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SOME COLLECTIONS OF FUNCTIONS DENSE IN AN ORLICZ SPACE

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Abstract. This paper presents sufficient conditions for a translation invariant subspace of $L_1(\mathbb{R}^n) \cap L_\Phi(\mathbb{R}^n)$ to be dense in the Orlicz space $L_\Phi(\mathbb{R}^n)$.

Introduction

Let φ be a function defined on $\mathbb{R}^n$ and a be a function defined on $\mathbb{Z}^n$. Their semi-discrete convolution [7] is defined by, for any $x \in \mathbb{R}^n$,

$$\varphi *' a(x) = \sum_{\alpha \in \mathbb{Z}^n} \varphi(x-\alpha) a(\alpha),$$

for which the series converges absolutely. Denote by $\ell_0(\mathbb{Z}^n)$ the space of all finitely supported functions on $\mathbb{Z}^n$ and $S_0(\varphi)$ the image of $\ell_0(\mathbb{Z}^n)$ under $\varphi *'$. If $\varphi \in C(\mathbb{R}^n)$ then $\varphi *' a \in C(\mathbb{R}^n)$.

A collection F of functions on $\mathbb{R}^n$ is called shift invariant [7] if for each $f \in F$, $\alpha \in \mathbb{Z}^n$, $f(. + \alpha) \in F$. Then $S_0(\varphi)$ is a linear span of the integer translates of φ and is shift invariant. A set F is called translation invariant if

$$\tau_t : f \longrightarrow f(. + t)$$

maps F into F for each $t \in \mathbb{R}^n$ and F is dilation invariant if

$$\sigma_h : f \longrightarrow f(h^{-1}.)$$

maps F into itself for each $h > 0$. Denote

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$$U_h = \bigcup_{j=1}^{\infty} \sigma_h^j S_0(\varphi).$$

The problem of finding sufficient conditions on a collection of functions generated by translations of a single function to be dense in $L_p(\mathbb{R}^n)$ or $C_0(\mathbb{R}^n)$ was studied by Kang Zhao in [7]. He showed that for a subspace which is generated by U_h, where φ satisfies some certain conditions, the spanU_h is dense in $L_p(\mathbb{R}^n)$ or $C_0(\mathbb{R}^n)$. This leads to the natural question under what conditions on the collection U_h and function φ, the linear span of U_h is dense in the Orlicz space $L_\Phi(\mathbb{R}^n)$?

In this paper, modifying the method of [7] we give some sufficient conditions for a collection of functions generated by translations of a single function in $L_1(\mathbb{R}^n) \cap L_\Phi(\mathbb{R}^n)$, to be dense in $L_\Phi(\mathbb{R}^n)$. We have to overcome some difficulties because $C_0^\infty(\mathbb{R}^n)$ is dense in $L_p(\mathbb{R}^n)$ but $C_0^\infty(\mathbb{R}^n)$ is not generally dense in $L_\Phi(\mathbb{R}^n)$. Moreover, the results of this paper are generalizations of the ones given by Kang Zhao in [7].

Results

Let $\Phi(t) : [0, +\infty) \longrightarrow [0, +\infty]$ be an arbitrary Young function, i.e., $\Phi(0) = 0$, $\Phi(t) \geq 0$, $\Phi(t) \not\equiv 0$ and $\Phi(t)$ is convex. We denote by $\overline{\Phi}(t)$ the Young conjugate function of $\Phi(t)$, i.e.,

$$\overline{\Phi}(t) = \sup_{s \geq 0} \big\{ ts - \Phi(s) \big\}$$

and by $L_\Phi(\mathbb{R}^n)$, the space of measurable functions $f(x)$ on $\mathbb{R}^n$ such that

$$\Big| \int_{\mathbb{R}^n} f(x) g(x) dx \Big| < \infty$$

for all $g(x)$ with $\rho(g, \overline{\Phi}) < \infty$, where

$$\rho(g, \overline{\Phi}) = \int_{\mathbb{R}^n} \overline{\Phi}(|g(x)|) dx.$$

Then $L_\Phi(\mathbb{R}^n)$ is a Banach space with respect to the Orlicz norm

$$\|f\|_\Phi = \sup \Big\{ \Big| \int_{\mathbb{R}^n} f(x) g(x) dx \Big| : \ \rho(g, \overline{\Phi}) \leq 1 \Big\}.$$

A Young function Φ is said to satisfy the Δ_2-condition if

$$\Phi(2x) \le K\Phi(x), \quad x \ge 0 \quad \text{for some absolute constant } K > 0 \text{ [see 5].}$$

We first recall some results on Orlicz spaces [5,4,1]. We have:

1. $\|f\|_\Phi = \sup\Big\{ \int\limits_{\mathbb{R}^n} |f(x)g(x)\Big|dx : \ \rho(g, \overline{\Phi}) \le 1 \Big\}.$

2. $L_\Phi(\mathbb{R}^n) \subset S'$, where S' is the dual of the space S of rapidly decreasing test functions.

