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ON NECESSARY OPTIMALITY CONDITIONS IN MULTIFUNCTION OPTIMIZATION WITH PARAMETERS

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ABSTRACT. We consider multifunction optimization problems with parameters. Necessary optimality conditions of the Fritz John and Kuhn-Tucker types are obtained with relaxed differentiability assumptions on the state variable and convexlikeness assumptions on the parameter.

1. INTRODUCTION

The range of application of multifunctions, i.e., set-valued functions is very large. An optimization theory involving multifunctions was established by Corley [1, 2, 3]. In [3] the Fritz John necessary optimality condition is extended to multifunction optimization. In [11, 12, 14] there are improvements of the above result in various cases.

This note is devoted to considering necessary optimality conditions for multifunction optimization problems with parameters of the following form.

Let X, Y, Z and W be Banach spaces, Y and Z being ordered by convex cones K and M, containing the origins and with $\operatorname{int} K \neq \emptyset$, $\operatorname{int} M \neq \emptyset$, respectively. Let U be an arbitrary set. Let F, G be multifunctions of $X \times U$ into Y and Z, respectively. Let p be a (single-valued) map of $X \times U$ into W. The problem under consideration is

$$\begin{aligned} &\min \; F(x,u),\\ &G(x,u) \subset -M, \qquad\qquad (P)\\ &p(x,u) = 0; \end{aligned}$$

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or

$$\begin{aligned} &\min\ F(x,u),\\ &G(x,u)\cap(-M)\neq\emptyset, \\ &p(x,u)=0. \end{aligned} \tag{$\tilde{P}$}$$

Here "min" means that we are looking for a (Pareto) minimum or a weak minimum. A multifunction $\mathcal{F} : X \rightsquigarrow Y$ is said to have a (*global*) *minimum* at $(x_0; f_0)$, where $f_0 \in \mathcal{F}(x_0)$, on a set $A \subset X$ if

$$\mathcal{F}(A) - f_0 \subset Y \setminus ((-K) \setminus K). \tag{1}$$

If (1) is replaced by

$$\mathcal{F}(A) - f_0 \subset Y \setminus (-\operatorname{int} K). \tag{2}$$

then $(x_0; f_0)$ is called a (*global*) *weak minimum of* $\mathcal{F}$ on A. If there is a neighborhood N of x_0 such that one has (1) or (2) with $\mathcal{F}(A)$ replaced by $\mathcal{F}(A\cap N)$, then $(x_0; f_0)$ is called a *local minimum* or *local weak minimum*, respectively, of $\mathcal{F}$.

Since U is an arbitrary set, on considering local minima or local weak minima of (P) and $(\tilde{P})$ we adopt for U the trivial topology consisting of only two sets $\emptyset$ and U, and for $X \times U$ the product topology.

Problems with parameters of the types (P) and $(\tilde{P})$ are often met in practical situations. Such situations usually require that the assumptions imposed on x and on u should be different. For instance, in control problems differentiability assumptions imposed on the control u should be much weaker than that on the state variable x. (P) coincides with $(\tilde{P})$ if G is single-valued. For both constraints $G(x,u) \subset -M$ and $G(x,u)\cap(-M) \neq \emptyset$ become then the inequality constrainst $G(x,u) \leq 0$. (For the sake of simplicity the notation $\leq$ is used commonly for various ordering in various spaces if no confusion may occur.) When F and G are single-valued, optimization problems with parameters were considered, e.g. in [5, 6, 7, 8, 9, 10, 13]. In the present note we restrict ourselves to the case where p is single-valued since in most applications equality constraints represent state equations (being often differential equations), initial and boundary conditions.

