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ON THE EXISTENCE OF BOUNDED SOLUTIONS FOR LOTKA-VOLTERRA EQUATIONS

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Abstract. This article deals with an extension of Ahmad's results in studying the existence and uniqueness of a solution bounded above and below by positive constants by mean of weakening the conditions imposed on the coefficients of competing species equations of random Lotka-Volterra type.

1. Introduction

In this article we are concerned with an extension of Ahmad's results in studying the existence and uniqueness of a solution bounded above and below by positive constants by mean of weakening the conditions imposed on the coefficients of competing species equations. The model to be studied is the random Lotka-Volterra system

$$\begin{cases} \dot{x} &= x\big(a(t) - b(t)x - c(t)y\big) \\ \dot{y} &= y\big(d(t) - e(t)x - f(t)y\big), \end{cases} \tag{1.1}$$

where a, b, c, d, e, f are random processes with continuous trajectories. We suppose that a, b, c, d, e, f are bounded above and below by positive random variables. This is a model of two competing species whose quantities at time t are $x(t)$ and $y(t)$, respectively, in a random environment. The processes a and d are the respective intrinsic growth rates; b and f measure the respective intraspecific competition within species x and y and the processes c, e measure the interspecific competitions between two species. The details of the ecological significance of such a system are discussed in [Go].

It is known that for system (1.1) the quadrant plane

$$R^2_+ = \{(u,v) : 0 < u < \infty; 0 < v < \infty\}$$

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is invariant, i.e., if $\mathrm{col}(x(t), y(t))$ is a solution of (1.1) with $x(t_0) > 0$, $y(t_0) > 0$ for some $t_0 \in R$ then $x(t) > 0$, $y(t) > 0$ for any $t \in (-\infty, \infty)$.

Let $g_M = \sup_{-\infty<t<\infty} g(t)$; $g_L = \inf_{-\infty<t<\infty} g(t)$ for any function $g(t)$. For the case where the coefficients a, b, c, d, e, f are deterministic continuous functions, Ahmad in [Ah2] has shown that under the condition

$$\frac{a_L}{c_M} > \frac{d_M}{f_L}; \quad \frac{d_L}{e_M} > \frac{a_M}{b_L}; \quad a_L,\ f_L,\ d_L,\ b_L > 0, \tag{1.2}$$

system (1.1) has a unique solution defined on $(-\infty, \infty)$ which is bounded above and below by positive constants. Furthermore, he has shown that if the coefficients are almost periodic functions then this unique solution is also almost periodic.

We note that every solution starting from t_0 is strictly positive (in the sense that it is bounded above and bellow by positive constants) on $[t_0, +\infty)$ for any $t_0 \in R$ (see Proposition 3). Therefore, the existence and uniqueness of the strictly positive solution of system (1.1) is determined by the behavior of the coefficients at $-\infty$. Thus, condition (1.2) seems to be strong for it is imposed on the coefficients for the whole interval $(-\infty, +\infty)$.

Following this suggestion, we want to weaken the Ahmad's condition by supposing that (1.2) is satisfied only at the infinity (see conditions (2.1) and (2.2) bellow). In this case, because condition (1.2) is not satisfied for all $t \in R$, we have to improve the proof in order to obtain the same result. Furthermore, our proof still works in the case where the coefficients are random or the competing system is described by Ito equations.

The article is organized as follows. In Section 2, we establish a new condition under which the existence of bounded above and below solution has been proved. In Section 3 we transfer the results to the case where the coefficients are stationary processes.

2. Main results

As mentioned in the introduction, we now realize the idea that condition (1.2) is satisfied at the infinity.

Hypotheses

(i) There exist two random variables ξ, η such that

$$P\{\xi > 0\} = P\{\eta > 0\} = 1,$$
$$P\{\xi < g < \eta\} = 1 \quad \text{for any} \quad g := a, b, c, d, e, f.$$

(ii) The following conditions are satisfied:

$$\limsup_{|t|\to\infty} \frac{a(t)}{b(t)} < \liminf_{|t|\to\infty} \frac{d(t)}{e(t)} \qquad \text{a.s.,} \tag{2.1}$$

$$\limsup_{|t|\to\infty} \frac{d(t)}{f(t)} < \liminf_{|t|\to\infty} \frac{a(t)}{c(t)} \qquad \text{a.s.} \tag{2.2}$$

By virtue of conditions (2.1) and (2.2), we can choose two random variables k_1 and k_2 satisfying

$$\limsup_{|t|\to\infty} \frac{a(t)}{b(t)} < k_1 < \liminf_{|t|\to\infty} \frac{d(t)}{e(t)},$$
$$\limsup_{|t|\to\infty} \frac{d(t)}{f(t)} < k_2 < \liminf_{|t|\to\infty} \frac{a(t)}{c(t)}.$$

Therefore, there exists a random variable $T > 0$ a.s. such that

$$\frac{a(t)}{b(t)} < k_1 < \frac{d(t)}{e(t)}; \quad \frac{d(t)}{f(t)} < k_2 < \frac{a(t)}{c(t)} \quad \text{a.s.} \tag{2.3}$$

for any t such that $|t| > T$.