3. If $f \in L_\Phi(\mathbb{R}^n)$ then $\|f(.+t)\|_\Phi = \|f\|_\Phi$ for each $t \in \mathbb{R}^n$.

4. Let $f \in L_\Phi(\mathbb{R}^n)$, $h \in L_1(\mathbb{R}^n)$ and $g \in L_{\overline{\Phi}}(\mathbb{R}^n)$. Then $\|f * h\|_\Phi \le \|f\|_\Phi \|h\|_1$ and

$$\int\limits_{\mathbb{R}^n} |f(x)g(x)|dx \le \|f\|_\Phi \|g\|_{\overline{\Phi}}.$$

Lemma 1. *Let Φ be a Young function satisfying Δ_2-condition. Then for each $f \in L_\Phi(\mathbb{R}^n)$, one has*

$$\lim_{t \to 0} \|f(.+t) - f\|_\Phi = 0, \qquad t \in \mathbb{R}^n. \tag{1}$$

Proof. We first prove that $f \in L^1_{\ell oc}(\mathbb{R}^n)$. For any $m = 1, 2, 3 \dots$, put $K_m = [-m, m]^n$. It follows from the convexity of Φ that

$$\Phi\Big(\frac{1}{\text{mes}K_m} \int\limits_{K_m} |f(x)|dx\Big) \le \frac{1}{\text{mes}K_m} \int\limits_{K_m} \Phi(|f(x)|)dx.$$

Since $f \in L_\Phi(\mathbb{R}^n)$, we have $\int_{\mathbb{R}^n} \Phi(|f(x)|)dx < \infty$. From the above inequality and the hypothesis of Φ, it follows that $\int_{K_m} |f(x)|dx < \infty$. Hence $f \in L^1_{\ell oc}(\mathbb{R}^n)$.

To prove the lemma, it suffices to show that for any sequence $\{t_k\} \subset \mathbb{R}^n$, if $t_k \to 0$, as $k \to \infty$, then

$$\lim_{k \to \infty} \|f(.+t_k) - f\|_\Phi = 0.$$

Assume to the contrary that there exists $\{t_k\} \subset \mathbb{R}^n$, $t_k \to 0$ such that

$$\|f(.+t_k) - f\|_\Phi \ge \varepsilon \qquad \text{for some } \ \varepsilon > 0. \tag{2}$$

As shown above, $f \in L^1_{\ell oc}(\mathbb{R}^n)$. For each K_m, we obtain

$$\int_{K_m} |f(x+t_k) - f(x)|dx \to 0, \qquad \text{as } k \to \infty.$$

Therefore, by [2, p.93, Theorem D], there exists a subsequence $\{t_{k_j}\} \subset \{t_k\}$ such that $f(.+t_{k_j}) \to f$ almost everywhere on K_m. Therefore, there exists a subsequence such that $\{f(.+t_{k_{j_h}})\} \to f$ a.e. on $\mathbb{R}^n$. For simplicity, we still denote it by $\{f(.+t_{k_j})\}$.

Since Φ is a convex function and satisfies Δ_2-condition, we have

$$\begin{aligned}\Phi(|f(x+t_{k_j}) - f(x)|) &\le \Phi(|f(x+t_{k_j})| + |f(x)|)\\ &\le \frac{1}{2}[\Phi(2|f(x+t_{k_j})|) + \Phi(2|f(x)|)]\\ &\le \frac{K}{2}[\Phi(|f(x+t_{k_j})|) + \Phi(|f(x)|)].\end{aligned}$$

Hence

$$0 \le \frac{K}{2}\big[\Phi(f(x+t_{k_j})) + \Phi(f(x))\big] - \Phi(|f(x+t_{k_j}) - f(x)|), \quad \forall x \in \mathbb{R}^n.$$