Let Z^* stand for the topological dual to Z. The dual cone M^* of M is

$$M^* := \big\{\mu \in Z^* : \langle \mu, z\rangle \geq 0\ \forall z \in M\big\}.$$

In later considered situations a feasible point (x_0, u_0) and some $g_0 \in G(x_0, u_0) \cap (-M)$ will be fixed. Then we set

$$M_0 := \{\gamma(z + g_0) : \gamma \in R_+, z \in M\},$$
$$M_0^* := \{\mu \in M^* : \langle \mu, g_0 \rangle = 0\} = (M_0)^*.$$

For a multifunction $\mathcal{F} : X \rightsquigarrow Y$ its graph is

$$\operatorname{gr} \mathcal{F} := \{(x, y) \in X \times Y : y \in \mathcal{F}(x)\}$$

and its domain is $\operatorname{dom} \mathcal{F} := \{x \in X : \mathcal{F}(x) \neq \emptyset\}$. We recall that the *Clarke derivative* of $\mathcal{F}$ at $(x_0; f_0) \in \operatorname{gr} \mathcal{F}$, denoted by $D\mathcal{F}(x_0; f_0)$, is a multifunction of X into Y whose graph is

$$\begin{aligned} \operatorname{gr} D\mathcal{F}(x_0; f_0) = \{(v, w) \in X \times Y : \forall (x_n, f_n) \to_{\mathcal{F}} (x_0, f_0),\ \forall t_n \to 0^+, \\ \exists (v_n, w_n) \to (v, w); \forall n, f_n + t_n w_n \in \mathcal{F}(x_n + t_n v_n)\}, \end{aligned}$$

where $\to_{\mathcal{F}}$ means that $(x_n, f_n) \in \operatorname{gr} \mathcal{F}$ and $(x_n, f_n) \to (x_0, f_0)$. Recall also that $D\mathcal{F}(x_0, f_0)$ is always a closed convex process, i.e. a multifunction whose graph is a nonempty closed convex cone. For a single-valued map $p : X \times U \to W$, $p_x(x_0, u_0)$ will stand for the Fréchet derivative of $p(., u_0)$ at x_0.

In our consideration only the following directional differentiability with respect to x is imposed on F and G.

Definition 1. Let X and Y be Banach spaces, Y being ordered by a convex cone K. A multifunction $\mathcal{F} : X \rightsquigarrow Y$ is called uniformly K-differentiable in the direction $\overline{x} \in X$ at $(x_0, f_0) \in \operatorname{gr} \mathcal{F}$ if for each neighborhood V of zero in Y there is a neighborhood N of $\overline{x}$ and a real $\gamma_0 > 0$ such that $\forall \gamma \in (0, \gamma_0)$, $\forall x \in N$, $\forall f \in \mathcal{F}(x_0 + \gamma x)$, $\forall f' \in D\mathcal{F}(x_0; f_0)\overline{x}$

$$\frac{1}{\gamma}(f - f_0) - f' \in V - K.$$

$\mathcal{F}$ is said to be uniformly K-differentiable at $(x_0; f_0)$ if this differentiability holds for all directions x in $\operatorname{dom} DF(x_0; f_0)$.

Note that the parameter set U in (P) and $(\tilde{P})$ is equipped with no structure. However, the following extension to multifunctions of the convexlikeness introduced by Fan [4] is needed.

Definition 2. A multifunction $\mathcal{F} : U \rightsquigarrow \mathcal{Y}$ where U is a set and $\mathcal{Y}$ is a vector space ordered by a convex cone $\mathcal{K}$, is said to be $\mathcal{K}$-convexlike in (U_1, U_2) with $U_1 \subset U$, $U_2 \subset U$ if $\forall (u_1, u_2) \in U_1 \times U_2$, $\forall f_1 \in \mathcal{F}(u_1)$, $\forall f_2 \in \mathcal{F}(u_2)$, $\forall \gamma \in [0, 1]$, $\exists u \in U$, $\exists f_u \in \mathcal{F}(u)$,

$$\gamma f_1 + (1 - \gamma) f_2 - f_u \in \mathcal{K}.$$

Definition 3. Let X, Y and $\mathcal{F}$ be as in Definition 1. Let $x_0 \in \operatorname{dom} \mathcal{F}$ and $T \subset \mathcal{F}(x_0)$ be nonempty. Then, $\mathcal{F}$ is called K-strong lower semicontinuous (K-s.l.s.c.) with T at x_0 if for each neighborhood V of zero in Y, there is a neighborhood N of x_0 such that $\forall x \in N$, $\exists f_x \in \mathcal{F}(x)$,

$$f_x - T \subset V - K.$$

Notice that, if $T = \mathcal{F}(x_0)$ and $\mathcal{F}$ is K-s.l.s.c. with T, then $\mathcal{F}(.) + K$ is l.s.c. in the usual sense (for multifunctions).

