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# STRONG POLYNOMIAL-TIME SOLVABILITY OF A MINIMUM CONCAVE COST NETWORK FLOW PROBLEM

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Abstract. A new simple proof is given of the strong polynomial-time solvability of the single source uncapacitated minimum concave cost network flow problem (SSUMCCNFP) with a fixed number of nonlinear arc costs.

## 1. The Problem

Let $\mathbf{G} = (N_G, A_G)$ be a directed graph consisting of a set $N_G$ of $N$ nodes and a set $A_G$ of $n$ ordered pairs of distinct nodes called arcs. With each arc $a_i$ we associate a *concave cost function* $g_i(t) : R_+ \to R_+$ and with each node $j$ a demand $d_j$ such that $\sum_{j=1}^{N} d_j = 0$. For each $j$ let $A_j^+$ ($A_j^-$, resp.) be the set of arcs entering (leaving, resp.) node $j$. One of the most challenging problems of combinatorial and global optimization is the following

$$\text{(1)} \qquad \text{MCCNFP} \quad \min \sum_{i:a_i \in A_G} g_i(x_i)$$

$$\text{(2)} \qquad \text{s.t.} \sum_{i:a_i \in A_j^+} x_i - \sum_{i:a_i \in A_j^-} x_i = d_j \quad j = 1, \ldots, N$$

$$\text{(3)} \qquad 0 \le x_i \le q_i \quad i = 1, \ldots, n.$$

Nodes with negative demands are called the *sources*, nodes with positive demands are the *sinks*. If $d_j < 0$ is the demand of a source then $s_j = -d_j$ is also called the *supply*. A vector $x = (x_i, a_i \in A_G)$ such that $0 \le x_i \le q_i$ $\forall a_i \in A_G$ is called a *flow* in $\mathbf{G}$. The component $x_i$ is the *value of the flow* on the arc $a_i$. A flow $x$ satisfying (3) is said to be *feasible*. So the

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MCCNFP is, for a given demand vector $d$, to find a feasible flow in **G** with minimum cost.

In view of its relevance to numerous applications in operations research, economics, engineering, etc. MCCNFP has been a subject of intensive research (see e.g. [1], [2], [3], [8], [9] and references therein).

MCCNFP is a concave minimization problem under network constraints. When $q_i = +\infty\ \forall i$ and there is only one single source, i.e. $d_j > 0$ for just one $j$, the problem is referred to as the single-source uncapacitated minimum concave cost network flow problem (SSU MCCNFP). It is well known that this special variant of MCCNFP is still NP-hard (see e.g. [9]). Since the arcs with nonlinear costs are the only nonlinear elements in SSU MCNFP, the complexity of this problem should critically depend on the number $k$ of these arcs.

For the sake of convenience, denote the problem SSU MCCNFP with a fixed number $k$ of nonlinear arc costs by FP($k$). While the general linearly constrained concave minimization problem is still NP-hard even when the number of nonlinear variables is fixed, FP($k$) has been proved to be solvable in polynomial time. Recall that in the complexity model generally adopted for problems with a nonlinear objective function (cf [4], it is assumed that there exists an oracle providing us with the required function values. An algorithm is then called *(strongly) polynomial-time* if both the *number of operations* (additions, multiplications, comparisons etc.) and the *number of objective function evaluations* it performs are (strongly) polynomial in the input length.

Actually the first strongly polynomial-time algorithm for FP($k$) was given in [14]. In a preliminary stage this algorithm reduces FP($k$) to a polynomially equivalent production-transportation problem with $r = k+1$ factories:

$$\begin{aligned} \mathrm{PTP}(r) \qquad & \text{minimize} \quad h(y_1,\dots,y_r) + \sum_{i,j} c_{ij}x_{ij} \\ & \text{subject to} \quad \sum_{j=1}^{m} x_{ij} = y_i, \quad i = 1,\dots,r \\ & \qquad \sum_{i=1}^{r} x_{ij} = d_j \quad j = 1,\dots,m \\ & \qquad x_{ij} \geq 0 \quad i = 1,\dots,r,\ j = 1,\dots,m \end{aligned}$$

where $h(y_1,\dots,y_r)$ is a continuous concave function on $R^r_+$. The main stage of the mentioned algorithm then solves PTP($r$) by a procedure requiring at most $P_r(m)$ elementary operations and $Q_r(m)$ evaluations of the nonlinear function $h(y)$, where $P_r(m)$ and $Q_r(m)$ are polynomials in $m$. Although strongly polynomial-time, the algorithm in [14] is rather complicated and has been established by a quite elaborate argument.

