ACTA MATHEMATICA VIETNAMICA
Volume 25, Number 2, 2000, pp. 181–193

AN INVARIANT PROPERTY OF INVEX FUNCTIONS AND APPLICATIONS

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ABSTRACT. Under directional differentiability assumptions, we prove that the minimum or the maximum of a finite family of invex functions is again an invex function. This invariant property of the invex functions is used to obtain first-order optimality conditions for a class of optimization problems.

1. INTRODUCTION

Hanson's paper [10] is the starting point of the theory of invex functions. The terms invex and cone-invex were introduced by Craven [3] who proved that the composition $g \circ f$, where g is a convex function and f is differentiable with f' having full rank, is an invex function.

The theory of invex functions have been extensively studied by many authors (see e.g. [1], [3]-[8], [10]-[14]). Craven and Glover [4] characterized invexity for quasidifferentiable functions in terms of Lagrange multipliers, and presented a number of classes of invex functions. Noted that Gâteaux differentiable functions satisfying Slater's condition are invex (see [4]).

In this paper we shall prove that the minimum (or the maximum) of a finite family of invex functions is an invex function. We shall show some applications of this invariant property to mathematical programming.

The above-mentioned property of invex functions is proved in Section 2. In Section 3, it is used to obtain first-order optimality conditions for a class of optimization problems.

2. A PROPERTY OF INVEX FUNCTIONS

Let f be a real-valued function defined on a Banach space X.

Received February 5, 1998; in revised form July 19, 1999.
1991 Mathematics Subject Classification. Primary 26 B 25, Secondary 90 C 30, 49 K 35
Key words and phrases. Invex function, directional derivative, Clarke's generalized derivative, Clarke's generalized gradient, regular function.

Recall that the directional derivative of f at x_0, with respect to d, is defined to be the limit

$$f'(x_0; d) = \lim_{\lambda \downarrow 0} \frac{f(x_0 + \lambda d) - f(x_0)}{\lambda},$$

if it exists.

The function f is called invex at x_0 if there exist a neighbourhood U of x_0 and a function $\omega(., x_0) : U \to X$ such that

$$f(x) - f(x_0) \geq f'(x_0; \omega(x, x_0)). \tag{1}$$

Note that a convex function f on a convex subset A in X is invex at each point $x_0 \in A$ with $\omega(x, x_0) = x - x_0$ (see e.g. [4]).

Following [2], a locally Lipschitz function at x_0 is called regular at x_0 if the directional derivative $f'(x_0; .)$ exists and

$$f'(x_0; .) = f^0(x_0; .),$$

where $f^0(x_0; .)$ denotes Clarke's generalized derivative of f at x_0.

If the function f is locally Lipschitz and regular at x_0 then (1) can be replaced by the condition

$$f(x) - f(x_0) \geq f^0(x_0; \omega(x, x_0))$$

or for every $\zeta \in \partial f(x_0)$,

$$f(x) - f(x_0) \geq \langle \zeta, \omega(x, x_0) \rangle,$$

where $\partial f(x_0)$ stands for Clarke's generalized gradient of f at x_0.

Let $f_1, f_2, \ldots, f_k$ be real-valued functions defined on a Banach space X. Define

$$f(x) = \min_{1 \leq i \leq k} f_i(x). \tag{2}$$

Note that when $f_1, f_2, \ldots, f_k$ are convex functions, the function f may be nonconvex.

For example, we consider the two functions

$$f_1(x) = (x + 1)^2, \quad f_2 = (x - 1)^2 \quad (x \in R).$$

It is obvious that $f_1(x)$ and $f_2(x)$ are convex functions while the following function is nonconvex:

$$f(x) = \min\{f_1(x), f_2(x)\} = \begin{cases} (x+1)^2, & \text{if } x \le 0, \\ (x-1)^2, & \text{if } x > 0. \end{cases}$$

However, the function f is invex at each point $x_0 \in R$ with respect to the function $\omega(x, x_0) = x - x_0$.

Motivated by this example we shall prove the following result.