Applying Fatou's lemma to the subsequence $\{\Phi(f(.+t_{k_j}))\}$ and using the equality

$$\lim_{j\to\infty} \int_{\mathbb{R}^n} \Phi(f(x+t_{k_j}))dx = \int_{\mathbb{R}^n} \Phi(f(x))dx,$$

we obtain

$$\begin{aligned}&K\int_{\mathbb{R}^n} \Phi(f(x))dx\\ &\le \lim_{j\to\infty}\inf \int_{\mathbb{R}^n} \Big[\frac{K}{2}[\Phi(f(x+t_{k_j})) + \Phi(f(x))] - \Phi(f(x+t_{k_j}) - f(x))\Big]dx\\ &= \lim_{j\to\infty} \frac{K}{2}\int_{\mathbb{R}^n} [\Phi(f(x+t_{k_j})) + \Phi(f(x))]dx\\ &\qquad - \lim_{j\to\infty}\sup \int_{\mathbb{R}^n} \Phi(f(x+t_{k_j}) - f(x))dx\end{aligned}$$

(3)

$$= K\int_{\mathbb{R}^n} \Phi(f(x))dx - \lim_{j\to\infty}\sup \int_{\mathbb{R}^n} \Phi(f(x+t_{k_j}) - f(x))dx.$$

By inequality (3),

$$\int_{\mathbb{R}^n} \Phi(f(x+t_{k_j}) - f(x))dx \to 0, \quad \text{as } j \to \infty.$$

By [5, Theorem 12], $\|f(.+t_{k_j}) - f\|_\Phi \to 0$, which contradicts (2). □

The subsequent two lemmas can be proved in a manner similar to that of Lemmas 2.1 and 2.2 of [7]. We include their proofs for the sake of completeness. They will be helpful for the understanding of the arguments that will be used in the sequel.

Denote by $\mathbb{R}^*$ the abelian group of all nonzero real numbers with the operation of ordinary multiplication and

$$\operatorname{dist}(\varphi, S)_\Phi = \min\{\|\varphi - f\|_\Phi,\ f \in S\}.$$

Lemma 2. *Let Φ be a Young function satisfying Δ_2-condition and $\varphi \in L_\Phi(\mathbb{R}^n)$. Assume that $\frac{1}{h}$ is an integer larger than 1. If $\varphi \in \overline{\operatorname{span} U_h}$, where $U_h = \bigcup_{j=1}^{\infty} \sigma_h^j S_0(\varphi)$, then $\overline{\operatorname{span} U_h}$ is translation invariant.*

Proof. By the definition of dilatation, for each $h > 0$, we have

$$\sigma_h^j f(x) = f(h^{-j}x) \qquad \text{for all} \quad j \geq 1.$$

Let f be an arbitrary in $\sigma_h^j S_0(\varphi)$, i.e. $f = \sigma_h^j g$ with $g \in S_0(\varphi)$. For any $\alpha \in \mathbb{Z}^n$, we get

$$\begin{aligned} f(x+\alpha) &= \sigma_h^j g(x+\alpha) = g(h^{-j}(x+\alpha)) \\ &= g(h^{-j}x + h^{-j}\alpha) = \sum_{\beta \in \mathbb{Z}^n} \varphi(h^{-j}x - \beta)a(\beta). \end{aligned}$$

Since $g \in S_0(\varphi)$, we have $g(.+\alpha) \in S_0(\varphi)$. Hence $f(.+\alpha) \in \sigma_h^j S_0(\varphi)$. This proves that $\sigma_h^j S_0(\varphi)$ is shift invariant. Therefore U_h is shift invariant and so is $\operatorname{span} U_h$.

For each $\beta \in \mathbb{Z}^n$, by Result 3, we have

$$\|\varphi(.-\beta) - f(.-\beta)\|_\Phi = \|\varphi - f\|_\Phi, \qquad \text{for all } f \in \operatorname{span} U_h.$$

Since $\text{span}U_h$ is shift invariant, we have

$$\text{dist}(\varphi(.-\beta), \text{span}U_h)_\Phi = \text{dist}(\varphi, \text{span}U_h)_\Phi.$$

By virtue of $\varphi \in \overline{\text{span}U_h}$ it follows that $\varphi(.-\beta) \in \overline{\text{span}U_h}$ for every $\beta \in \mathbb{Z}^n$. This implies that $S_0(\varphi) \subset \overline{\text{span}U_h}$. Note that $U_h \bigcup S_0(\varphi) = \sigma_h^{-1} U_h$. Then

$$\overline{\text{span}U_h} = \overline{\text{span}\sigma_h^{-1} U_h}. \tag{4}$$

On the other hand, we have

$$\sigma_h^k \text{span}U_h = \text{span}\sigma_h^k U_h = \sigma_h^{k+1} \text{span}\sigma_h^{-1} U_h. \tag{5}$$