In the present paper we shall provide a new and much simpler strongly polynomial-time algorithm for PTP($r$) and thereby for FP($k$). This new algorithm turns out to be a direct extension of a very efficient algorithm earlier proposed for PTP(2), i.e. FP(1), in [6] and [13]. For small values of $r$ it should also be much more practical than the one given earlier in [14].

## 2. Equivalent Parametric Problem

As usual, we assume that $c_{ij} \geq 0\ \forall i, j$, and $h(y)$ is increasing on $R^r_+$, i.e. $h(y') \geq h(y)$ whenever $y' \geq y$. By substituting $\sum_{j=1}^{n} x_{ij}$ for $y_i$ in $h(y)$ we can reformulate PTP($r$) as

$$\text{PTP}(r) \qquad \min h\Big(\sum_j x_{1j}, \dots, \sum_j x_{rj}) + \sum_{i,j} c_{ij}x_{ij}$$

$$\text{s.t.} \quad \sum_{i=1}^{r} x_{ij} = d_j \quad j = 1, \dots, m$$

$$x_{ij} \geq 0 \quad i = 1, \dots, r,\ j = 1, \dots, m$$

To this problem we associate the parametric program

$$\text{P}(t) \qquad \min \sum_{i=1}^{r} t_i \sum_{j=1}^{m} x_{ij} + \sum_{i,j} c_{ij}x_{ij}$$

$$\text{s.t.} \quad \sum_{i=1}^{r} x_{ij} = d_j \quad j = 1, \dots, m$$

$$x_{ij} \geq 0 \quad i = 1, \dots, r,\ j = 1, \dots, m$$

where $t \in R^r_+$. It is well known that the parameter domain $R^r_+$ can then be partitioned into a finite collection $\mathcal{P}$ of polyhedrons ("cells"), such that $\cup\big\{\Pi \big| \Pi \in \mathcal{P}\big\} = R^n_+$ and for each $\Pi \in \mathcal{P}$ there is a basic solution $x^\Pi$ which is optimal to $P(t)$ for all $t \in \Pi$. If $\mathcal{P}$ is such a collection of cells, then

**Proposition 1.** *An optimal solution of* PTP($r$) *is* $x^{\Pi^*}$ *where*

$$(4) \qquad \Pi^* \in \text{argmin}\Big\{h\Big(\sum_{j=1}^{m} x^\Pi_{1j}, \dots, \sum_{j=1}^{m} x^\Pi_{kj}\Big) + \sum_{i,j} c_{ij}x^\Pi_{ij} \big|\ \Pi \in \mathcal{P}\Big\}.$$

*Proof.* This result can be derived from a general theorem on monotonic optimization (see [7], [12]). For completeness we give here a direct proof. To simplify the notation we rewrite PTP($r$) and P($t$) as

$$(5)\qquad \min\Big\{h(Dx)+\langle c,x\rangle\Big|\ x\in G\Big\},\quad \min\Big\{\langle t,Dx\rangle+\langle c,x\rangle\Big|\ x\in G\Big\}$$

where $x \in R^{rm}$, $G$ is a polytope in $R^{rm}_+$, $c \in R^{rm}_+$, $D \in R^{r\times(rm)}$, and $h : R^r_+ \to R$ a continuous quasiconcave function such that $h(y') \geq h(y)$ whenever $y \in R^r_+$, $y' \geq y$. Let $f(x) = h(Dx) + \langle c, x\rangle$. For each $(t, \lambda) \in R^r_+ \times R_+$ let $x^{(t,\lambda)}$ be an arbitrary basic optimal solution of the parametric problem

$$(6)\qquad \min\Big\{\langle t,Dx\rangle+\lambda\langle c,x\rangle\Big|\ x\in G\Big\}.$$