Theorem 1. *Assume that the functions $f_1, f_2, \dots, f_k$ are continuous, and directionally differentiable at x_0. Suppose, in addition, that the functions $f_1, f_2, \dots, f_k$ are invex at x_0 with respect to a function ω. Then the function f defined by* (2) *is invex at x_0 with respect to the function ω.*

Proof. In order to prove the function f defined by (2) is invex at x_0 we shall begin by showing that

$$f'(x_0; d) = \min_{i \in I(x_0)} f_i'(x_0; d), \tag{3}$$

where

$$I(x_0) := \Big\{ i \in \{1, 2, \dots, k\} : f_i(x_0) = \min_{1 \le j \le k} f_j(x_0) \Big\}.$$

Note that the formula (3) war proved for finite-dimensional case in [9]. Here it is proved for the infinite case.

Indeed, for $i \in I(x_0)$ one has

$$f_i(x_0) < f_j(x_0) \quad (\forall j \notin I(x_0)).$$

By setting

$$\varepsilon = \min_{j \notin I(x_0)} f_j(x_0) - f_i(x_0) \quad (i \in I(x_0))$$

we get $\varepsilon > 0$. Since $f_i(x_0) = f(x_0)$ for every $i \in I(x_0)$, ε does not depend on the choice of i.

Since the functions $f_1, f_2, \dots, f_k$ are continuous, there is a neighbourhood U of 0 such that for every $x \in U$, $i \in I(x_0)$, $j \notin I(x_0)$,

$$f_i(x_0 + x) < f_i(x_0) + \frac{\varepsilon}{3} < f_j(x_0) - \frac{\varepsilon}{3} < f_j(x_0 + x),$$

which implies that

$$\min_{1 \le i \le k} f_i(x_0 + x) = \min_{i \in I(x_0)} f_i(x_0 + x) \quad (\forall x \in U). \tag{4}$$

On the other hand, it follows from (4) that

$$\begin{aligned}
f'(x_0; d) &= \lim_{t\downarrow 0} \frac{\min\limits_{1\le i\le k} f_i(x_0+td) - \min\limits_{1\le i\le k} f_i(x_0)}{t} \\
&= \lim_{t\downarrow 0} \frac{\min\limits_{i\in I(x_0)} f_i(x_0+td) - \min\limits_{i\in I(x_0)} f_i(x_0)}{t} \\
&= \lim_{t\downarrow 0} \frac{\min\limits_{i\in I(x_0)} \big\{ f_i(x_0+td) - \min\limits_{i\in I(x_0)} f_i(x_0)\big\}}{t} \\
&= \lim_{t\downarrow 0} \min_{i\in I(x_0)} \frac{f_i(x_0+td) - f_i(x_0)}{t}
\end{aligned}$$

Since for each $i \in I(x_0)$, the function

$$\varphi_i(t) := \frac{f_i(x_0+td) - f_i(x_0)}{t}$$

(with $\varphi_i(0) = f_i'(\overline{x}; d)$) is continuous at $t = 0$, the function $\min\limits_{i\in I(x_0)} \varphi_i(t)$ is also continuous at 0. Hence

$$\begin{aligned}
f'(x_0; d) &= \min_{i\in I(x_0)} \lim_{t\downarrow 0} \frac{f_i(x_0+td) - f_i(x_0)}{t} \\
&= \min_{i\in I(x_0)} f_i'(x_0; d).
\end{aligned}$$

By the hypotheses, the functions $f_1, f_2, \ldots, f_k$ are invex with respect to the same function ω. Hence, for some neighbourhood U of 0 and for each $i \in I(x_0)$, we have

$$f_i(x) - f_i(x_0) \ge f_i'(x_0; \omega(x, x_0)) \quad (\forall x \in x_0 + U),$$

whence

$$f_i(x) - f_i(x_0) \ge \min_{i\in I(x_0)} f_i'(x_0; \omega(x, x_0)) \quad (\forall x \in x_0 + U).$$

Therefore, for every $x \in x_0 + U$ and $i \in I(x_0)$,

$$f_i(x) - \min_{i\in I(x_0)} f_i(x_0) \ge \min_{i\in I(x_0)} f_i'(x_0; \omega(x, x_0)).$$