Combining (4) and (5), we conclude that

$$\overline{\text{span}\sigma_h^k U_h} = \overline{\text{span}\sigma_h^{k+1} U_h}, \quad \text{for any } k \geq 1.$$

Therefore

$$\overline{\text{span}U_h} = \bigcap_{j=1}^{\infty} \overline{\text{span}\sigma_h^j U_h}. \tag{6}$$

For each $\alpha \in \mathbb{Z}^n$, $\sigma_h^k S_0(\varphi)$ is $h^k\alpha$-translation invariant and so is $\overline{\text{span}\sigma_h^k U_h}$. From (6) it follows that $\overline{\text{span}U_h}$ is $h^k\alpha$-translation invariant.

Since $\bigcup_{j=k}^{\infty} h^j \mathbb{Z}^n$ is dense in $\mathbb{R}^n$, for each $k \geq 1$, we have, by using Lemma 1,

$$\lim_{t \to 0} \|g(.+t) - g\|_\Phi = 0, \qquad \text{for all } g \in \text{span}U_h.$$

Hence $\overline{\text{span}U_h}$ is translation invariant. □

Lemma 3. *Let Φ be a Young function satisfying Δ_2-condition. Assume that $\varphi \in L_\Phi(\mathbb{R}^n)$ and G is a subgroup of $\mathbb{R}^*$. If*

$$\lim_{h \in G \ h \to 0} \text{dist}(\varphi, \sigma_h S_0(\varphi))_\Phi = 0,$$

then $\overline{\bigcup_{j=1}^{\infty} \sigma_h^j S_0(\varphi)}$ *is a translation invariant subspace of* $L_\Phi(\mathbb{R}^n)$*, for any sequence* $\{h_j\} \subset G$ *with* $\lim_{j\to\infty} h_j = 0$.

Proof. For any $\beta \in \mathbb{Z}^n$, $h \neq 0$ then $h^{-1}\beta = \alpha + \xi$, with $\alpha \in \mathbb{Z}^n$, $\xi \in [0,1)^n$. Fix a function $a \in \ell_0(\mathbb{Z}^n)$. We have

$$\|\varphi(.-\beta) - \sigma_h(\varphi *' a)\|_\Phi = \|\varphi(.-h\xi) - \sigma_h(\varphi *' b)\|_\Phi, \tag{7}$$

with $a(\beta) = b(\beta - \alpha)$. From (1) and the hypothesis

$$\lim_{h\to 0} \operatorname{dist}(\varphi, \sigma_h S_0(\varphi))_\Phi = 0,$$

it follows that

$$\lim_{h\to 0} \|\varphi(.-h\xi) - \sigma_h(\varphi *' b)\|_\Phi = 0.$$

From (7) we conclude that $\varphi(.-\beta) \in \overline{\bigcup_{j=1}^{\infty} \sigma_{h_j} S_0(\varphi)}$, for all $\beta \in \mathbb{Z}^n$. Therefore

$$S_0(\varphi) \subset \overline{\bigcup_{j=1}^{\infty} \sigma_{h_j} S_0(\varphi)}. \tag{8}$$

On the other hand, we have

$$\operatorname{dist}(\sigma_{h_1} f, \sigma_{h_j} S_0(\varphi))_\Phi \leq \operatorname{dist}(f, \sigma_{h_j h_1^{-1}} S_0(\varphi))_\Phi, \quad f \in S_0(\varphi).$$

Therefore

$$\sigma_{h_1} S_0(\varphi) \subset \overline{\bigcup_{j=2}^{\infty} \sigma_{h_j} S_0(\varphi)}.$$

From the hypothesis that G is a subgroup and (8), we get

$$S_0(\varphi) \subset \overline{\bigcup_{j=m}^{\infty} \sigma_{h_j/h_k} S_0(\varphi)}$$

for every $m \geq k \geq 1$. By an argument similar to the previous one, we obtain

$$\sigma_{h_k} S_0(\varphi) \subset \overline{\bigcup_{j=m}^{\infty} \sigma_{h_j} S_0(\varphi)} \tag{9}$$

for every $m \geq k \geq 1$. Hence

$$\overline{\bigcup_{j=1}^{\infty} \sigma_{h_j} S_0(\varphi)} = \bigcap_{k=1}^{\infty} \overline{\bigcup_{j=k}^{\infty} \sigma_{h_j} S_0(\varphi)}. \tag{10}$$

For each $\alpha \in \mathbb{Z}^n$, $\sigma_{h_k} S_0(\varphi)$ is $h_k\alpha$-translation invariant and so is $\overline{\bigcup_{j=m}^{\infty} \sigma_{h_j} S_0(\varphi)}$ for every $m \geq k \geq 1$. From (10) it follows that $\overline{\bigcup_{j=1}^{\infty} \sigma_{h_j} S_0(\varphi)}$ is $h_k\alpha$-translation invariant. For any sequence $\{h_j\} \subset G$, $h_j \to 0$ and $t \in \mathbb{R}^n$ there exists a point $\alpha \in \mathbb{Z}^n$ such that $\|t - \alpha h_j\|$ (the ordinary norm on $\mathbb{R}^n$) is sufficiently small, for some $j \geq 1$.