It is obvious that if Ahmad's condition (1.2) holds then conditions (2.1) and (2.2) are satisfied. But it is easy to give an example showing that conditions (2.1) and (2.2) are weaker than (1.2).

Example. Let $a = 2 + \cos t$, $b = 2 + \cos t$, $d = \frac{3}{2}(2 + \cos t)$, $e = 2 + \cos t$ and c, f be chosen conveniently. Then

$$\limsup_{|t|\to\infty} \frac{a(t)}{b(t)} = 1 \ < \ \liminf_{|t|\to\infty} \frac{d(t)}{e(t)} = 3/2,$$

but

$$\frac{a_M}{b_L} = 3 \ > \ 1/2 = \frac{d_L}{e_M} \cdot .$$

Thus Ahmad's condition (1.2) is not true.

From condition (2.3) we can take an $\delta > 0$ such that

$$\begin{aligned} &\delta < \min\{k_1, k_2\}, \\ &a(t) - b(t) \cdot \delta - c(t) \cdot k_2 > 0, \\ &d(t) - e(t) \cdot k_1 - f(t) \cdot \delta > 0, \end{aligned} \tag{2.4}$$

for any $t : |t| > T$.

Proposition 1. (The comparison of solutions of (1.1), see [Fa, Lemma 4.4.1]) *If* $\mathrm{col}(x_1(t), y_1(t))$ *and* $\mathrm{col}(x_2(t), y_2(t))$ *are two solutions of (1.1) then for any* $t_0 \in R$,

a) If $x_1(t_0) < x_2(t_0)$; $y_1(t_0) > y_2(t_0)$ then $x_1(t) < x_2(t)$; $y_1(t) > y_2(t)$ for all $t \geq t_0$;

b) If $x_1(t_0) < x_2(t_0)$; $y_1(t_0) < y_2(t_0)$ then $x_1(t) < x_2(t)$; $y_1(t) < y_2(t)$ for all $t \leq t_0$.

Proof. The item a) can be referred to Lemma 4.4.1 of [Fa]. We only have to prove b). By the continuity of the solutions, there is an $\bar{t} \in R$ such that $x_1(t) < x_2(t)$; $y_1(t) < y_2(t)$ for all $\bar{t} < t < t_0$ and either $x_1(\bar{t}) = x_2(\bar{t})$ or $y_1(\bar{t}) = y_2(\bar{t})$. Suppose that $x_1(\bar{t}) = x_2(\bar{t})$. By simple calculation we get

$$\dot{x}_1(\bar{t}) - \dot{x}_2(\bar{t}) = -x_1(\bar{t}) \cdot c(\bar{t}) \cdot \big(y_1(\bar{t}) - y_2(\bar{t})\big) > 0.$$

On the other hand, from $x_1(t) < x_2(t); x_1(\bar{t}) = x_2(\bar{t})$ it follows that $\dot{x}_1(\bar{t}) - \dot{x}_2(\bar{t}) \leq 0$. This is a contradiction. So $x_1(t) < x_2(t)$; $y_1(t) < y_2(t)$ for all $t \leq t_0$. $\square$

Let $t_0 \in R$ arbitrary. If we consider equation (1.1) for $t > t_0$ and every solution started from $x(t_0)$ at t_0, then we call it by forward equation. In the case when the solution starts at t_0 but the system is considered only for $t < t_0$ we call it by backward equation. Using the time transformation t by $-t$, the backward equation turns into the following forward one:

$$(2.5) \qquad \begin{cases} \dot{x} = x\big(-a(-t) + b(-t)x + c(-t)y\big), \\ \dot{y} = y\big(-d(-t) + e(-t)x + f(-t)y\big). \end{cases} \qquad t \geq -t_0.$$

We show that under conditions (2.1) and (2.2), every solution of forward equation (1.1) is strictly positive. We need the following lemma:

Lemma 2. *Let* $G(t)$ *and* $F(t)$ *be two differentiable functions defined on* $(0, \infty)$ *such that* $\lim_{t\to\infty} G(t) = \lim_{t\to\infty} F(t) = +\infty$ *then*

$$\limsup_{t\to\infty} \frac{G(t)}{F(t)} \leq \limsup_{t\to\infty} \frac{G'(t)}{F'(t)}; \quad \liminf_{t\to\infty} \frac{G(t)}{F(t)} \geq \liminf_{t\to\infty} \frac{G'(t)}{F'(t)}.$$

Proof. By the Cauchy theorem for differential functions, for any $t_1,\ t_2 > 0$ there is an $\theta \in (t_1, t_2)$ such that

$$\frac{G'(\theta)}{F'(\theta)} = \frac{G(t_1) - G(t_2)}{F(t_1) - F(t_2)} = \frac{G(t_2)}{F(t_2)} \times \frac{1 - \dfrac{G(t_1)}{G(t_2)}}{1 - \dfrac{G(t_1)}{F(t_2)}}\,.$$

Letting t_1 and $t_2 \to \infty$ such that $\lim \dfrac{G(t_1)}{G(t_2)} = \lim \dfrac{F(t_1)}{F(t_2)} = 0$ we get the result. □

Proposition 3. *For the forward equation (1.1) with $T > 0$ the domain*

$$S = \{\delta \leq x \leq k_1;\ \delta \leq y \leq k_2\}$$

is an attractor, i.e., for any solution $\operatorname{col}(x(t), y(t))$*, there exists a random variable $t_0 > T$ such that* $\operatorname{col}(x(t), y(t)) \in S \quad \forall t \geq t_0$*.*

Proof. From the inequality

$$\dot{x} = x(a - bx - cy) < x(a - bx)$$

it follows that

$$x(t) \leq \frac{x(T).\exp\{A(t)\}}{1 + x(T)\int\limits_T^t \exp\{A(s)\}b(s)\,ds}; \quad A(t) = \int\limits_T^t a(s)\,ds.$$

Since $a(t)$ and $b(t)$ are bounded below by positive random variables,

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty} \int\limits_T^t \exp\{A(s)\}b(s)ds = \infty.$$

Therefore, by Lemma 2 we have

$$\limsup_{t\to\infty} x(t) \leq \limsup_{t\to\infty} \frac{a(t)}{b(t)} < k_1.$$

Thus, there exists $t_1 > T$ such that

$$x(t) < k_1 \quad \text{for all} \ \ t \geq t_1 \quad \text{a.s.}$$

Similarly, there is an $t_2 > T$ such that

$$y(t) < k_2 \quad \text{for all} \ \ t \geq t_2 \quad \text{a.s.}$$

On the other hand, for any $t \geq t_2$

$$\begin{aligned}\dot{x}(t) &= x(a - bx - cy) \geq x(a - bx - c.k_2)\\ &= x\big(h(t) - b(t)x\big),\end{aligned}$$

where $h(t) = a(t) - c(t)\cdot k_2$ and $h(t)$ is bounded below by positive random variable. Hence

$$x(t) \geq \frac{x(t_2).\exp\{H(t)\}}{1 + x(t_2)\int\limits_{t_1}^{t}\exp\{H(s)\}b(s)\,ds}; \quad H(t) = \int\limits_{t_2}^{t} h(s)\,ds.$$

By Lemma 2 and by (2.4) it is easy to see that

$$\liminf_{t\to\infty} x(t) \geq \liminf_{t\to\infty}\frac{h(t)}{b(t)} > \delta.$$

Thus, there exists an $t_3 > t_2$ such that

$$x(t) \geq \delta; \quad \forall t \geq t_3.$$

By the same argument we can prove that there is an t_4 such that

$$y(t) \geq \delta; \quad \forall t \geq t_4.$$

Taking $t_0 = \max\{t_1, t_2, t_3, t_4\}$ we have $\mathrm{col}(x(t), y(t)) \in S$ for any $t \geq t_0$. Proposition 3 is proved. $\square$

Proposition 4. *If there exists an* $t_0 \leq -T$ *such that* $x(t_0) = k_1$ *or* $y(t_0) = k_2$ *then the solution* $\mathrm{col}(x(t), y(t))$ *of equation* (1.1) *is exploded, i.e., this solution can not be extended on* $(-\infty, -T)$.