Hence

$$\min_{i\in I(x_0)} f_i(x) - \min_{i\in I(x_0)} f_i(x_0) \geq \min_{i\in I(x_0)} f_i'(x_0;\omega(x,x_0)). \tag{5}$$

Substituting (3) and (4) into (5) yields

$$\min_{1\leq i\leq k} f_i(x) - \min_{1\leq i\leq k} f_i(x_0) \geq f'(x_0;\omega(x,x_0)) \quad (\forall x\in x_0+U).$$

Consequently, f is a invex function at x_0. This completes the proof. □

Corollary 1. *Assume that $f_1, f_2, \dots, f_k$ are convex, locally Lipschitz functions at x_0. Then, the function $f(x)$ defined by* (2) *is invex at x_0, with respect to the function $\omega(x,x_0) = x - x_0$.*

Proof. By the hypotheses the functions $f_1, f_2, \dots, f_k$ are directionally differentiable at x_0 and invex with respect to the same function $\omega(x,x_0) = x - x_0$. The conclusion follows from Theorem 1. □

Let $g_1, g_2, \dots, g_k$ be real-valued functions defined on X. Let us consider the function

$$g(x) = \max_{1\leq i\leq k} g_i(x). \tag{6}$$

By an argument analogous to that used for the proof of Theorem 1 we get the following.

Theorem 2. *Assume that the function $g_1, g_2, \dots, g_k$ are continuous, and directionally differentiable at x_0. Suppose, furthermore, that the functions $g_1, g_2, \dots, g_k$ are invex at x_0, with respect to a function ω. Then, the function g defined by* (6) *is invex at x_0, with respect to the function ω.*

3. Applications in mathematical programming

Let $\varphi_1, \varphi_2, \dots, \varphi_k, \psi_1, \psi_2, \dots, \psi_m$ be real-valued functions defined on a Banach space X. Let C be a nonempty subset of X. In this section we shall be concerned with the following problem

$$(P) \qquad \begin{cases} \displaystyle\min_{1\leq i\leq k} \varphi_i(x) \to \min, \\ \text{s.t.} \\ \displaystyle\min_{1\leq j\leq m} \psi_j(x) \leq 0, \\ \qquad x\in C, \end{cases}$$

Denote by M the feasible set of Problem (P).

Recall that Clarke's tangent cone to C at x_0 is defined as follows

$$T_C(x_0) := \{v \in X : d_C^0(x_0; v) = 0\},$$

where $d_C^0(x_0; v)$ denotes Clarke's directional derivative of the distance function $d_C(.)$ at x_0 with respect to the direction v.

Clarke's normal cone to C at x_0 is defined by

$$N_C(x_0) := \{\xi \in X^* : \langle \xi, v\rangle \le 0,\ \forall v \in T_C(x_0)\},$$

where X^* is the set of all continuous linear functionals defined on X.

We begin by establishing a necessary condition of the Kuhn-Tucker type for Problem (P).

Theorem 3. *Let x_0 be a local minimum of Problem (P). Assume that the set C is closed, and the functions $\varphi_1, \varphi_2, \dots, \varphi_k, \psi_1, \psi_2, \dots, \psi_m$ are locally Lipschitz at x_0. Then, there exist numbers $\bar{\gamma} \ge 0$, $\bar{\alpha} \ge 0$, $\bar{\lambda_i} \ge 0$ $(i \in I_1(x_0))$ with $\sum_{i\in I_1(x_0)} \bar{\lambda_i} = 1$ and $\bar{\mu_j} \ge 0$ $(j \in I_2(x_0))$ with $\sum_{j\in I_2(x_0)} \bar{\mu_j} = 1$, $\bar{\gamma}$ and $\bar{\alpha}$ are not both equal to zero such that*

$$0 \in \bar{\gamma} \sum_{i\in I_1(x_0)} \bar{\lambda_i} \partial\varphi_i(x_0) + \bar{\alpha} \sum_{j\in I_2(x_0)} \bar{\mu_j}\partial\psi_j(x_0) + N_C(x_0), \tag{7}$$