Let $f \in \overline{\bigcup_{j=1}^{\infty} \sigma_{h_j} S_0(\varphi)}$ and $\varepsilon > 0$. Then there exists a function $g \in \bigcup_{j=1}^{\infty} \sigma_{h_j} S_0(\varphi)$ such that

$$\|f(. + t) - g(. + t)\|_\Phi < \varepsilon. \tag{11}$$

Clearly $g \in \sigma_{h_k} S_0(\varphi)$, for some $k \geq 1$. From (9) we conclude that $g \in \overline{\bigcup_{j=m}^{\infty} \sigma_{h_j} S_0(\varphi)}$ for every integer m larger than 1. From (1) we have

$$\|g(. + t) - g(. + h_j\alpha)\|_\Phi < \varepsilon. \tag{12}$$

A combination of (11) and (12) yields $f(. + t) \in \overline{\bigcup_{j=1}^{\infty} \sigma_{h_j} S_0(\varphi)}$. Thus $\overline{\bigcup_{j=1}^{\infty} \sigma_{h_j} S_0(\varphi)}$ is translation invariant.

Let $f, g \in S_0(\varphi)$, $r, t \geq 1$. We have

$$\begin{aligned} &\operatorname{dist}(\sigma_{h_r} f + \sigma_{h_t} g, \sigma_h S_0(\varphi))_\Phi \\ &\leq \operatorname{dist}(f, \sigma_{hh_r^{-1}} S_0(\varphi))_\Phi + \operatorname{dist}(g, \sigma_{hh_t^{-1}} S_0(\varphi))_\Phi. \end{aligned}$$

This implies that if $f, g \in \overline{\bigcup_{j=1}^{\infty} \sigma_{h_j} S_0(\varphi)}$ and $\lambda, \gamma \in \mathbb{R}$, then $\lambda f + \gamma g \in \overline{\bigcup_{j=1}^{\infty} \sigma_{h_j} S_0(\varphi)}$. Hence $\overline{\bigcup_{j=1}^{\infty} \sigma_{h_j} S_0(\varphi)}$ is a translation invariant subspace. $\square$

The spectrum of a function g, denoted by $\mathrm{sp}(g)$, is defined to be the support of $\widehat{g}$, the Fourier transform of g [1].

Lemma 4. [1] *Let* $\Phi(t) > 0$ *for* $t > 0$, $f \in L_\Phi(\mathbb{R}^n), f(x) \not\equiv 0$ *and* $\xi^0 \in \mathrm{sp}(f)$ *be an arbitrary point. Then the restriction of* $\widehat{f}$ *on any neighbourhood of* ξ^0 *cannot concentrate on any finite number of hyperplanes.*

The following lemma, which will be used in the sequel, is the analog for Orlicz spaces of [6, Theorem 9.3], and has a similar proof.

Lemma 5. *Let* $(\Phi, \overline{\Phi})$ *be a complementary pair of Young functions. Assume that* $f \in L_1(\mathbb{R}^n) \cap L_\Phi(\mathbb{R}^n)$ *and* $g \in L_{\overline{\Phi}}(\mathbb{R}^n)$. *If* $f * g = 0$ *then*

$$\mathrm{sp}(g) \subset Z(f) := \{t \in \mathbb{R}^n \ : \ \widehat{f}(t) = 0 \ \}.$$

Proof. For each t in the complement of $Z(f)$, we have $\widehat{f}(t) \neq 0$. Without loss of generality, we may assume that $\widehat{f}(t) = 1$. By Lemma 9.2 [6] there exists $h \in L_1(\mathbb{R}^n)$ with $\|h\|_1 < 1$ such that $\widehat{h}(s) = 1 - \widehat{f}(s)$ for all s in some neighbourhood V of t.