Let $\gamma = \inf\big\{f(x^{(t,\lambda)})\big|\ t \in R^r_+, \lambda \geq 0\big\}$. We first show that

$$(7)\qquad \min\big\{f(x)\big|\ x\in G\big\}=\gamma.$$

Denote by $E$ the convex hull of the set $\big\{x^{(t,\lambda)}\big|\ t \in R^r_+, \lambda \geq 0\big\}$. This set is finite because it is contained in the vertex set of $G$, so $E$ is a polytope and if we define $K = \big\{x|\langle c,x\rangle \geq 0, Dx \geq 0\big\}$, then $M := E + K$ is a polyhedral set ([10], Corollary 19.3.2). Furthermore, for any $x \in M$. i.e. $x = y + z$ with $y \in E$, $z \in K$ one has $Dz \geq 0$, $\langle c, z\rangle \geq 0$, hence, $f(x) = h(D(y + z)) + \langle c, y + z\rangle \geq h(Dy) + \langle c, y\rangle = f(y)$. Therefore, $f(x) \geq \gamma\ \forall x \in M$. Now suppose [7] is not true, so that there exists $\bar{x} \in G \setminus M$. Since $\bar{x} \notin M$ one can find $p \in R^n$ and $x^1 \in \partial M$ (the boundary of M) satisfying

$$(8)\qquad \langle p,x-x^1\rangle\geq 0\quad \forall x\in M;\quad \langle p,\bar{x}-x^1\rangle<0.$$

([10], Corollary 11.6.2). For any $y \in K$, we have $x^1 + y \in M + K = M$, hence $\langle p, y\rangle \geq 0$, and therefore, $p = \lambda c + D^T t$, with $\lambda \geq 0$, $t \in R^r_+$. Since $x^{(t,\lambda)}$ is an optimal solution of (6) while $\overline{x} \in G$, we have $\langle p, x^{(t,\lambda)}\rangle \leq \langle p, \bar{x}\rangle < \langle p, x^1\rangle$, (by the right inequality (8)), hence $\langle p, x^{(t,\lambda)} - x^1\rangle < 0$, conflicting with the left inequality (8) because $x^{(t,\lambda)} \in M$. This contradiction proves (7), and so

$$(9)\qquad \min\Big\{f(x)\big|x\in G\Big\}=\min\Big\{f(x^{(t,\lambda)})\Big|\,t\in R^n_+,\ \lambda\geq 0\Big\}.$$

Further, since for every $t \in R^n_+$ we have $x^{(t,0)} = \lim_{\nu\to\infty} x^{(t,1/\nu)}$, the continuity of $f(x)$ implies that

$$\begin{aligned}\inf\left\{f(x^{(t,\lambda)})\big|\, t \in \mathbb{R}^n_+,\ \lambda \geq 0\right\} &= \inf\left\{f(x^{(t,\lambda)})\big|\, t \in R^n_+,\ \lambda > 0\right\}\\ &= \inf\left\{f(x^{(t,\lambda)})\big|\, t \in R^n_+\right\}.\end{aligned} \tag{10}$$

Now if $\mathcal{P}$ is a collection of cells covering $R^n_+$ then the relation (4) follows from (9) and (10) by taking $x^{(t,1)} = x^\Pi$ for all $t \in \Pi$ □

Thus, to solve PTP($r$) it suffices to generate a collection $\mathcal{P}$ of cells covering the whole $R^r_+$. We show that for fixed $r$ such a collection $\mathcal{P}$ exists whose cardinality is bounded by a polynomial in $m$.

## 3. Construction of the collection $\mathcal{P}$

Observe that the dual of P($t$) is

$$\mathrm{P}^*(t) \qquad \max \sum_{j=1}^{m} d_j u_j$$
$$\text{s.t.} \quad u_j \leq t_i + c_{ij}, \quad i = 1,\dots,r \quad j = 1,\dots,m.$$

Also for any fixed $t \in R^r_+$, a basic solution of P($t$) is a vector $x^t$ such that for every $j = 1,\dots,m$ there is an $i_j$ satisfying

$$x^t_{ij} = \begin{cases} d_j & i = i_j \\ 0 & i \neq i_j. \end{cases} \tag{11}$$

By the duality theorem of linear programming, $x^t$ defined by (11) is a basic optimal solution of P($t$) if and only if there exists a feasible solution $u = (u_1,\dots,u_m)$ of P$^*(t)$ satisfying

$$u_j \begin{cases} = t_i + c_{ij} & i = i_j \\ \leq t_i + c_{ij} & i \neq i_j \end{cases} \tag{12}$$

or, alternatively, if and only if for every $j = 1,\dots,m$:

$$i_j \in \operatorname{argmin}_{i=1,\dots,r}\{t_i + c_{ij}\}. \tag{13}$$

Now let $I_*^2$ be the set of all pairs $(i_1, i_2)$ such that $i_1 < i_2 \in \{1, \dots, r\}$. Define a cell to be a polyhedron $\Pi \subset R_+^r$ which is the solution set of a linear system formed by taking, for every pair $(i_1, i_2) \in I_*^2$ and every $j = 1, \dots, m$, one of the following inequalities:

$$t_{i_1} + c_{i_1 j} \leq t_{i_2} + c_{i_2 j}, \qquad t_{i_1} + c_{i_1 j} \geq t_{i_2} + c_{i_2 j}. \tag{14}$$