$$\bar{\alpha} \min_{1\le j\le m} \psi_j(x_0) = 0, \tag{8}$$

where

$$I_1(x_0) := \Big\{i \in \{1, 2, \dots, k\} :\ \varphi_i(x_0) = \min_{1\le j\le k} \varphi_j(x_0)\Big\},$$
$$I_2(x_0) := \Big\{j \in \{1, 2, \dots, m\} :\ \psi_j(x_0) = \max_{1\le \ell\le m} \psi_\ell(x_0)\Big\}.$$

Proof. By virtue of Theorem 6.1.1 of [2] there exist numbers $\bar{\gamma} \ge 0$, $\bar{\alpha} \ge 0$, not both zero, such that

$$0 \in \bar{\gamma}\partial\Big(\min_{1\le i\le k} \varphi_i\Big)(x_0) + \bar{\alpha}\partial\Big(\min_{1\le j\le m} \psi_j\Big)(x_0) + N_C(x_0), \tag{9}$$

$$\bar{\alpha} \min_{1\leq j\leq m} \psi_j(x_0) = 0.$$

It is clear that

$$I_1(x_0) := \Big\{ i \in \{1,2,\ldots,k\} :\ -\varphi_i(x_0) = \max_{1\leq j\leq k}(-\varphi_j(x_0)) \Big\},$$
$$I_2(x_0) := \Big\{ j \in \{1,2,\ldots,m\} :\ -\psi_j(x_0) = \max_{1\leq \ell\leq m}(-\psi_\ell(x_0)) \Big\}.$$

Taking account of Proposition 2.3.12 of [2] one gets

$$\begin{aligned}
\partial\Big(\min_{1\leq i\leq k}\varphi_i\Big)(x_0) &= \partial\Big(-\max_{1\leq i\leq k}(-\varphi_i))(x_0) \\
&= -\partial\Big(\max_{1\leq i\leq k}(-\varphi_i))(x_0) \\
&\subset -co\big\{\partial(-\varphi_i)(x_0) : i \in I_1(x_0)\big\} \\
&= co\big\{\partial(\varphi_i)(x_0) : i \in I_1(x_0)\big\},
\end{aligned}$$

where *co* indicates the convex hull.

Hence

$$\partial\Big(\min_{1\leq i\leq k}\varphi_i\Big)(x_0) \subset \Big\{ \sum_{i\in I_1(x_0)} \lambda_i \partial\varphi_i(x_0),\ \lambda_i \geq 0,\ \sum_{i\in I_1(x_0)} \lambda_i = 1 \Big\} \tag{10}$$

By an argument similar to the previous one, we get

$$\partial\Big(\min_{1\leq j\leq m}\psi_j\Big)(x_0) \subset \Big\{ \sum_{j\in I_2(x_0)} \mu_j \partial\psi_j(x_0), \mu_j \geq 0, \sum_{j\in I_2(x_0)} \mu_j = 1 \Big\} \tag{11}$$

Combining (9), (10), and (11) yields the existence of numbers $\bar{\lambda_i} \geq 0$ $(i \in I_1(x_0))$ with $\sum_{i\in I_1(x_0)} \bar{\lambda_i} = 1$ and $\bar{\mu_j} \geq 0$ $(j \in I_2(x_0))$ with $\sum_{j\in I_2(x_0)} \bar{\mu_j} = 1$ such that

$$0 \in \bar{\gamma} \sum_{i\in I_1(x_0)} \bar{\lambda_i}\partial\varphi_i(x_0) + \bar{\alpha} \sum_{j\in I_2(x_0)} \bar{\mu_j}\partial\psi_j(x_0) + N_C(x_0).$$

The proof is now complete. $\square$

Remark 1. Let us consider the following problem

$$(P_1) \qquad \begin{cases} \min\limits_{1\le i\le k} \varphi_i(x) \to \max, \\ \text{s.t.} \\ \min\limits_{1\le j\le m} \psi_j(x) \ge 0, \\ \qquad x \in C, \end{cases}$$

where $\varphi_1, \dots, \varphi_k, \psi_1, \dots, \psi_m$, C are as in Problem (P).