2. Main results

First we present necessary optimality conditions for local weak minima of $(\tilde{P})$. Of course these necessary conditions hold also for local minima.

Theorem 1 (Fritz John necessary condition). *Assume that*

(i) $p(., u_0)$ *is continuously differentiable at* x_0 *and* $p_x(x_0, u_0)$ *is onto, where* (x_0, u_0) *is feasible for* $(\tilde{P})$*;*

(ii) $F(., u_0)$ *and* $G(., u_0)$ *are uniformly* K*-differentiable at* $(x_0, u_0; f_0)$ *and uniformly* M*-differentiable at* $(x_0, u_0; g_0)$*, respectively, where* $f_0 \in F(x_0, u_0)$*,* $g_0 \in G(x_0, u_0) \cap (-M)$*;*

(iii) *for each* $u \neq u_0, F(., u)$ *(*$G(., u)$*, respectively) is* K*-s.l.s.c. with* $F(x_0, u)$ *(*M*-s.l.s.c. with* $G(x_0, u)$*) at* x_0*. Moreover,* $F(., u_0)$ *(*$G(., u_0)$*) is* $-K$*-s.l.s.c. with* f_0 *(*$-M$*-s.l.s.c. with* g_0*, respectively) at* x_0*;*

(iv) *for each* x *in a neighhorhood* V *of* x_0*,* $(F, G, p)(x, .)$ *is* $K \times M \times \{0\}$*-convexlike in* $(U, \{u_0\})$*. Moreover,* $(F, G, p)(x_0, .)$ *is* $K \times M \times \{0\}$*-convexlike in* (U, U)*.*

If $(x_0, u_0; f_0)$ *is a local weak minimum of* $(\tilde{P})$*, then there exists*

$$(\lambda_0, \mu_0, \nu_0) \in K^* \times M_0^* \times W^* \setminus \{0\}$$

such that, for all $(x, u) \in X \times U$*,*

$$\begin{aligned}&\langle \lambda_0, D_x F(x_0, u_0; f_0)x + F(x_0, u) - f_0\rangle \\ &\quad + \langle \mu_0, D_x G(x_0, u_0; g_0)x + G(x_0, u) - g_0\rangle \\ &\quad + \langle \nu_0, p_x(x_0, u_0)x + p(x_0, u)\rangle \subset R_+. \end{aligned}\tag{3}$$

Proof. It is clear that (3) is true for

$$x \notin \operatorname{dom} D_x F(x_0, u_0; f_0) \cap \operatorname{dom} D_x G(x_0, u_0; g_0)$$

since the leff-hand side of (3) is empty. So we may assume that x belongs to this intersection. However, for the simplicity of notation, we still write $x \in X$.

Let C be the set of all $(y, z, w) \in Y \times Z \times W$ such that $\exists (x, u) \in X \times U$, $\exists f'_x \in D_x F(x_0, u_0; f_0)x$, $\exists f^u_{x_0} \in F(x_0, u)$, $\exists g'_x \in D_x G(x_0, u_0; g_0)x$, $\exists g^u_{x_0} \in G(x_0, u)$,

$$f'_x + f^u_{x_0} - f_0 - y \in -\operatorname{int} K, \tag{4}$$

$$g'_x + g^u_{x_0} - g_0 - z \in -\operatorname{int} M, \tag{5}$$

$$p_x(x_0, u_0)x + p(x_0, u) = w. \tag{6}$$

We claim that C is convex. Indeed, if $(y_i, z_i, w_i) \in C$, $i = 1, 2$, then, with the notations defined similarly as the terms in (4)-(6), for all $\gamma \in [0, 1]$ one has

