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DEFICIENCIES OF COMPOSITE ENTIRE FUNCTIONS

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Abstract. In this paper we generalize a result of Goldstein.

1. Introduction

Goldstein [4] proved the following result:

Let $g(z)$ be a polynomial and $f(z)$ a meromorphic function such that $T(r,f) = O((\log r)^\alpha)$ for some $\alpha > 1$. Then for any value of a, we have

$$\delta(a, f(g)) = \delta(a, f).$$

In this paper we generalize the condition $T(r,f) = O((\log r)^\alpha)$ to the form $T(r,f) = O(e^{(\log r)^\alpha})$ and replace the $g(z)$ by a function g with $T(r,g) = O((\log r)^\beta)$ $(0 < \alpha < 1,\ \beta > 1,\ \alpha\beta < 1)$ and obtain the following result:

Theorem 1. *Let f and g be two transcendental entire functions with $T(r,f) = O(e^{(\log r)^\alpha})$ and $T(r,g) = O((log r)^\beta)$, where $0 < \alpha < 1$, $\beta > 1$ and $\alpha\beta < 1$. Then for any value of $a \neq \infty$ we have*

$$\delta(a, f(g)) = \delta(a, f).$$

2. Preliminaries

We shall need the following lemmas

Lemma 1 [5]. *Let $f(z)$ be an entire function. For $0 \le r < R < \infty$, we have*

$$T(r,f) \le \log^+ M(r,f) \le \frac{R+r}{R-r} T(R,f).$$

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Lemma 2 [6]. *Let $f(z)$ and $g(z)$ be two entire functions with $g(0) = 0$. Then for all $r > 0$, we have*

$$T(r, f(g)) \leq T(M(r, g), f).$$

Lemma 3 [2]. *Let f and g be two entire functions and $g(0) = 0$. Then*

$$M(r, f(g)) \geq M((1 - o(1))M(r, g), f) \quad (r \to \infty,\ r \notin E),$$

where E is a set of finite logarithmic measure of r.

Lemma 4 [1]. *Let f be an entire function of order zero and $z = re^{i\theta}$. Then for any $\zeta > 0$, $\eta > 0$, there exist $R_0 = R_0(\zeta, \eta)$, $k = k(\zeta, \eta)$ such that for all $R > R_0$,*

$$log|f(re^{i\theta})| - N(2R) - log|c| > -kQ(2R), \quad \zeta R \leq r \leq R,$$

except in a set of circles enclosing the zeros of f, the sum of whose radii is at most ηR, where

$$Q(r) = r \int_r^\infty \frac{n(t, 1/f)}{t^2} dt,$$

$$N(r) = \int_0^r \frac{n(t, 1/f)}{t} dt.$$

Lemma 5. *Let $T_1(r)$ and $T_2(r)$ be two nonnegative nondecreasing functions of r with*

$$T_1(r) = O(T_2(r)) \quad (r \to \infty).$$

Then

(i) $$\overline{\lim_{r\to\infty}} \frac{\log^+ T_1(r)}{\log r} \leq \overline{\lim_{r\to\infty}} \frac{\log^+ T_2(r)}{\log r},$$

(ii) $$\underline{\lim_{r\to\infty}} \frac{\log^+ T_1(r)}{\log r} \leq \underline{\lim_{r\to\infty}} \frac{\log^+ T_2(r)}{\log r}.$$

Lemma 6. *Let $\phi(r)$ and $H(r)$ be two positive nondecreasing and continuous functions which tend to ∞ as $r \to \infty$, $A = A(r) > 1$, and*

$\frac{\phi(Ar)}{\phi(r)} \to c(r \to \infty)$ $(c \geq 1)$. Let $\overline{\lim_{r\to\infty}} \frac{log^+ \phi(r)}{log\, r} = 0$. If $H(r) = O(\phi(r))$, there exists $r_0 > 1$ such that $\frac{H(Ar)}{H(r)}$ is upper bounded in $[r_0, +\infty)$.

Proof. Suppose that $\frac{H(Ar)}{H(r)}$ is not bounded from above in $[r_0, +\infty)$. Then there exists a sequence $\{r_n\}$ such that $r_n \to \infty$ $(n \to \infty)$. For arbitrary $G > 0$ there exists a natural number n_0 such that for $n > n_0$ we have

$$\frac{H(Ar_n)}{H(r_n)} > G. \tag{1}$$

Put $\overline{\lim_{r\to\infty}} \frac{H(r)}{\phi(r)} = k$. Then $0 \leq k < +\infty$. We distinguish two cases.