Then for every $j \in \{1, \dots, m\}$ the order of magnitude of the sequence

$$t_i + c_{ij}, \quad i = 1, \dots, r$$

remains unchanged as $t$ varies over a cell $\Pi$. Hence the index $i_j$ satisfying (12) and (13) remains the same for all $t \in \Pi$, in other words, $x^t$ (basic optimal solution of P$(t)$) equals a constant vector $x^\Pi$ for all $t \in \Pi$. Let $\mathcal{P}$ be the collection of all cells defined that way. Since every $t \in R_+^r$ satisfies one of the inequalities (14) for every $(i_1, i_2) \in I_*^2$ and every $j = 1, \dots, m$, the collection $\mathcal{P}$ covers all of $R_+^r$. Let us estimate an upper bound of the number of cells in $\mathcal{P}$.

Observe that for any fixed pair $(i_1, i_2) \in I_*^2$ we have $t_{i_1} + c_{i_1 j} \leq t_{i_2} + c_{i_2 j}$ if and only if $t_{i_1} - t_{i_2} \leq c_{i_2 j} - c_{i_1 j}$. Let us sort the numbers $c_{i_2 j} - c_{i_1 j}$, $j = 1, \dots, m$, in increasing order

$$c_{i_2 j_1} - c_{i_1 j_1} \leq c_{i_2 j_2} - c_{i_1 j_2} \leq \cdots \leq c_{i_2 j_m} - c_{i_1 j_m} \tag{15}$$

and let $\nu_{i_1, i_2}(j)$ be the position of $c_{i_2 j} - c_{i_1 j}$ in this ordered sequence.

**Proposition 2.** *A cell $\Pi$ is characterized by a mapping $\ell_\Pi : I_*^2 \to \{1, \dots, m, m+1\}$ such that $\Pi$ is the solution set of the linear system*

$$t_{i_1} + c_{i_1 j} \leq t_{i_2} + c_{i_2 j} \quad \forall (i_1, i_2) \in I_*^2, \forall j \in \{j |\ \nu_{i_1, i_2}(j) \geq \ell_\Pi(i_1, i_2)\} \tag{16}$$

$$t_{i_1} + c_{i_1 j} \geq t_{i_2} + c_{i_2 j} \quad \forall (i_1, i_2) \in I_*^2, \forall j \in \{j |\ \nu_{i_1, i_2}(j) < \ell_\Pi(i_1, i_2)\} \tag{17}$$

*Proof.* Let $\Pi \subset R_+^r$ be a cell. For every pair $(i_1, i_2)$ with $i_1 < i_2$ denote by $J_\Pi^{i_1 i_2}$ the set of all $j = 1, \dots, m$ such that the left inequality (14) holds for all $t \in \Pi$, and define

$$\ell_\Pi(i_1, i_2) = \begin{cases} \min \left\{ \nu_{i_1, i_2}(j) \middle|\ j \in J_\Pi^{i_1 i_2} \right\} & \text{if } J_\Pi^{i_1, i_2} \neq \emptyset \\ m + 1 & \text{if } J_\Pi^{i_1, i_2} = \emptyset. \end{cases} \tag{18}$$