$$\gamma f'_{x_1} + (1-\gamma) f'_{x_2} + \gamma f^{u_1}_{x_0} + (1-\gamma) f^{u_2}_{x_0} - f_0 \leq \gamma y_1 + (1-\gamma) y_2, \tag{7}$$

$$\gamma g'_{x_1} + (1-\gamma) g'_{x_2} + \gamma g^{u_1}_{x_0} + (1-\gamma) g^{u_2}_{x_0} - g_0 \leq \gamma z_1 + (1-\gamma) z_2, \tag{8}$$

$$\begin{aligned}&p_x(x_0, u_0)(\gamma x_1 + (1-\gamma) x_2) + \gamma p(x_0, u_1) + (1-\gamma) p(x_0, u_2) \\ &\qquad = \gamma w_1 + (1-\gamma) w_2.\end{aligned}\tag{9}$$

By the convexity of the Clarke derivative, setting $\overline{x} := \gamma x_1 + (1-\gamma) x_2$ one sees that

$$\begin{aligned}f'_{\overline{x}} &:= \gamma f'_{x_1} + (1-\gamma) f'_{x_2} \in D_x F(x_0, u_0; f_0)\overline{x}, \\ g'_{\overline{x}} &:= \gamma g'_{x_1} + (1-\gamma) g'_{x_2} \in D_x G(x_0, u_0; g_0)\overline{x}.\end{aligned}$$

The $K \times M \times \{0\}$-convexlikeness of $(F, G, p)(x_0, .)$ in (U, U) yields a $\overline{u} \in U$, $f^{\overline{u}}_{x_0} \in F(x_0, \overline{u})$ and $g^{\overline{u}}_{x_0} \in G(x_0, \overline{u})$ such that

$$f_{x_0}^{\overline{u}} \le \gamma f_{x_0}^{u_1} + (1-\gamma) f_{x_0}^{u_2},$$
$$g_{x_0}^{\overline{u}} \le \gamma g_{x_0}^{u_1} + (1-\gamma) g_{x_0}^{u_2},$$
$$p(x_0, \overline{u}) = \gamma p(x_0, u_1) + (1-\gamma) p(x_0, u_2).$$

Consequently, (7)-(9) together show that

$$(\overline{y}, \overline{z}, \overline{w}) := \gamma(y_1, z_1, w_1) + (1-\gamma)(y_2, z_2, w_2)$$

belongs to C, i.e. C is convex.

Next we show that $\operatorname{int} C \neq \emptyset$. Assume that $(\overline{y}, \overline{z}, \overline{w}) \in C$. Then for a small and fixed $\varepsilon > 0$,

$$f'_{\overline{x}} + f_{x_0}^u - f_0 - \overline{y} + \varepsilon B_Y \subset -\operatorname{int} K, \tag{10}$$
$$g'_{\overline{x}} + g_{x_0}^u - g_0 - \overline{z} + \varepsilon B_Z \subset -\operatorname{int} M, \tag{11}$$
$$p_x(x_0, u_0)\overline{x} + p(x_0, u) = \overline{w}, \tag{12}$$

where B_X, B_Y and B_Z are open unit balls in X, Y and Z, respectively.

By the uniform K-differentiability of $F(., u_0)$ assumed in (ii), $\exists \delta > 0$, $\exists \gamma_0 > 0$, $\forall x \in \overline{x} + \delta B_X$, $\forall \gamma \in (0, \gamma_0)$, $\forall f \in F(x_0 + \gamma x, u_0)$,

$$\frac{1}{\gamma}(f - f_0) - f'_{\overline{x}} \in \frac{\varepsilon}{4} B_Y - K. \tag{13}$$