To prove the lemma, it suffices to show that $\widehat{g} = 0$ in V, i.e., for every $\psi \in S$ such that $\widehat{\psi}$ (the Fourier transform of ψ) has its support in V, one proves that $\widehat{g}(\widehat{\psi}) = 0$. By the definition of the Fourier transform of a distribution $g \in S'$ [3,6], we have

$$\widehat{g}(\widehat{\psi}) = g(\check{\psi}) = (g * \psi)(0),$$

where $\check{\psi}(x) = \psi(-x)$.

We shall prove that $(g * \psi)(x) = 0$ for all $x \in \mathbb{R}^n$. Fix a function ψ in S. Put $g_0 = \psi, \quad g_j = h * g_{j-1}$, for $j \geq 1$ and $G = \sum_{j=0}^{\infty} g_j$. It is clear that $G \in L_1(\mathbb{R}^n)$. In fact, by Young's inequality for convolution products, we get

$$\|g_j\|_1 \leq \|h\|_1 \|g_{j-1}\|_1 \leq \|h\|_1^j \|\psi\|_1.$$

This implies that

$$\|G\|_1 \leq \sum_{j=1}^{\infty} \|h\|_1^j \|\psi\|_1 < \infty.$$

Since $\widehat{h}(s) = 1 - \widehat{f}(s)$ on the support of $\widehat{\psi}$, we have

$$[1 - \widehat{h}(s)]\widehat{\psi}(s) = \widehat{f}(s)\widehat{\psi}(s) = \widehat{f}(s)\widehat{g}_0(s)$$
$$[\widehat{h}(s) - \widehat{h}^2(s)]\widehat{\psi}(s) = \widehat{f}(s)\widehat{\psi}(s)\widehat{h}(s) = \widehat{f}(s)\widehat{g}_1(s)$$
$$\cdots\cdots$$

Therefore, $\widehat{\psi}(s) = \widehat{f}(s)\widehat{G}(s)$ for all $s \in V$. It is clear that

$$\widehat{\psi} = \widehat{f}\widehat{G} = \widehat{f * G}. \tag{13}$$

From (13) we get $\psi = G * f$. Since $f, G \in L_1(\mathbb{R}^n)$ and $g \in L_{\overline{\Phi}}(\mathbb{R}^n)$, one can see that

$$\|\psi * g\|_{\overline{\Phi}} = \|(G * f) * g\|_{\overline{\Phi}} \leq \|G * f\|_1 \|g\|_{\overline{\Phi}} \leq \|G\|_1 \|f\|_1 \|g\|_{\overline{\Phi}} < \infty.$$

Since the convolution product is associative, we get

$$\psi * g = (G * f) * g = G * (f * g) = 0.$$

Hence $\widehat{g}(\widehat{\psi}) = 0$. It is clear that $\widehat{g} = 0$ in the neighbourhood V. This implies that t in the complement of the spectrum of g. Therefore $\mathrm{sp}(g) \subset Z(f)$. □

Theorem 1. *Let $(\Phi, \overline{\Phi})$ be a complementary pair of Young functions, Φ satisfy Δ_2-condition and $\overline{\Phi}(t) > 0$ for $t > 0$. Assume that Y is a translation invariant subspace of $L_1(\mathbb{R}^n) \cap L_\Phi(\mathbb{R}^n)$. If for each $\xi \in Z(Y) := \bigcap_{f \in Y} \{t \in \mathbb{R}^n : \widehat{f}(t) = 0\}$ there is a neighbourhood V of ξ such that $V \cap Z(Y)$ is contained in a finite number of hyperplanes, then Y is dense in $L_\Phi(\mathbb{R}^n)$.*

Proof. Assume to the contrary that Y is not dense in $L_\Phi(\mathbb{R}^n)$. Then, by the Hahn-Banach theorem, there exists a nonzero functional $g \in L_{\overline{\Phi}}(\mathbb{R}^n)$ such that

$$\int_{\mathbb{R}^n} f(x)g(x)dx = 0, \qquad \text{for all } f \in \overline{Y},$$

by [5, Corollary 6] $(L_\Phi(\mathbb{R}^n))^* = L_{\overline{\Phi}}(\mathbb{R}^n)$. Since Y is a translation invariant subspace, we have

$$\int_{\mathbb{R}^n} f(y - x)g(x)dx = 0, \qquad \text{for all } f \in Y.$$

In other words, $f * g = 0$ for all $f \in Y$. By Lemma 5, we obtain

$$\mathrm{sp}(g) \subset \{t \in \mathbb{R}^n : \widehat{f}(t) = 0\}, \quad \text{for all } f \in Y.$$

Hence, $\mathrm{sp}(g) \subset Z(Y)$.