1. $k \neq 0$. By (1) we obtain

$$\frac{H(Ar_n)}{\phi(Ar_n)} > G \frac{H(r_n)}{\phi(r_n)} \frac{\phi(r_n)}{\phi(Ar_n)}\,.$$

Take $G = 2c$. Since $\overline{\lim_{r\to\infty}} \frac{\log^+ \phi(r)}{\log r} = 0$, by lemma 5 we have

$$\overline{\lim_{r\to\infty}} \frac{\log^+ H(r)}{\log r} = 0.$$

So $\phi(r)$ and $H(r)$ are of regular growth. In addition, $\phi(r)$ and $H(r)$ are two nondecreasing continuous functions. Thus

$$\overline{\lim_{n\to\infty}} \frac{H(Ar_n)}{\phi(Ar_n)} \geq G \overline{\lim_{n\to\infty}} \frac{H(r_n)}{\phi(r_n)} \lim_{n\to\infty} \frac{\phi(r_n)}{\phi(Ar_n)}\,,$$

i.e., $k \geq 2c(1/c)k = 2k$. This is a contradiction.

2. $k = 0$. Then $\lim_{r\to\infty} \frac{H(r)}{\phi(r)} = 0$. Let $G = 4c$. By (1) we get $\frac{H(Ar_n)}{H(r_n)} > 4c$ for $n > n_0$. So, for arbitrary natural number m, we have

$$H(A^m r_n) > 4cH(A^{m-1} r_n) > \cdots > (4c)^m H(r_n). \tag{2}$$

Since $\lim_{n\to\infty} \frac{\phi(Ar_n)}{\phi(r_n)} = c$, taking $\varepsilon_0 = c > 0$, there exists $n_1 > n_0$ such that for $n > n_1$ we obtain

$$\left| \frac{\phi(Ar_n)}{\phi(r_n)} - c \right| < \varepsilon_0 = c,$$

i.e. $\phi(Ar_n) < 2c\phi(r_n)$.

Thus, for arbitrary natural number m, we get

$$\phi(A^m r_n) < 2c\phi(A^{m-1} r_n) < \cdots < (2c)^m \phi(r_n). \tag{3}$$

Take $n = n_2 > n_1 > n_0$. By (2) and (3) we have

$$\frac{H(A^m r_{n_2})}{\phi(A^m r_{n_2})} > \frac{(4c)^m H(r_{n_2})}{(2c)^m \phi(r_{n_2})} = 2^m \frac{H(r_{n_2})}{\phi(r_{n_2})} \to \infty \quad (m \to \infty) \tag{4}$$

As $m \to \infty$, we obtain $A^m r_{n_2} \to \infty$. So, (4) is a contradiction to $\lim_{r\to\infty} \frac{H(r)}{\phi(r)} = 0$.

This completes the proof of the Lemma 6. □

Lemma 7. *Let f be a transcendental entire function with $T(r,f) = O(e^{(\log r)^\alpha})$ $(0 < \alpha < 1)$. Then*

(i) $T(r,f) \sim \log M(r,f) \quad (r \to \infty,\ r \notin E)$,

(ii) $T(\delta r,f) \sim T(r,f) \quad (r \to \infty,\ b \geq 2,\ r \notin E)$,

where E is a set of finite logarithmic measure.

Proof. We may assume that $f(0) = 1$. Otherwise, we only need to make a transformation

$$F(z) = f(z) - f(0) + 1.$$

By Jeesen's theorem

$$N(r,1/f) = \int_0^r \frac{n(t,1/f)}{t} dt = \frac{1}{2\pi} \int_0^{2\pi} \log|f(re^{i\theta})| d\theta \leq \log M(r,f), \tag{5}$$

for $r > 1$ and $A > 1$. By (5) we have

$$n(r,1/f) \log A \leq \int_r^{Ar} \frac{n(t,1/f)}{t} dt \leq N(Ar,1/f) \leq \log M(Ar,f).$$