Now for each $x \in \overline{x} + \dfrac{\delta}{2} B_X$, from $f'_x \in D_x F(x_0, u_0; f_0)x$ it follows that, for the given ε, $\exists \gamma \in (0, \gamma_0)$, $\exists x' \in x + \dfrac{\delta}{2} B_X$, $\exists f \in F(x_0 + \gamma x', u_0)$,

$$f'_x - \frac{1}{\gamma}(f - f_0) \in \frac{\varepsilon}{4} B_Y. \tag{14}$$

(13) and (14) together show that for all $x \in \hat{x} + \dfrac{\delta}{2} B_X$,
$\forall f'_x \in D_x F(x_0, u_0; f_0)x$, $\exists \gamma \in (0, \gamma_0)$, $\exists x' \in x + \dfrac{\delta}{2} B_X$, $\exists f \in F(x_0 + \gamma x', u_0)$,

$$f'_x - f'_{\overline{x}} = f'_x - \frac{1}{\gamma}(f - f_0) + \frac{1}{\gamma}(f - f_0) - f'_{\overline{x}} \in \frac{\epsilon}{2} B_Y - K. \tag{15}$$

Using (10), (15) one sees that, for all $x \in \overline{x} + \dfrac{\delta}{2} B_X$ and
$f'_x \in D_x F(x_0, u_0; f_0)x$,

$$f'_x + f^u_{x_0} - f_0 - \overline{y} + \frac{\varepsilon}{2} B_Y \subset -K - \operatorname{int} K \subset -\operatorname{int} K.$$

Similarly, for all $x \in \overline{x} + \frac{\delta}{2} B_X$ and $g'_x \in D_x G(x_0, u_0; g_0)x$,

$$g'_x + g^u_{x_0} - g_0 - \overline{z} + \frac{\varepsilon}{2} B_Y \subset -\operatorname{int} M.$$

Considering (12), one sees from assumption (i) that $p_x(x_0, u_0)\Big(\frac{\delta}{2} B_X\Big) + p(x_0, u)$ contains an open neighborhood $\overline{w} + \varepsilon_1 B_W$ of $\overline{w}$. Now it is easy to check that

$$(\overline{y} + \frac{\varepsilon}{2} B_Y) \times \Big(\overline{z} + \frac{\varepsilon}{2} B_Z\Big) \times (\overline{w} + \varepsilon_1 B_W) \subset C,$$

i.e. $\operatorname{int} C \neq \emptyset$. If

$$C \cap \{(-\operatorname{int} K) \times (-\operatorname{int} M_0^{**}) \times \{0\})\} = \emptyset, \tag{16}$$

then by a standard separation theorem we obtain (3).

So, it remains to prove (16). Suppose to the contrary that there are $(\hat{x}, \hat{u}) \in X \times U$, $f'_{\hat{x}} \in D_x F(x_0, u_0; f_0)\hat{x}$, $g'_{\hat{x}} \in D_x G(x_0, u_0; g_0)\hat{x}$, $f^{\hat{u}}_{x_0} \in F(x_0, \hat{u})$ and $g^{\hat{u}}_{x_0} \in G(x_0, \hat{u})$ such that

$$f'_{\hat{x}} + f^{\hat{u}}_{x_0} - f_0 \in -\operatorname{int} K, \tag{17}$$

$$g'_{\hat{x}} + g^{\hat{u}}_{x_0} - g_0 \in -\operatorname{int} M_0^{**}, \tag{18}$$

$$p_x(x_0, u_0)\hat{x} + p(x_0, \hat{u}) = 0. \tag{19}$$

Defining a new map $\mathcal{P} : X \times R \to W$ by

$$\mathcal{P}(x, \alpha) := \alpha p(x_0 + x, \hat{u}) + (1 - \alpha) p(x_0 + x, u_0), \tag{20}$$

we see that

$$\mathcal{P}(0, 0) = 0,\ \mathcal{P}'(0, 0)(x, \alpha) = p_x(x_0, u_0)x + \alpha p(x_0, \hat{u}) \text{ and } \mathcal{P}'(0, 0)(\hat{x}, 1) = 0.$$