Proof. For the forward equation (2.5), from the inequalities

$$\begin{aligned}\dot{x} &= x\big(-a(-t) + b(-t)x + c(-t)y\big) \geq x\big(-a(-t) + b(-t)x\big)\\ \dot{y} &= y\big(-d(-t) + e(-t)x + f(-t)y\big) \geq y\big(-d(-t) + e(-t)x\big), \quad t \geq T,\end{aligned}$$

it follows that

$$x(t) > \frac{x(-t_0)\cdot\exp\{A(t)\}}{1 - x(-t_0)\int\limits_{-t_0}^{t}\exp\{A(s)\}b(-s)ds}, \quad A(t) = -\int\limits_{-t_0}^{t} a(-s)ds,$$

$$y(t) > \frac{y(-t_0)\cdot\exp\{D(t)\}}{1 - y(-t_0)\int\limits_{-t_0}^{t}\exp\{D(s)\}f(-s)ds}, \quad D(t) = \int\limits_{-t_0}^{t} d(-s)ds;\ t \geq -t_0.$$

Since

$$\int_{-t_0}^{\infty} \exp\{A(s)\}b(-s)ds > \frac{1}{k_1} \cdot \int_{-t_0}^{\infty} \exp\{A(s)\}a(-s)ds = \frac{1}{k_1}\,,$$

there exists $\beta_1 > -t_0$ such that

$$\int_{-t_0}^{\beta_1} \exp\{A(s)\}b(-s)ds = 1.$$

Therefore, if $x(-t_0) = k_1$ we have

$$\lim_{t \to \beta_1} x(t) = +\infty.$$

Similarly, from

$$\int_{-t_0}^{\infty} \exp\{D(s)\}f(-s)ds > \frac{1}{k_2} \cdot \int_{-t_0}^{\infty} \exp\{D(s)\}d(-s)ds = \frac{1}{k_2}\,,$$

it follows that there is $\beta_2 > t_2$ such that

$$\int_{-t_0}^{\beta_2} \exp\{D(s)\}f(-s)ds = 1.$$

Hence, if $y(t_0) = k_2$ then

$$\lim_{t \to \beta_2} y(t) = +\infty. \quad \square$$

Proposition 5. *Suppose that $y(t) \le k_2$ for all $t \le -T$ (respectively $x(t) \le k_1$ for all $t \le -T$) and the solution $\mathrm{col}(x(t), y(t))$ is defined on $(-\infty, \infty)$. If there is an $t_0 \le -T$ such that $x(t_0) \le \delta$ (respectively, $y(t_0) \le \delta$) then $\lim_{t \to \infty} x(t) = 0$ (respectively, $\lim_{t \to \infty} y(t) = 0$).*

Proof. We consider once more forward equation (2.5). Let $\operatorname{col}(x(t), y(t))$ be a solution of (2.5) such that $y(t) \leq k_2$ for all $t \geq T$ and $x(t_0) \leq \delta$ for an $t_0 \geq T$. From inequality (2.3) we get

$$\begin{aligned}\dot{x} &= x\big(-a(-t) + b(-t)x + c(-t)y\big)\\ &\leq x\big(-a(-t) + b(-t)x + c(-t)\cdot k_2\big)\\ &= x(-h(t) + b(-t)x),\end{aligned}$$

where $h(t) = a(-t) - c(-t)\cdot k_2$ is bounded above and below by positive random variables. Hence

$$x(t) \leq \frac{x(t_0)\cdot \exp\{-H(t)\}}{1 - \int_{t_0}^{t} \exp\{-H(s)\}b(-s)ds}, \quad H(t) = \int_{t_0}^{t} h(s)ds.$$

On the other hand, by (2.4)

$$\int_{t_0}^{\infty} \exp\{-H(s)\}b(-s)ds < \frac{1}{\delta}\cdot \int_{t_0}^{t} \exp\{-H(s)\}h(s)\,ds = \frac{1}{\delta}\cdot$$

Therefore, we get $\lim_{t\to\infty} x(t) = 0$. Similarly we can prove the second case. □

Thus, Propositions 4 and 5 tell us that if there exists a bounded above and below solution defined on $(-\infty, \infty)$ then it is necessarily that $\operatorname{col}(x(t), y(t)) \in S$ for any $t \in (-\infty, -T] \cup [T, \infty)$. We now pass to the proof of the existence.