So

$$n(r,1/f) \leq \frac{\log M(Ar,f)}{\log A}. \tag{6}$$

Since $T(r,f) = O(e^{(\log r)^\alpha})$ $(0 < \alpha < 1)$, by Lemma 1 we get

$$\log M(r,f) = O(e^{(\log r)^\alpha}). \tag{7}$$

Take $A = r^{\sigma(r)}$ and $\sigma(r) = \dfrac{1}{(\log r)^\alpha}$. By (6) we obtain

$$n(r, 1/f) \leq \frac{\log M(r^{1+\sigma(r)}, f)}{\sigma(r)\log r}\,. \tag{8}$$

Therefore, putting $r = e^u$ we have

$$\begin{aligned} \frac{e^{(\log r^{1+\sigma(r)})^\alpha}}{r^{1/2}\sigma(r)\log r} &= \frac{e^{(1+\frac{1}{(\log r)^\alpha})^\alpha(\log r)^\alpha}}{r^{1/2}(\log r)^{1-\alpha}} \\ &= \frac{e^{(1+1/u^\alpha)^\alpha u^\alpha}}{(e^u)^{1/2}u^{1-\alpha}} \\ &= \frac{1}{u^{1-\alpha}e^{u^\alpha(\frac{1}{2}u^{1-\alpha}-(1+1/u^\alpha)^\alpha)}}\,. \end{aligned} \tag{9}$$

Since $0 < \alpha < 1$, for sufficiently large value of u we have $\dfrac{1}{2}u^{1-\alpha} - (1 + 1/u^\alpha)^\alpha > 0$ and $\dfrac{1}{2}u^{1-\alpha} - (1 + 1/u^\alpha)^\alpha)$ increases. By (9), for sufficiently large of r, $\dfrac{e^{(\log r^{1+\sigma(r)})^\alpha}}{r^{1/2}\sigma(r)\log r}$ decreases, and by (8) and (7) we have

$$\begin{aligned} Q(r) &= r\int_r^{+\infty} \frac{n(t, 1/f)}{t^2}dt \\ &\leq r\int_r^{+\infty} \frac{\log M(t^{1+\sigma(t)}, f)}{t^2\sigma(t)\log t}dt \\ &= \lim_{b\to+\infty} r\int_r^{b} \frac{O(e^{(\log t^{1+\sigma(t)})^\alpha})}{t^2\sigma(t)\log t}dt \\ &= \lim_{b\to+\infty} O\Big(r\int_r^{b} \frac{e^{(\log t^{1+\sigma(t)})^\alpha})}{t^2\sigma(t)\log t}dt\Big) \\ &\leq \frac{r^{1/2}O(e^{(\log r^{1+\sigma(r)})^\alpha})}{\sigma(r)\log r}\int_r^{+\infty} t^{-3/2}dt \\ &= \frac{2\log M(r^{1+\sigma(r)}, f)}{\sigma(r)\log r}\,. \end{aligned} \tag{10}$$

Let $\phi(r) = e^{(\log r)^\alpha}$ $(0 \le \alpha < 1,\ \beta > 1)$, $A = r^{\sigma(r)}$ and $\sigma(r) = \dfrac{1}{(\log r)^\alpha}$.
Then

$$\begin{aligned}
\frac{\phi(Ar)}{\phi(r)} &= \frac{e^{(\log r^{1+\sigma(r)})^\alpha}}{e^{(\log r)^\alpha}} \\
&= e^{(\log r)^\alpha[(1+\sigma(r))^\alpha - 1]} \\
&= e^{(\log r)^\alpha\, \alpha\sigma(r)(1+o(1))} \\
&= e^{(\log r)^\alpha\, \alpha \frac{1}{(\log r)^\alpha}(1+o(1))} \longrightarrow e^\alpha (\ge 1) \quad (r \to \infty).
\end{aligned} \tag{11}$$

By (7), $\log M(r,f) = O(\phi(r))$ with $\overline{\lim\limits_{r\to\infty}} \frac{\log^+ \phi(r)}{\log r} = 0$ and $\log M(r,f)$ is a nondecreasing continuous functions. By (10), (11) and Lemma 5 there exists $L > 0$ for $r > r_1 > r_0$ such that

$$\begin{aligned}
\frac{Q(r)}{\log M(r,f)} &\le \frac{2\log M(r^{1+\sigma(r)}, f)}{\sigma(r)\log r\ \log M(r,f)} \\
&= \frac{\log M(Ar,f)}{\log M(r,f)}\ \frac{2}{\sigma(r)\log r} \\
&\le \frac{2L}{\sigma(r)\log r} \\
&= \frac{2L}{(\log r)^{1-\alpha}} \to 0 \quad (r \to \infty).
\end{aligned}$$