Hence, by the Lusternik theorem, there exist $t^0 > 0$ and maps $t \to \hat{x}(t)$, $t \to \hat{\alpha}(t)$ of $[0, t^0]$ into X and R, respectively, such that $\hat{x}(t) \to 0$ and $\hat{\alpha}(t) \to 0$ as t does and, for all $t \in [0, t^0]$,

$$\mathcal{P}(t(\hat{x}+\hat{x}(t), t(1+\hat{\alpha}(t)))=0. \tag{21}$$

Setting $x(t) := x_0 + t(\hat{x}+\hat{x}(t))$, from (20) and (21) we obtain

$$t(1+\hat{\alpha}(t))p(x(t),\hat{u}) + (1-t(1+\hat{\alpha}(t)))p(x(t),u_0) = 0.$$

Hence, by (iv), for all t small enough, there exists $u(t) \in U$ such that

$$p(x(t),u(t)) = 0.$$

A contradiction to the minimality of $(x_0, u_0; f_0)$ will be achieved if we can show that for all sufficiently small t, there is $f_t \in F(x(t),u(t))$ such that

$$\begin{aligned}
&G(x(t),u(t)) \cap (-M) \neq \emptyset,\\
&f_t - f_0 \in -\operatorname{int} K.
\end{aligned}$$

These two facts should be shown by the following common argument, which we write down explicitly only for G. By virtue of the assumption (iii), for all sufficiently small t and ε, there are $g^{\hat{u}}_{x(t)} \in G(x(t),\hat{u})$ and $g^{u_0}_{x(t)} \in G(x(t),u_0)$ such that

$$g^{\hat{u}}_{x(t)} - g^{\hat{u}}_{x_0} \in \varepsilon B_Z - M, \tag{22}$$

$$-g^{u_0}_{x(t)} + g_0 \in \varepsilon B_Z - M. \tag{23}$$

In turn, the uniform M-differentiability of $G(.,u_0)$ assumed in (ii) ensures that, for all δ and t small enough,

$$g^{u_0}_{x(t)} \in g_0 + tg'_{\hat{x}} + \frac{t\delta}{2}B_Z - M. \tag{24}$$

According to (18) we choose δ so small that

$$g'_{\hat{x}} + g^{\hat{x}}_{x_0} - g_0 + \delta B_Z - M \subset -\operatorname{int} M_0^{**}. \tag{25}$$

Estimating $g^{u_0}_{x(t)} + t(1+\hat{\alpha}(t))(g^{\hat{u}}_{x(t)} - g^{u_0}_{x(t)})$ we get, by (22), (23) and the fact $\hat{\alpha}(t) \to 0$ that

$$\begin{aligned}
&t(1+\hat{\alpha}(t))(g^{\hat{u}}_{x(t)} - g^{u_0}_{x(t)})\\
&= t(1+\hat{\alpha}(t))(g^{\hat{u}}_{x(t)} - g^{\hat{u}}_{x_0} - g^{u_0}_{x(t)} + g_0) + t(g^{\hat{u}}_{x_0} - g_0) + t\hat{\alpha}(t)(g^{\hat{u}}_{x_0} - g_0)\\
&\subset t(1+\hat{\alpha}(t))(2\epsilon B_Z - M) + t(g^{\hat{u}}_{x_0} - g_0) + t\epsilon B_Z\\
&\subset t(g^{\hat{u}}_{x_0} - g_0 + \frac{\delta}{2}B_Z - M)
\end{aligned}$$

for all $\varepsilon > 0$ small enough. This together with (24) give $b_\delta^t \in \delta B_Z$ and $m^t \in M$ such that

(26) $$g^{u_0}_{x(t)} + t(1+\hat{\alpha}(t))(g^{\hat{u}}_{x(t)} - g^{u_0}_{x(t)}) = g_0 + t(g'_{\hat{x}} + g^{\hat{u}}_{x_0} - g_0 + b^t_\delta - m^t).$$

Hence, in view of assumption (iv), there is $g_t \in G(x(t), u(t))$ such that

(27) $$g_0 + t(g'_{\hat{x}} + g^{\hat{u}}_{x_0} - g_0 + b^t_\delta - m^t) \in g_t + M.$$