Under the hypotheses stated in Theorem 3, we obtain the following Kuhn-Tucker necessary conditions:

$$0 \in -\bar{\gamma} \sum_{i\in I_1(x_0)} \bar{\lambda_i} \partial\varphi_i(x_0) - \bar{\alpha} \sum_{j\in I_2(x_0)} \bar{\mu_j} \partial\psi_j(x_0) + N_C(x_0), \tag{12}$$

$$\bar{\alpha} \min_{1\le j\le m} \psi_j(x_0) = 0. \tag{13}$$

Remark 2. If a condition of Slater type or a condition of Mangasarian-Fromowitz type is satisfied, then we get $\bar{\gamma} > 0$ in (7) and (12), and we may assume that $\bar{\gamma} = 1$.

Now we shall deal with a sufficient condition for optimality.

Theorem 4. *Let x_0 be a feasible point of Problem (P). Let the functions $\varphi_1, \dots, \varphi_k, \psi_1, \dots, \psi_m$ be locally Lipschitz and regular at x_0; the functions $\min\limits_{1\le i\le k} \varphi_i(x)$ and $\min\limits_{1\le j\le m} \psi_j(x)$ are regular at x_0. Assume that there are a neighbourhood V of x_0 and a function $\omega : M \times M \to T_C(x_0)$ such that the functions $\varphi_1, \dots, \varphi_k, \psi_1, \dots, \psi_m$ are invex at x_0 on $M \cap V$, with respect to the function ω. Suppose, furthermore, that there exist numbers $\bar{\alpha} \ge 0$, $\bar{\lambda_i} \ge 0$ $(i \in I_1(x_0))$ with $\sum\limits_{i\in I_1(x_0)} \bar{\lambda_i} = 1$ and $\bar{\mu_j} \ge 0$ $(j \in I_2(x_0))$ with $\sum\limits_{j\in I_2(x_0)} \bar{\mu_j} = 1$ such that*

$$0 \in \sum_{i\in I_1(x_0)} \bar{\lambda_i} \partial\varphi_i(x_0) + \bar{\alpha} \sum_{j\in I_2(x_0)} \bar{\mu_j} \partial\psi_j(x_0) + N_C(x_0), \tag{14}$$

$$\bar{\alpha} \min_{1\le j\le m} \psi_j(x_0) = 0, \tag{15}$$

Then, x_0 is a local minimum of Problem (P).

Proof. It follows from the condition (14) that there are $\bar{\zeta}_i \in \partial\varphi_i(x_0)$ $(i \in I_1(x_0))$, $\bar{\eta}_j \in \partial\psi_j(x_0)$ $(j \in I_2(x_0))$, $\bar{s} \in N_C(x_0)$, such that

$$\sum_{i\in I_1(x_0)} \bar{\lambda}_i\bar{\zeta}_i + \bar{\alpha}\sum_{j\in I_2(x_0)} \bar{\mu}_j\bar{\eta}_j + \bar{s} = 0. \tag{16}$$

We shall prove that for every $x \in M \cap V$,

$$\min_{1\le i\le k}\varphi_i(x) - \min_{1\le i\le k}\varphi_i(x_0) \ge 0.$$

According to Theorem 1, the functions $\min\limits_{1\le i\le k}\varphi_i(x)$ and $\min\limits_{1\le j\le m}\psi_i(x_0)$ are invex at x_0 with respect to $\omega(x,x_0)$. Hence, taking x to be a feasible point of Problem (P) and $x \in V$, it follows from (15) that

$$\begin{aligned}
&\min_{1\le i\le k}\varphi_i(x) - \min_{1\le i\le k}\varphi_i(x_0)\\
&\ge \Big[\min_{1\le i\le k}\varphi_i(x) + \bar{\alpha}\min_{1\le j\le m}\psi_j(x)\Big] - \Big[\min_{1\le i\le k}\varphi_i(x_0) + \bar{\alpha}\min_{1\le j\le m}\psi_j(x_0)\Big]\\
&\ge \Big(\min_{1\le i\le k}\varphi_i\Big)'(x_0;\omega(x,x_0)) + \bar{\alpha}\Big(\min_{1\le j\le m}\psi_j\Big)'(x_0;\omega(x,x_0)).
\end{aligned}$$