Proposition 6. *If* $t_0 < -T$ *and* $\operatorname{col}(x_1(t), y_1(t))$ *and* $\operatorname{col}(x_2(t), y_2(t))$ *are two solutions of* (1.1) *such that*

$$\begin{aligned}\operatorname{col}(x_1(t_0), y_1(t_0)) &= (\delta, k_2),\\ \operatorname{col}(x_2(t_0), y_2(t_0)) &= (k_1, \delta),\end{aligned}$$

then

$$\delta \leq x_1(t) \leq x_2(t) \leq k_1, \quad \delta \leq y_2(t) \leq y_1(t) \leq k_2,$$

for $t_0 < t \leq -T$.

Proof. See [Ah2, Lemma 1] □

Proposition 7. *Under the conditions* (2.1) *and* (2.2), *Equation* (1.1) *has two solutions, namely* $\mathrm{col}(x^*(t), y^*(t))$ *and* $\mathrm{col}(x_*(t), y_*(t))$, *which are bounded above and below on* $(-\infty, +\infty)$ *by two positive random variables* ε, Δ, *i.e.*,

$$\varepsilon \leq x_* \leq x^* \leq \Delta, \quad \varepsilon \leq y^* \leq y_* \leq \Delta \quad \text{a.s.} \tag{2.6}$$

Proof. The proof of the theorem is somewhat similar to that of Lemma 1 in [Ah2] with some slight improvements of the technique. For every $n \in N$ we denote by $\mathrm{col}(x_n^*(t), y_n^*(t))$ and $\mathrm{col}(x_{*n}(t), y_{*n}(t))$ the solutions of (1.1) satisfying the initial conditions

$$x_{*n}(-n) = y_n^*(-n) = \delta; \quad x_n^*(-n) = k_1; \;\; y_{*n}(-n) = k_2; \quad t \geq -n.$$

By Proposition 6 we have

$$\delta \leq x_{*n} \leq x_n^* \leq k_1, \quad \delta \leq y_n^* \leq y_{*n} \leq k_2, \quad -n \leq t \leq -T.$$

Hence

$$\begin{aligned} \delta = x_{*n}(-n) &\leq x_{*(n+1)}(-n) \leq x_{n+1}^*(-n) \leq x_n^*(-n) = k_1, \\ \delta = y_n^*(-n) &\leq y_{n+1}^*(-n) \leq y_{*(n+1)}(-n) \leq y_{*n}(-n) = k_2. \end{aligned}$$

Therefore, by Proposition 1 and Proposition 6

$$\begin{aligned} \delta &\leq x_{*n}(t) \leq x_{*(n+1)}(t) \leq x_{n+1}^*(t) \leq x_n^*(t) \leq k_1, \\ \delta &\leq y_n^*(t) \leq y_{n+1}^*(t) \leq y_{*(n+1)}(t) \leq y_{*n}(t) \leq k_2 \end{aligned}$$

for any $t \in [-n, -T]$. In particular, the sequences of random variables $(x_{*n}(-T))$ and $(y_n^*(-T))$ are increasing in n and the sequences $(x_n^*(-T))$ and $(y_{*n}(-T))$ are decreasing in n. Thus, there almost surely exist the limits

$$\begin{aligned} \lim_{n\to\infty} x_n^*(-T) &= \xi^*; \quad \lim_{n\to\infty} x_{*n}(-T) = \xi_*; \\ \lim_{n\to\infty} y_n^*(-T) &= \eta^*; \quad \lim_{n\to\infty} y_{*n}(-T) = \eta_*; \end{aligned}$$

Moreover,

$$\xi_* \leq \xi^*; \quad \eta^* \leq \eta_* \quad \text{a.s.}$$

Let $\mathrm{col}(x^*(t), y^*(t))$ and $\mathrm{col}(x_*(t), y_*(t))$ be two solutions of the equation (2.1) with $x^*(-T) = \xi^*$, $x_*(-T) = \xi_*$; $y^*(-T) = \eta^*$, $y_*(-T) = \eta_*$. As in [Ah2, Lemma 1] we can show that the solutions $\mathrm{col}(x^*(t), y^*(t))$ and $\mathrm{col}(x_*(t), y_*(t))$ are defined on $(-\infty, -T]$ (so in $(-\infty, \infty)$ by Proposition 3), i.e., they are unexploded. Furthermore,

$$\delta \le x_*(t) \le x^*(t) \le k_1, \quad y^*(t) \le y_*(t) \le k_2, \quad t \in (-\infty, -T].$$