So

$$Q(r) = o(\log M(r,f)). \tag{12}$$

Since $T(r,f) = O(e^{(\log r)^\alpha})$, the order ρ of f is equal to zero, $n(r, 1/f) = o(r)$ and

$$\begin{aligned}
\log M(r,f) &\le \log \prod_{n=1}^{+\infty} (1 + r/r_n) \\
&= \int_0^{+\infty} \log(1 + r/t)\, dn(t, 1/f) \\
&\le \int_0^{+\infty} \frac{r}{t}\, dn(t, 1/f)
\end{aligned}$$

$$= r \int_0^{+\infty} \frac{n(t,1/f)}{t(t+r)}\, dt$$

$$= r\Big(\int_0^{r} + \int_r^{+\infty} \Big) \frac{n(t,1/f)}{t(t+r)} dt$$

$$\leq r\, \frac{1}{r} \int_0^{r} \frac{n(t,1/f)}{t} dt + r \int_r^{+\infty} \frac{n(t,1/f)}{t^2} dt$$

$$= N(r) + Q(r). \tag{13}$$

So, by Lemma 4, (12) and (13) we obtain

$$\begin{aligned} \log|f(re^{i\theta})| &> N(2R) - kQ(2R) \quad (\zeta R \leq r \leq R,\ r \notin E) \\ &= N(2R) + Q(2R) - (k+1)Q(2R) \\ &\geq \log M(2R,f) - (k+1) \circ (\log M(2R,f)) \\ &= \log M(2R,f)(1-o(1)) \qquad (14) \\ &\geq \log M(r,f)(1-o(1)), \qquad (15) \end{aligned}$$

where E is a set finite logarithmic measure.

On the other hand,

$$\log|f(z)| \leq \log M(r,f) \leq \log M(\delta r, f) \quad (|z| = r,\ \delta \geq 2). \tag{16}$$

In (14), let $2R = \delta r$, $\delta \geq 2$. Then from (14), (15) and (16) it follows that

$$\log|f(z)| \sim \log M(\delta r, f) \quad (r \to \infty,\ r \notin E), \tag{17}$$

$$\log|f(z)| \sim \log M(r, f) \quad (r \to \infty,\ r \notin E). \tag{18}$$

By (18), for sufficiently large value of r, we have

$$\begin{aligned} m(r,f) &= \frac{1}{2\pi} \int_0^{2\pi} \log^+|f(re^{i\theta})| d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log M(r,f)(1+o(1)) d\theta \\ &= \log M(r,f)(1+o(1)) \quad (r \to \infty,\ t \notin E) \end{aligned}$$

So

$$\lim_{r\to\infty}\frac{T(r,f)}{\log M(r,f)}=1\quad (r\notin E). \tag{19}$$

by (17) and (18) we get

$$\log M(\delta r,f)\sim\log M(r,f)\quad (r\to\infty,\ r\notin E,\ \delta\geq 2) \tag{20}$$

By (19) and (20) Lemma 7 is proved. □

3. Proof of Theorem 1

By Lemma 2 we have

$$T(r,f(g))\leq T(M(r,g),f)=O(e^{\log M(r,g))^{\alpha}}). \tag{21}$$

Since $T(r,g)=O((\log r)^{\beta})$, by Lemma 1 we obtain

$$\log M(r,g)=O((\log r)^{\beta}).$$

So

$$T(r,f(g))\leq O(e^{(\log M(r,g))^{\alpha}})=O(e^{(O((\log r)^{\beta}))^{\alpha}})=O(e^{O((\log r)^{\alpha\beta})}).$$

Since $O((\log r)^{\alpha\beta})\leq K(\log r)^{\alpha\beta}$ $(K>0)$, there exists $r_0>1$ and $\mu>0$ $(\alpha\beta<\mu<1)$ such that for $r>r_0$ we get $K(\log r)^{\alpha\beta}<(\log r)^{\mu}$. So

$$T(r,f(g))\leq O(e^{O((\log r)^{\alpha\beta})})<O(e^{(\log r)^{\mu}}),$$

where $\alpha\beta<\mu<1$, i.e.

$$T(r,f(g))=O(e^{(\log r)^{\mu}})\quad (0<\alpha\beta<\mu<1).$$

Thus, by Lemma 7 we have

$$T(r,f(g))\sim\log M(r,f(g))\quad (r\to\infty,\ r\notin E), \tag{22}$$

where E is a set of finite logarithmic measure, and

$$\lim_{r\to\infty}T\Big(\frac{1}{8}M(r,g),f\Big)/T(M(r,g),f)=1\quad (r\notin E). \tag{23}$$