By the regularity assumption we have

$$\min_{1\le i\le k}\varphi_i(x) - \min_{1\le i\le k}\varphi_i(x_0) \ge \langle\zeta,\omega(x,x_0)\rangle + \bar{\alpha}\langle\eta,\omega(x,x_0)\rangle$$

for all $\zeta \in \partial\Big(\min\limits_{1\le i\le k}\varphi_i\Big)(x_0)$ and $\eta \in \partial\Big(\min\limits_{1\le j\le m}\psi_j\Big)(x_0)$. Taking

$$\begin{aligned}
\bar{\zeta} &= \sum_{i\in I_1(x_0)}\bar{\lambda}_i\bar{\zeta}_i \in \partial\Big(\min_{1\le i\le k}\varphi_i\Big)(x_0),\\
\bar{\eta} &= \sum_{j\in I_2(x_0)}\bar{\mu}_j\bar{\eta}_j \in \partial\Big(\min_{1\le j\le m}\psi_j)(x_0)
\end{aligned}$$

and $\bar{s} \in N_C(x_0)$, by virtue of (16) one gets

$$\begin{aligned}
&\min_{1\le i\le k}\varphi_i(x) - \min_{1\le i\le k}\varphi_i(x_0)\\
&\ge \sum_{i\in I_1(x_0)}\bar{\lambda}_i\langle\bar{\zeta}_i,\omega(x,x_0)\rangle + \sum_{j\in I_2(x_0)}\bar{\mu}_j\langle\bar{\eta}_j,\omega(x,x_0)\rangle + \langle s,\omega(x,x_0)\rangle\\
&= 0.
\end{aligned}$$

This completes the proof. $\square$

In the case of convex functions we get the following

Theorem 5. *Let x_0 be a feasible point of Problem (P) and C be a closed convex subset of X. Let the functions $\varphi_1, \varphi_2, \dots, \varphi_k, \psi_1, \psi_2, \dots, \psi_m$ be convex and locally Lipschitz at x_0, and the functions $\min\limits_{1\le i\le k} \varphi_i(x)$ and $\min\limits_{1\le j\le m} \psi_j(x)$ are regular at x_0. Assume that there are numbers $\bar{\alpha} \ge 0$, $\bar{\lambda_i} \ge 0 \quad (i \in I_1(x_0))$ with $\sum\limits_{i\in I_1(x_0)} \bar{\lambda_i} = 1$ and $\bar{\mu_j} \ge 0 \quad (j \in I_2(x_0))$ with $\sum\limits_{j\in I_2(x_0)} \bar{\mu_j} = 1$ such that (14) and (15) are fulfilled. Then x_0 is a local minimum of Problem (P).*

Proof. It follows from proposition 2.2.7 of [2] that the functions $\varphi_1, \varphi_2, \dots, \varphi_k, \psi_1, \psi_2, \dots, \psi_m$ are regular at x_0. Moreover, these functions are invex with respect to the same function $\omega(x, x_0) = x - x_0$. Because C is a convex set, the normal cone $N_C(x_0)$ coincides with the one in the sense of convex analysis:

$$N_C(x_0) = \big\{\zeta \in X : \langle\zeta, x - x_0\rangle 0, \ \forall x \in C\big\}.$$

By an argument analogous to that used for the proof of Theorem 4, the conclusion follows. □

Now we turn back to Problem (P_1) and establish the following sufficient condition.