By hypothesis, it follows that for each $\xi \in \mathrm{sp}(g)$ there is a neighbourhood V of ξ such that $V \cap \mathrm{sp}(g)$ is contained in a finite number of hyperplanes. Applying Lemma 4, we have $g = 0$. This is impossible. □

Corollary 1. *Let $(\Phi, \overline{\Phi})$ be a complementary pair of Young functions, Φ satisfy Δ_2-condition and $\overline{\Phi}(t) > 0$ for $t > 0$. Assume that Y is a translation invariant subspace of $L_1(\mathbb{R}^n) \cap L_\Phi(\mathbb{R}^n)$. If $Z(Y)$ is contained in a finite number of hyperplanes, then Y is dense in $L_\Phi(\mathbb{R}^n)$.*

Remark 1. In the previous theorem, the assumption $\overline{\Phi}(t) > 0$ for $t > 0$ can be dropped when we replace the hypothesis that for each $\xi \in Z(Y)$ there is a neighbourhood V of ξ such that $V \cap Z(Y)$ is contained in a finite number of hyperplanes by the hypothesis that $Z(Y) = \emptyset$.

Theorem 2. *Let $(\Phi, \overline{\Phi})$ be a complementary pair of Young functions and let Φ satisfy Δ_2-condition. Assume that $\varphi \in L_1(\mathbb{R}^n) \cap L_\Phi(\mathbb{R}^n)$ with $\widehat{\varphi}(0) \neq 0$ and $\dfrac{1}{h}$ is an integer larger than 1. If $\varphi \in \overline{\mathrm{span} U_h}$, where $U_h = \bigcup\limits_{j=1}^{\infty} \sigma_h^j S_0(\varphi)$, then $\overline{\mathrm{span} U_h} = L_\Phi(\mathbb{R}^n)$.*

Proof. For any $g \in L_{\overline{\Phi}}(\mathbb{R}^n)$ satisfying

$$\int\limits_{\mathbb{R}^n} f(x)g(x)dx = 0,$$

for all $f \in \overline{\mathrm{span} U_h}$, we will prove that $g = 0$. By virtue of Lemma 2, we get

$$\int\limits_{\mathbb{R}^n} \sigma_h^j \varphi(y-x)g(x)dx = 0, \quad \forall j \geq 1, \quad \text{for all } y \in \mathbb{R}^n.$$

Since $g \in L_{\overline{\Phi}}(\mathbb{R}^n)$ and $\sigma_h^j \varphi \in L_1(\mathbb{R}^n)$, we have

$$(g * \sigma_h^j \varphi)(y) = g(\sigma_h^j \varphi(y - .)) = 0,$$

for all $y \in \mathbb{R}^n$. Note that the Fourier transform of $\sigma_h^j \varphi(x)$ is $h^{jn}\widehat{\varphi}(h^j t)$. It follows from Lemma 5 that

$$\mathrm{sp}(g) \subset \bigcap_{j=1}^{\infty} \{t \in \mathbb{R}^n : \widehat{\varphi}(h^j t) = 0\} = Z(\varphi). \tag{14}$$

Since $\varphi \in L_1(\mathbb{R}^n)$ and $\widehat{\varphi}(0) \neq 0$, we have $\widehat{\varphi}(h^j t) \neq 0$ for each $t \in \mathbb{R}^n$ and j sufficiently large. This shows that $Z(\varphi) = \emptyset$. From (14) we conclude that $\widehat{g} = 0$. Hence $g = 0$. By the Hahn-Banach theorem, we have $\overline{\mathrm{span} U_h} = L_\Phi(\mathbb{R}^n)$. □

Theorem 3. *Let $(\Phi, \overline{\Phi})$ be a complementary pair of Young functions, Φ satisfy Δ_2-condition and $\overline{\Phi}(t) > 0$ for $t > 0$. Assume that $\frac{1}{h}$ is an integer > 1. Suppose $\varphi \in L_1(\mathbb{R}^n) \cap L_\Phi(\mathbb{R}^n)$ and $\varphi \in \overline{\mathrm{span} U_h}$, where $U_h = \bigcup_{j=1}^{\infty} \sigma_h^j S_0(\varphi)$. If for each $\xi \in Z(\varphi)$ there is a neighbourhood V of ξ such that $V \cap Z(\varphi)$ is contained in a finite number of hyperplanes, then* $\mathrm{span} U_h$ *is dense in $L_\Phi(\mathbb{R}^n)$.*