On the other hand, by Proposition 3, there exists $t_0 \ge T$ such that $\mathrm{col}(x_*(t), y_*(t)) \in S$ and $\mathrm{col}(x^*(t), y^*(t)) \in S$ for any $t \ge t_0$. Put

$$\varepsilon = \min\Big\{\delta, \min_{-T\le t\le t_0} y^*(t), \min_{-T\le t\le t_0} x_*(t), \Big\},$$
$$\Delta = \max\Big\{k_1, k_2, \max_{-T\le t\le t_0} x^*(t), \max_{-T\le t\le t_0} y_*(t)\Big\}.$$

Then it follows that

$$\varepsilon \le x_*(t) \le x^*(t) \le \Delta,$$
$$\varepsilon \le y^*(t) \le y_*(t) \le \Delta.$$

Proposition 7 is proved. □

Proposition 8. *If* $\mathrm{col}(\hat{x}(t), \hat{y}(t))$ *is a bounded above and below solution of* (1.1) *then*

$$\begin{array}{l} \delta \le x_*(t) \le \hat{x}(t) \le x^*(t) \le k_1, \\ \delta \le y^*(t)) \le \hat{y}(t) \le y_*(t) \le k_2, \end{array} \quad t \in (-\infty, -T].$$

Proof. The proof is quite similar as in [Ah2, Lemma 1]. We omit it here. □

We turn to the uniqueness. We have shown in Proposition 3 that every solution of forward equation (1.1) is bounded above and below. So the uniqueness (if it has) is determined by the backward equation on $(-\infty, -T]$. From condition (2.1) and (2.2) it is easy to prove that there exists a random positive variable γ such that $\dfrac{b(t)}{e(t)} > \dfrac{c(t)}{f(t)} + \gamma$ for any $t : |t| \ge T$ but we need a further hypothesis.

Hypothesis.

$$\liminf_{t\to\infty} \frac{b(t)}{e(t)} > \limsup_{t\to\infty} \frac{c(t)}{f(t)} \cdot \tag{2.7}$$

It is easy to check that Ahmad's condition implies this condition.

Proposition 9. *Under the hypotheses* (2.1), (2.2) *and* (2.7), *the solution of equation* (2.1) *bounded above and below by positive constants is unique.*

Proof. By Proposition 8 it suffices to show that

$$x_*(t) = x^*(t), \quad y_*(t) = y^*(t).$$

By virtue of (2.7) and without loss of generality we may assume that there exist three positive random variables α, β and γ such that

$$\frac{b(t)}{e(t)} > \frac{\alpha}{\beta} + \gamma, \quad \frac{c(t)}{f(t)} < \frac{\alpha}{\beta} - \gamma; \quad \text{for all} \quad t \le -T. \tag{2.8}$$

Dividing both sides of (1.1) by x^*, x_* and y^*, y_* respectively and subtracting them we get

$$\begin{aligned} \frac{\dot{x}^*}{x^*} - \frac{\dot{x}_*}{x_*} &= -b(x^* - x_*) - c(y^* - y_*), \\ \frac{\dot{y}^*}{y^*} - \frac{\dot{y}_*}{y_*} &= -e(x^* - x_*) - f(y^* - y_*), \end{aligned}$$

or, equivalently,

$$\begin{aligned} \Big(\ln \frac{x^*}{x_*}\Big)' &= -b(x^* - x_*) - c(y^* - y_*), \\ \Big(\ln \frac{y^*}{y_*}\Big)' &= -e(x^* - x_*) - f(y^* - y_*). \end{aligned}$$

Putting

$$\begin{aligned} &U(t) = \ln \frac{x^*(t)}{x_*(t)} \ge 0, \qquad V(t) = \ln \frac{y^*(t)}{y_*(t)} \le 0, \\ &X(t) = x^*(t) - x_*(t) \ge 0, \quad Y(t) = y^*(t) - y_*(t) \le 0, \end{aligned}$$

we have

$$\begin{aligned} \dot{U}(t) &= -b(t)X(t) - c(t)Y(t), \\ \dot{V}(t) &= -e(t)X(t) - f(t)Y(t). \end{aligned} \tag{2.9}$$

By multiplying the first equation of (2.9) by β and the second one by α and subtracting them we obtain

$$\alpha\dot{U}(t) - \beta\dot{V}(t) = \big(-\beta b(s) + \alpha e(s)\big)X(s) + \big(-\beta c(s) + \alpha f(s)\big)Y(s). \tag{2.10}$$

Since $U(t)$ and $V(t)$ are bounded above and below by positive constants, it follows that

$$\int_{-\infty}^{-T} (-\beta b(s) + \alpha e(s))X(s)ds + \int_{-\infty}^{-T} \big(-\beta c(s) + \alpha f(s)\big)Y(s)ds < \infty.$$