Theorem 6. *Let x_0 be a feasible point of Problem (P_1). Let the functions $-\varphi_1, -\varphi_2, \dots, -\varphi_k, -\psi_1, -\psi_2, \dots, -\psi_m$ be locally Lipschitz and regular at x_0. Assume that there are a neighbourhood V of x_0 and a function $\omega : M \times M \to T_C(x_0)$ such that the functions $-\varphi_1, -\varphi_2, \dots, -\varphi_k, -\psi_1, -\psi_2, \dots, -\psi_m$ are invex at x_0 on $M \cap V$ with respect to the same function ω. Suppose, in addition, that there exist numbers $\bar{\alpha} \ge 0$, $\bar{\lambda_i} \ge 0 \quad (i \in I_1(x_0))$ with $\sum\limits_{i\in I_1(x_0)} \bar{\lambda_i} = 1$ and $\bar{\mu_j} \ge 0 \quad (j \in I_2(x_0))$ with $\sum\limits_{j\in I_2(x_0)} \bar{\mu_j} = 1$ such that*

$$0 \in -\sum_{i\in I_1(x_0)} \bar{\lambda_i}\partial\varphi_i(x_0) - \bar{\alpha} \sum_{j\in I_2(x_0)} \bar{\mu_j}\partial\psi_j(x_0) + N_C(x_0), \tag{17}$$

$$\bar{\alpha} \min_{1\le j\le m} \psi_j(x_0) = 0, \tag{18}$$

Then x_0 is a local maximum of Problem (P_1).

Proof. As in the proof of Theorem 4 it follows that there are $\bar{\zeta}_i \in \partial\varphi_i(x_0)$ $(i \in I_1(x_0))$, $\bar{\eta}_j \in \partial\psi_j(x_0)$ $(j \in I_2(x_0))$ and $\bar{s} \in N_C(x_0)$, such that

$$-\sum_{i\in I_1(x_0)} \bar{\lambda}_i\bar{\zeta}_i - \bar{\alpha} \sum_{j\in I_2(x_0)} \bar{\mu}_j\bar{\eta}_j + \bar{s} = 0. \tag{19}$$

To prove that x_0 is a local maximum of Problem (P_1) we shall prove that for every $x \in M \cap V$,

$$\max_{1\le i\le k}(-\varphi_i)(x) - \max_{1\le i\le k}(-\varphi_i)(x_0) \ge 0,$$

that is the function $\max_{1\le i\le k}(-\varphi_i)(x)$ reaches a local minimum at x_0. Taking $x \in M \cap V$, it follows from (18) that

$$\begin{aligned}
&\max_{1\le i\le k}(-\varphi_i)(x) - \max_{1\le i\le k}(-\varphi_i)(x_0)\\
&\ge \Big[\max_{1\le i\le k}(-\varphi_i)(x) - \bar{\alpha}\min_{1\le j\le m}\psi_j(x)\Big]\\
&\quad - \Big[\max_{1\le i\le k}(-\varphi_i)(x_0) - \bar{\alpha}\min_{1\le j\le m}\psi_j(x_0)\Big]\\
&= \Big[\max_{1\le i\le k}(-\varphi_i)(x) + \bar{\alpha}\max_{1\le j\le m}(-\psi_j)(x)\Big]\\
&\quad - \Big[\max_{1\le i\le k}(-\varphi_i)(x_0) + \bar{\alpha}\max_{1\le j\le m}(-\psi_j)(x_0)\Big]
\end{aligned} \tag{20}$$

Since the functions $-\varphi_1, -\varphi_2, \ldots, -\varphi_k, -\psi_1, -\psi_2, \ldots, -\psi_m$ are regular at x_0, it follows from Proposition 2.3.12 of [2] that the functions $\max_{1\le i\le k}(-\varphi_i)$ and $\max_{1\le j\le m}(-\psi_j)$ are regular at x_0, that is

$$\Big(\max_{1\le i\le k}(-\varphi_i)\Big)'(x_0; .) = \Big(\max_{1\le i\le k}(-\varphi_i)\Big)^0(x_0; .),$$
$$\Big(\max_{1\le j\le m}(-\psi_j)\Big)'(x_0; .) = \Big(\max_{1\le j\le m}(-\psi_j)\Big)^0(x_0; .).$$