On the other hand, we may assume $g(0) = b$, $G(z) = g(z) - b$, $F(z) = f(z + b)$. Then

$$G(0) = g(0) - b = 0$$

$$F(G(z)) = f(G(z) + b) = f(g(z)).$$

By (22), (23), Lemma 3 and Lemma 7, for sufficiently large values of r, we obtain

$$\begin{aligned}
T(r, f(g)) &= T(r, F(G)) \\
&= \log M(r, F(G))(1 + o(1)) \\
&\geq \log M((1 - o(1))M(r, G), F)(1 + o(1)) \\
&\geq \log M\Big(\frac{1}{4}M(r, G), F\Big)(1 + o(1)) \\
&= \log M\Big(\frac{1}{4}M(r, g - b), F\Big)(1 + o(1)) \\
&\geq \log M\Big(\frac{1}{8}M(r, g), f\Big)(1 + o(1)) \\
&= T\Big(\frac{1}{8}M(r, g), f\Big)(1 + o(1)) \\
&= T(M(r, g), f)(1 + o(1)) \quad (r \notin E). \qquad (24)
\end{aligned}$$

Thus, by (21) and (24) we get

$$T(r, f(g)) \sim T(M(r, g), f) \quad (r \to \infty,\ r \notin E).$$

Hence, for arbitrary small $\varepsilon > 0$, there exists $r_0 > 0$ such that for $r > r_0$, we have

$$(1 - \varepsilon)T(M(r, g), f) < T(r, f(g)) < (1 + \varepsilon)T(M(r, g), f). \qquad (25)$$

By Nevanlinna Theory, except in a set of capacity zero, for arbitrary complex number a we get

$$N\Big(r, \frac{1}{f(g) - a}\Big) \sim T(r, f(g)) \quad (r \to \infty),$$

and

$$N\Big(M(r, g), \frac{1}{f - a}\Big) \sim T(M(r, g), f) \quad (r \to \infty).$$

So there exists $r_1 > r_0 > 0$ such that for $r > r_1$ we obtain

$$(1-\varepsilon)T(r, f(g)) < N\Big(r, \frac{1}{f(g)-a}\Big) < (1+\varepsilon)T(r, f(g)), \tag{26}$$

and

$$\begin{aligned}(1-\varepsilon)N\Big(M(r,g), \frac{1}{f-a}\Big) &< T(M(r,g), f)\\ &< (1+\varepsilon)N\Big(M(r,g), \frac{1}{f-a}\Big)\cdot\end{aligned} \tag{27}$$

Thus, by (26), (25) and (27), there exists $r_2 > r_1 > 0$ such that for $r > r_2$, we have

$$\begin{aligned}(1-\varepsilon')N\Big(M(r,g), \frac{1}{f-a}\Big) &< N\Big(r, \frac{1}{f(g)-a}\Big)\\ &< (1+\varepsilon')N\Big(M(r,g), \frac{1}{f-a}\Big),\end{aligned} \tag{28}$$

where $\varepsilon' = \varepsilon'(\varepsilon)$ and $0 < \varepsilon < \varepsilon' < 1$. Since ε, ε' are arbitrary, by (25) and (28) we get

$$\begin{aligned}\overline{\lim_{r\to\infty, r\notin E}} \frac{N\Big(r, \frac{1}{f(g)-a}\Big)}{T(r, f(g))} &\le \overline{\lim_{r\to\infty, r\notin E}} \frac{(1+\varepsilon')N\Big(M(r,g), \frac{1}{f-a}\Big)}{(1-\varepsilon)T(M(r,g), f)}\\ &= 1-\delta(a, f),\end{aligned}$$

and

$$\begin{aligned}\overline{\lim_{r\to\infty, r\notin E}} \frac{N\Big(r, \frac{1}{f(g)-a}\Big)}{T(r, f(g))} &\ge \overline{\lim_{r\to\infty, r\notin E}} \frac{(1-\varepsilon')N\Big(M(r,g), \frac{1}{f-a}\Big)}{(1+\varepsilon)T(M(r,g), f)}\\ &= 1-\delta(a, f).\end{aligned}$$

So we have

$$\delta(a, f(g)) = \delta(a, f).$$

This is the proof of Theorem 1. $\square$

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