From (2.8) it follows that $-\beta b(s)+\alpha e(s)$ and $-\beta c(s)+\alpha f(s)$ are bounded above and below by positive constants. Hence

$$\int_{-\infty}^{-T} X(s)ds < \infty; \quad -\int_{-\infty}^{-T} Y(s)ds < \infty.$$

Because $X(t)$ and $Y(t)$ are bounded together with their derivatives we get

$$\lim_{t\to-\infty} X(t) = \lim_{t\to-\infty} Y(t) = 0.$$

Furthermore, $x_*(t)$, $x^*(t)$, $y_*(t)$, $y^*(t)$ are also bounded below. Therefore,

$$\lim \frac{x^*(t)}{x_*(t)} = 1, \quad \lim \frac{y^*(t)}{y_*(t)} = 1$$

which implies that

$$\lim_{t\to-\infty} U(t) = \lim_{t\to-\infty} V(t) = 0. \tag{2.11}$$

Hence

$$\int_{-\infty}^{-T} b(t)X(t)\,dt + \int_{-\infty}^{-T} c(t)f(t))Y(t)\,dt \le 0,$$

$$\int_{-\infty}^{T} e(t)X(t)\,dt + \int_{\infty}^{T} f(t)Y(t)\,dt \ge 0.$$

From these inequalities we obtain

$$\frac{\int_{-\infty}^{-T} b(t)X(t)\,dt}{\int_{-\infty}^{T} e(t)X(t)\,dt} \le \frac{\int_{-\infty}^{-T} c(t)f(t))Y(t)\,dt}{\int_{\infty}^{T} f(t)Y(t)\,dt}\,. \tag{2.12}$$

On the other hand, if $\int_{-\infty}^{-T} b(s)X(s)ds \neq 0$ and $\int_{-\infty}^{-T} c(s)Y(s)ds \neq 0$ then by the mean value of integrals it follows that

$$\frac{\int_{-\infty}^{-T} b(s)X(s)ds}{\int_{-\infty}^{-T} e(s)X(s)ds} \geq \inf_{t<-T} \frac{b(t)}{e(t)} > \sup_{t<-T} \frac{c(t)}{f(t)} \geq \frac{\int_{-\infty}^{-T} c(s)Y(s)ds}{\int_{-\infty}^{-T} f(s)Y(s)ds}$$

which contradicts (2.12). Thus $\int_{-\infty}^{-T} b(s)X(s)ds = 0$ and $\int_{-\infty}^{-T} c(s)Y(s)ds = 0$. Hence, it is easy to see that $X(t) \equiv 0$, $Y(t) \equiv 0$. Proposition 9 is proved. □

Finally, we show that the above strictly positive solution attracts every solution on $[t_0, +\infty)$.

Proposition 10. *If* $\mathrm{col}(x_1(t), y_1(t))$ *and* $\mathrm{col}(x_2(t), y_2(t))$ *are two solutions of* (1.1) *defined on* $[0, \infty)$ *with* $x_2(0) \geq x_1(0)$; $y_2(0) \leq y_1(0)$, *then*

$$\lim_{t\to+\infty} (x_2(t) - x_1(t)) = 0; \quad \lim_{t\to+\infty} (y_2(t) - y_1(t)) = 0. \tag{2.13}$$

Proof. From Proposition 1 it follows that

$$x_2(t) \geq x_1(t); \quad y_2(t) \leq y_1(t) \quad \text{for any} \quad t \in [0, \infty).$$

Hence, $X(t) := x_2(t) - x_1(t) \geq 0$ and $Y(t) := y_2(t) - y_1(t) \leq 0$. As in the proof of Proposition 9 we can show that

$$\int_T^\infty X(t)\,dt < \infty; \quad -\int_T^\infty Y(t)\,dt < \infty.$$

Since $X(t)$ and $Y(t)$ are bounded together with their derivatives, it implies that

$$\lim_{t\to\infty} X(t) = 0; \quad \lim_{t\to\infty} Y(t) = 0.$$

The proposition is proved. □.

Proposition 11. *The unique bounded above and below solution mentioned in Proposition 9 is attractive.*

Proof. Let $\mathrm{col}(x_*(t), y_*(t))$ be the unique solution bounded above and below on $(-\infty, +\infty)$. Suppose that $\mathrm{col}(x(t), y(t))$ is an arbitrary positive solution of (1.1). If either $x_*(0) \le x(0)$; $y_*(0) \ge y(0)$ or $x_*(0) \ge x(0)$; $y_*(0) \le y(0)$ then the proof follows from Proposition 10. Therefore, we only have to consider the case $x_*(0) \le x(0)$; $y_*(0) \le y(0)$.