To verify that $G(x(t), u(t)) \cap (-M) \neq \emptyset$ for all $t > 0$ small enough, we assume to the contrary that $\exists t_n \to 0^+$, $\exists \mu_n \in M^*$, $\|\mu_n\| = 1$ (then assume that μ_n^*-weakly tends to $\overline{\mu} \in M^*$), $\langle \mu_n, g_{t_n}\rangle \geq 0$. $\overline{\mu}$ must belong to M_0^*. Indeed, if $\overline{\mu} \notin M_0^*$, there would be $\beta > 0$ such that $\langle \overline{\mu}, g_0\rangle < -\beta$. On the other hand, it follows from (27) that

(28) $$\langle \mu_n, g_{t_n}\rangle \leq \langle \mu_n, g_0\rangle + t_n\langle \mu_n, g'_{\hat{x}} + g^{\hat{u}}_{x_0} - g_0 + b^{t_n}_\delta - m^{t_n}\rangle.$$

Therefore, for sufficiently small t_n, $\langle \mu_n, g_{t_n}\rangle < 0$, which is a contradiction.

Now since $\overline{\mu} \in M_0^\star$, a glance at (25) yields, for large n,

$$t_n\langle \mu_n, g'_{\hat{x}} + g^{\hat{u}}_{x_0} - g_0 + b^{t_n}_\delta - m^{t_n}\rangle < 0.$$

Therefore (28) contradicts the fact that $\langle \mu_n, g_{t_n}\rangle \geq 0$.

The above argument applied to F instead of G will give $f_t \in F(x(t), u(t))$ such that (similarly as (27), (25))

$$f_t - f_0 \in -K + t(f'_{\hat{x}} + f^{\hat{u}}_{x_0} - f_0 + b^t_\delta - k^t) \subset -K - \operatorname{int} K = -\operatorname{int} K.$$

Thus, (16) must hold and the proof is complete.

Remark. The necessary condition (3) has a form of the classical multiplier rule. We can write this condition in the following equivalent form

(a) $$\langle \lambda_0, D_x F(x_0, u_0; f_0)x\rangle + \langle \mu_0, D_x G(x_0, u_0; g_0)x\rangle + \langle \nu_0, p_x(x_0, u_0)x\rangle \subset R_+ \quad \forall\, x \in X;$$

(b) $$\langle \lambda_0, f_0\rangle + \langle \mu_0, g_0\rangle + \langle \nu_0, p_0\rangle = \min_{u\in U}\big\{f(\lambda_0, x_0, u) + g(\mu_0, x_0, u) + p(\nu_0, x_0, u)\big\},$$

where $p_0 = p(x_0, u_0) = 0$,

$$f(\lambda_0, x_0, u) = \min\big\{\langle\lambda_0, y\rangle :\ y \in F(x_0, u)\big\},$$
$$g(\mu_0, x_0, u) = \min\big\{\langle\mu_0, z\rangle :\ z \in G(x_0, u)\big\},$$

and $p(\nu_0, x_0, u) = \langle\nu_0, p(x_0, u)\rangle$. This equivalent form was used in [5, 7, 8, 9, 10] instead of (3).

Adding a constraint qualification of the Slater type we get from Theorem 1 an extension of the Kuhn-Tucker necessary condition as follows.

Theorem 2. *In additon to the assumptions of Theorem 1, assume that* $p_x(x_0, u_0)X + p(x_0, U)$ *contains a neighborhood of zero in* W *and that there are* $(\tilde{x}, \tilde{u}) \in X \times U$, $g'_{\tilde{x}} \in D_xG(x_0, u_0; g_0)\tilde{x}$ *and* $g^{\tilde{u}}_{x_0} \in G(x_0, \tilde{u})$ *such that*

$$g'_{\tilde{x}} + g^{\tilde{u}}_{x_0} - g_0 \in -int\, M_0^{**},$$
$$p_x(x_0, u_0)\tilde{x} + p(x_0, \tilde{u}) = 0.$$

Then, $\lambda_0 \neq 0$.