Taking account of Theorem 2 we see that the functions $\max_{1\le i\le k}(-\varphi_i)(x)$ and $\max_{1\le j\le m}(-\psi_j)(x)$ are invex at x_0 on $M \cap V$ with respect to the function $\omega(x, x_0)$. From (20) it follows that

$$\max_{1\le i\le k}(-\varphi_i)(x) - \max_{1\le i\le k}(-\varphi_i)(x_0) \ge \langle\zeta, \omega(x, x_0)\rangle + \bar{\alpha}\langle\eta, \omega(x, x_0)\rangle \tag{21}$$

$$\Big(\forall \zeta \in \partial\Big(\max_{1\le i\le k}(-\varphi_i)\Big)(x_0),\ \forall \eta \in \partial\Big(\max_{1\le j\le m}(-\psi_j)\Big)(x_0)\Big)$$

For $\bar{\zeta}_i \in \partial\varphi_i(x_0)$ $(i \in I_1(x_0))$ and $\bar{\eta}_j \in \partial\psi_j(x_0)$ $(j \in I_2(x_0))$, observe that

$$\sum_{i\in I_1(x_0)} \bar{\lambda}_i\bar{\zeta}_i \in \partial\Big(\min_{1\le i\le k}\varphi_i\Big)(x_0)$$

$$\sum_{j\in I_2(x_0)} \bar{\mu}_j\bar{\eta}_j \in \partial\Big(\min_{1\le j\le m}\psi_j\Big)(x_0).$$

Then we have

$$-\sum_{i\in I_1(x_0)} \bar{\lambda}_i\bar{\zeta}_i \in \partial\Big(\max_{1\le i\le k}(-\varphi_i))(x_0), \tag{22}$$

$$-\sum_{j\in I_2(x_0)} \bar{\mu}_j\bar{\eta}_j \in \partial\Big(\max_{1\le j\le m}(-\psi_j)\Big)(x_0). \tag{23}$$

Note that for every $x \in M\cap V$, since $\omega(x,x_0) \in T_C(x_0)$ and $\bar{s} \in N_C(x_0)$,

$$\langle \bar{s}, \omega(x,x_0)\rangle \le 0. \tag{24}$$

Combining (19), (21) - (24) yields that for every $x \in M \cap V$,

$$\begin{aligned}
&\max_{1\le i\le k}(-\varphi_i)(x) - \max_{1\le i\le k}(-\varphi_i)(x_0)\\
&\ge \Big\langle -\sum_{i\in I_2(x_0)} \bar{\lambda}_i\bar{\zeta}_i, \omega(x,x_0)\Big\rangle + \bar{\alpha}\Big\langle -\sum_{i\in I_2(x_0)} \bar{\mu}_j\bar{\eta}_j, \omega(x,x_0)\Big\rangle\\
&\ge -\sum_{i\in I_1(x_0)} \bar{\lambda}_i\langle\bar{\zeta}_i, \omega(x,x_0)\rangle - \bar{\alpha}\sum_{j\in I_2(x_0)} \bar{\mu}_j\langle\bar{\eta}_j, \omega(x,x_0)\rangle + \langle\bar{s}, \omega(x,x_0)\rangle\\
&= 0.
\end{aligned}$$

The proof is now complete. □

Remark 3. From Theorem 6 we can see that if the functions $\varphi_1, \varphi_2, \dots, \varphi_k, \psi_1, \psi_2, \dots, \psi_m$ are convex and locally Lipschitz at a feasible point x_0 of (P_1), C is a closed convex set, and there exist $\bar{\alpha} \ge 0$, $\bar{\lambda}_i \ge 0$ $(i \in I_1(x_0))$ with $\sum_{i\in I_1(x_0)} \bar{\lambda}_i = 1$ and $\bar{\mu}_j \ge 0$ $(j \in I_2(x_0))$ with $\sum_{j\in I_2(x_0)} \bar{\mu}_j = 1$ such that (17) and (18) hold, then x_0 is a maximum of Problem (P_1).

Acknowledgement

The authors thank the referee for his valuable comments.

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