Let $\mathrm{col}(\overline{x}(t),\ \overline{y}(t))$ be the solution of (1.1) satisfying $\overline{x}(0) = x_*(0)$; $\overline{y}(0) = y(0)$. Since $\overline{x}(0) = x_*(0)$; $\overline{y}(0) \ge y_*(0)$, we have

$$\lim_{t\to\infty} (\overline{x}(t) - x_*(t)) = 0; \quad \lim_{t\to\infty} (\overline{y}(t) - y_*(t)) = 0. \tag{2.14}$$

On the other hand $\overline{x}(0) \le x(0)$; $\overline{y}(0) = y(0)$. Therefore,

$$\lim_{t\to\infty} (\overline{x}(t) - x(t)) = 0; \quad \lim_{t\to\infty} (\overline{y}(t) - y(t)) = 0. \tag{2.15}$$

Combining (2.14) with (2.15) gives the proof of Proposition 11. □

3. Existence of stationary solutions

We now suppose that the processes a, b, c, d, e, f are stationary. We shall show that the unique bounded above and below solution in Propositions 8 and 9 is a stationary process too. The following Lemma can be referred in [Aw].

Lemma 12. *Suppose that $\xi(t)$ is a stationary process and the system*

$$\dot{X} = g\big(\xi(t), X(t)\big); \quad t \in (-\infty, \infty), \ \ X \in R^d \tag{3.1}$$

has a solution bounded in probability in the following sense: for any $\gamma > 0$, there is a compact set $K_\gamma \subset R^d$ such that

$$\lim_{t\to\infty} \frac{1}{2t} \int_{-t}^{t} P\{X(t) \in K_\gamma\}\, dt > 1 - \gamma.$$

Then (3.1) *possesses a stationary solution, namely $\eta(t)$, such that $P\{\eta(t) \in K_\gamma\} \ge 1 - \gamma$ for any γ.*

Proposition 13. *The solution col$(x(t), y(t))$ mentioned in Propositions 8 and 9 is a stationary process*

Proof. From (2.7) it follows that for any $\gamma > 0$,

$$P\{\varepsilon \le x(t) \le \Delta;\ \varepsilon \le y(t) \le \Delta\} = 1.$$

Let K be a compact $[\alpha,\beta]\times[\alpha,\beta]\subset R^2$ such that $P\{\alpha<\varepsilon;\Delta>\beta\}\geq 1-\gamma$ then

$$P\big\{\alpha\leq x(t)\leq\beta;\ \alpha\leq y(t)\leq\beta\big\}\geq 1-\gamma.$$

Therefore, the conclusion follows from the hypothesis of Lemma 12. □

Open Problem. The author is not sure whether the obtained result remains true if conditions (2.1) and (2.3) are replaced by an averaging condition (similar as in the periodic cases):

$$\limsup_{|T|\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{a(t)}{b(t)}\,dt<\liminf_{|T|\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{d(t)}{e(t)}\,dt\quad\text{a.s.}\tag{3.2}$$

$$\limsup_{|T|\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{d(t)}{f(t)}\,dt<\liminf_{|T|\to\infty}\frac{1}{2T}\int_{-T}^{T}\frac{a(t)}{c(t)}\,dt\quad\text{a.s.}\tag{3.3}$$

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References

[Ah1] S. Ahmad, *On almost periodic solutions of the competing species problem*, Proc. Amer. Math. Soc. **102** (1988), 855-861.

[Ah2] S. Ahmad, *On the autonomous Volterra-Lotka competing equations*, Proc. Amer. Math. Soc. **117** (1993), 199-124.

[Ar] L. Arnold, *Random Dynamical Systems*, Preliminary Version 1994.

[Aw] L. Arnold and V. Wihstutz, *Stationary solutions of linear system with additive and multiplicative noise*, *Stochastics* **7** (1982), 133-155.

[Fa] M. Farkas, *Periodic Motions*, Springer-Verlag, 1994.

[Go] K. Gopalsamy, *Global asymptotic stability in a periodic Lotka-Volterra system*, J. Australian Math. Soc. Ser. **B27** (1988), 66-72.

[Ko] A. N. Kolmogorov and S. V. Formin, *Introductory Real Analysis,* Prentice-Hall, 1970.

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