It is easy to see that $\Pi$ is then the solution set of the system (16)-(17). Indeed, let $t \in \Pi$. If $\nu_{i_1,i_2}(j) \geq \ell_\Pi(i_1,i_2)$ then $\ell_\Pi(i_1,i_2) \neq m+1$, so $\ell_\Pi(i_1,i_2) = \nu_{i_1,i_2}(l)$ for some $l \in J_\Pi^{i_1,i_2}$. Then $t_{i_1} + c_{i_1 l} \leq t_{i_2} + c_{i_2 l}$, hence $t_{i_1} - t_{i_2} \leq c_{i_2 l} - c_{i_1 l}$ and since the relation $\nu_{i_1,i_2}(j) \geq \nu_{i_1,i_2}(l)$ means that $c_{i_2 j} - c_{i_1 j} \geq c_{i_2 l} - c_{i_1 l}$ it follows that $t_{i_1} - t_{i_2} \leq c_{i_2 j} - c_{i_1 j}$, i.e. $t_{i_1} + c_{i_1 j} \leq t_{i_2} + c_{i_2 j}$. Therefore (16) holds. On the other hand, if $\nu_{i_1,i_2}(j) < \ell_\Pi(i_1,i_2)$ then by definition $j \notin J_\Pi^{i_1,i_2}$, hence (17) holds, too (since from the definition of a cell, any $t \in \Pi$ must satisfy one of one of inequalities (14). Thus, every $t \in \Pi$ is a solution of the system (16)-(17). Conversly, if $t$ satisfies (16)-(17) then for every $(i_1,i_2) \in I_*^2$, $t$ satisfies the left inequality (14) for $j \in J_\Pi^{i_1,i_2}$ and the right inequality for $j \notin J_\Pi^{i_1.i_2}$, hence $t \in \Pi$. Therfore, each cell $\Pi$ is determined by a mapping $\ell_\pi : I_*^2 \to \{1,\ldots.m+1\}$. Furthermore, it is easily seen that $\ell_\Pi \neq \ell_{\Pi'}$ for two different cells $\Pi, \Pi'$. Indeed, if $\Pi \neq \Pi'$ then at least for some $(i_1,i_2) \in I_*^2$ and some $j = 1,\ldots,m$, one has $j \in J_\Pi^{i_1 i_2} \setminus J_{\Pi'}^{i_1 i_2}$. Then $\ell_\Pi(i_1,i_2) \leq \nu_{i_1,i_2}(j)$ but $\ell_{\Pi'}(i_1,i_2) > \nu_{i_1,i_2}(j)$. □

**Corollary 1.** *The total number of cells is bounded above by* $(m+1)^{r(r-1)/2}$.

*Proof.* The number of cells does not exceed the number of different mappings $\ell : I_2^* \to \{1,\ldots,m+1\}$ and there are $(m+1)^{r(r-1)/2}$ such mappings. □

In particular, for $r = 2$ there are at most $m+1$ cells, as was proved in [12]. In fact the above method is a direct (but far from trivial) extension of the method in the latter paper.

*Remark.* The formulation of PTP($r$) in Section 2 as a concave minimization problem over a polytope indicates that an optimal solution can be sought among the $(r+1)^m$ vertices of the feasible polytope

$$G = \Big\{x = (x_{ij}) \geq 0 \Big| \sum_{i=1}^{r} x_{ij} = d_j, j = 1,\ldots,m\Big\}.$$

The above approach shows that only $(m+1)^{r(r-1)/2}$ of these vertices should be investigated. Furthermore, as can be seen from the proof of Proposition 1, this approach is still valid even if the function $h(y)$ is quasiconcave but $f(x) = h(Dx) + \langle c, x\rangle$ is not.

For every cell $\Pi$ (defined by a mapping $\ell_\Pi : I_*^2 \to \{1,\ldots,m+1\}$) the associated basic solution $x^\Pi$ can be computed as follows: for every $j = 1,\ldots,m$, use the relations ($i_1 < i_2$):

$$t_{i_1} + c_{i_1 j} \leq t_{i_2} + c_{i_2 j} \quad \text{if and only if} \quad \nu_{i_1,i_2}(j) \geq \ell_\Pi(i_1,i_2)$$

to define the index $i_j$ satisfying (11) (i.e. (13)). Then $x^\Pi$ is the vector such that (see (11)):

$$x^\Pi_{i_j j} = d_j, \quad x^\Pi_{ij} = 0 \text{ for } i \neq i_j.$$

To sum up, the proposed algorithm for solving PTP($r$) involves the following steps:

1) Ordering the sequences $c_{i_2 j} - c_{i_1 j}$, $j = 1, \dots, m$ for every pair $(i_1, i_2) \in I^2_*$, so as to determine $\nu_{i_1, i_2}(j)$, $j = 1, \dots, m$, $(i_1, i_2) \in I^2_*$.

2) Computing the vector $x^\Pi$ for every cell $\Pi \in \mathcal{P}$ ($\mathcal{P}$ is the collection of cells defined by the mappings $\ell_\Pi : I^*_2 \to \{1, \dots, m+1\}$).

3) Computing the values $f(x^\Pi)$ and select $\Pi^*$ according to (4).

The steps 1) and 2) require obviously a number of elementary operations bounded by a polynomial in $m$, while the step 3) requires $m^{r(r-1)/2}$ evaluations of $f(x)$.

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