a quasi-projective variety V as follows.

For any points $P, Q \in V, \overline{D}_V(P,Q) = \inf \sum_{i=1}^{k} d_{C_i}(p_{i-1}, p_i)$, where the infimum is taken over all chains of points $p_0 = P, p_1, \ldots, p_k = Q$ and irreducible algebraic curves $C_1, C_2, \ldots, C_k$ in V such that $p_0 \in C_1, p_1 \in C_1 \cap C_2, \ldots, p_{k-1} \in C_{k-1} \cap C_k, p_k \in C_k$. It is easy to see that $D_V(P,Q) \geq \overline{D}_V(P,Q)$ for all $P, Q \in V$. Therefore, from Theorem 3.2 we get the following

Corollary 3.2. [DLS, Corollary 1.4]. *Let V be a quasi-projective variety. Then*

$$D_V(P,Q) = \overline{D}_V(P,Q) = d_V(P,Q) \text{ for all } P, Q \in V.$$

From the well-known properties of the Kobayashi pseudo-distance we obtain the following

Corollary 3.3. *If $D_V(P,Q)$ is a distance, then it is an inner distance and its topology is the same as the underlying differential topology.*

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