Proof. Assume to the contrary that then there exists a nonzero functional $g \in (L_\Phi(\mathbb{R}^n))^*$ such that

$$\int_{\mathbb{R}^n} f(x) g(x) dx = 0, \qquad \text{for all } f \in \overline{\mathrm{span} U_h}.$$

Then $g \in L_{\overline{\Phi}}(\mathbb{R}^n) = (L_\Phi(\mathbb{R}^n))^*$ by a result of [5]. By virtue of Lemma 2, $\overline{\mathrm{span} U_h}$ is translation invariant, and hence,

$$\int_{\mathbb{R}^n} \sigma_h^j \varphi(y - x) g(x) dx = 0, \quad \forall j \geq 1, \quad \forall y \in \mathbb{R}^n. \tag{15}$$

Since $\varphi \in L_1(\mathbb{R}^n)$ and $g \in L_{\overline{\Phi}}(\mathbb{R}^n)$, from (15) we get $(g * \sigma_h^j \varphi)(y) = 0$ for all $y \in \mathbb{R}^n$. The Fourier transform of $(g * \sigma_h^j \varphi)(x)$ is $h^{nj} \widehat{\varphi}(h^j t) \widehat{g}(t)$. Therefore

$$\widehat{(\sigma_h^j \varphi * g)} = h^{nj} \widehat{\varphi}(h^j .) \widehat{g} = 0, \ \forall j \geq 1. \tag{16}$$

Lemma 5 and (16) imply that

$$\mathrm{sp}(g) \subset \bigcap_{j=1}^{\infty} \{t \in \mathbb{R}^n : \widehat{\varphi}(h^j t) = 0\} = Z(\varphi).$$

By the hypothesis and Lemma 4, we get $g = 0$. This leads to a contradiction. Therefore $\overline{\mathrm{span} U_h} = L_\Phi(\mathbb{R}^n)$. □

Corollary 2. *Let* $(\Phi, \overline{\Phi})$ *be a complementary pair of Young functions,* Φ *satisfy* Δ_2*-condition and* $\overline{\Phi}(t) > 0$ *for* $t > 0$*. Assume that* $\dfrac{1}{h}$ *is an integer* > 1*. Suppose* $\varphi \in L_1(\mathbb{R}^n) \cap L_\Phi(\mathbb{R}^n)$ *and* $\varphi \in \overline{\mathrm{span} U_h}$*, where* $U_h = \bigcup_{j=1}^{\infty} \sigma_h^j S_0(\varphi)$*. If* $Z(\varphi)$ *is contained in a finite number of hyperplanes, then* $\mathrm{span} U_h$ *is dense in* $L_\Phi(\mathbb{R}^n)$*.*

Remark 2. The conclusions of Theorem 2, Theorem 3 and Corollary 2 remain valid if the condition $\varphi \in \overline{\mathrm{span} U_h}$ is replaced by the condition $\lim\limits_{h \to 0, h \in G} \mathrm{dist}(\varphi, \sigma_h S_0(\varphi))_\Phi = 0$.

Using the same argument as in the proof of [7, Proposition 6.1], we obtain the following two corollaries:

Corollary 3. *Let* $(\Phi, \overline{\Phi})$ *be a complementary pair of Young functions and* Φ *satisfy* Δ_2*-condition. Assume that* $\varphi \in L_1(\mathbb{R}^n) \cap L_\Phi(\mathbb{R}^n)$ *with* $\widehat{\varphi}(0) \neq 0$ *and* $\frac{1}{h}$ *is an integer larger than 1. If* $\varphi \in \overline{\sigma_h S_0(\varphi)}$*, then*

$$\lim_{j \to \infty} \mathrm{dist}(f, \sigma_h^j S_0(\varphi))_\Phi = 0, \qquad \forall f \in L_\Phi(\mathbb{R}^n).$$

Corollary 4. *Let* $(\Phi, \overline{\Phi})$ *be a complementary pair of Young functions,* Φ *satisfy* Δ_2*-condition and* $\overline{\Phi}(t) > 0$*, for* $t > 0$*. Assume that* $\varphi \in L_1(\mathbb{R}^n) \cap L_\Phi(\mathbb{R}^n)$ *and* $\frac{1}{h}$ *is an integer larger than 1. If* $Z(\varphi)$ *is contained in a finite number of hyperplanes and* $\varphi \in \overline{\sigma_h S_0(\varphi)}$*, then*

$$\lim_{j \to \infty} \mathrm{dist}(f, \sigma_h^j S_0(\varphi))_\Phi = 0, \qquad \forall f \in L_\Phi(\mathbb{R}^n).$$

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