Proof. Suppose $\lambda_0 = 0$. If $\mu_0 = 0$, then $\nu_0 \neq 0$. Since $p_x(x_0, u_0)X + p(x_0, U)$ includes a neighborhood of zero, one can choose $(x_1, u_1) \in X \times U$ such that

$$\langle\nu_0, p_x(x_0, u_0)x_1 + p(x_0, u_1)\rangle < 0,$$

contradicting (3). So μ_0 must be non zero. But, then $(\tilde{x}, \tilde{u})$ does not satisfy (3). Thus, $\lambda_0 \neq 0$.

Now we consider the problem (P). We shall see that with a small modification in the assumption (iv) the two theorems are still valid for (P). We need the following definition

Definition 4. Let F, G and p be defined as in the problem (P). Then $(F, G, p)(x, .)$ is $K \times M \times \{0\}$-convexlike in $(U, \{u_0\})$ strongly with respect to G if $\forall u_1 \in U$, $\forall (f^1, g^1) \in F(x, u_1) \times G(x, u_1)$, $\forall (f^0, g^0) \in F(x, u_0) \times G(x, u_0)$, $\forall \gamma \in [0, 1]$, $\exists u \in U$, $\forall g^u \in G(x, u)$, $\exists f^u \in F(x, u)$,

$$\gamma f^1 + (1-\gamma)f^0 - f^u \in K,$$
$$\gamma g^1 + (1-\gamma)g^0 - g^u \in M,$$
$$\gamma p(x, u_1) + (1-\gamma)p(x, u_0) = p(x, u).$$

It is easy to check that Theorems 1 and 2 still hold for (P) if the convexlikeness in $(U, \{u_0\})$ assumed in (iv) is replaced by the convexlikeness in $(U, \{u_0\})$ strong with respect to G.

3. Remarks

We discuss the assumptions of Theorems 1 and 2. Observe first that (i) is rather strict but it is probably inevitable (in a certain sense) because it is usually imposed on the equality constraint in order to apply the Lusternik theorem and it is satisfied in various practial situations. The uniform differentiability on (ii) is an extension to multifunctions of the corresponding notion of Ioffe-Tihomirov [5]. The example below gives a multifunction that satisfies both (ii) and (iii). The $K \times M \times \{0\}$-convexlikeness assumed in (iv) is of course much weaker than the $K \times M \times \{0\}$-convexity often imposed in convex cases.

Example. Let $X = R$, $U = R$ and $x_0 \in X$. Let $G : X \times U \rightsquigarrow R$ defined by

$$G(x, u) := (x - x_0)^2(|u|[0, 1] - 1).$$

We verify (ii) and (iii) for $(x_0, u_0) := (x_0, -1)$.

(ii) For given $\varepsilon > 0$ and $\overline{x} \in X$ we will find $\delta > 0$ such that, $\forall x \in (x_0 - \delta, x_0 + \delta)$, $\forall g' \in D_xG(x_0, u_0; g_0)\overline{x}$, $\forall \gamma > 0$,

$$G(x_0 + \gamma x, u_0) - g_0 - \gamma g' \subset \gamma(-\epsilon, \epsilon) - R_+, \tag{29}$$

where $g_0 \in G(x_0, u_0) = \{0\}$. First we show that $D_xG(x_0, u_0; 0)\overline{x} = \{0\}$ $\forall \overline{x} \in X$. By definition, $g' \in D_xG(x_0, u_0; 0)\overline{x}$ means that $\forall g_n \to 0$, $\forall x_n \to x_0$, $\forall t_n \to 0^+$, $\exists \overline{x}_n \to \overline{x}$, $\exists g'_n \to g'$, $\forall n$,

$$g_n + t_n g'_n \in G(x_n + t_n\overline{x}_n, u_0) \equiv [-(x_n + t_n\overline{x}_n - x_0)^2, 0].$$

Choosing $g_n \equiv 0, x_n \equiv x_0$ yields $g'_n \in [-t_n\overline{x}_n^2, 0]$ $\forall n$. Therefore, since $g'_n \to g'$, $t_n \to 0^+$ and $\overline{x}_n \to \overline{x}$, $g' = 0$. Then (29) becomes

$$[-\gamma^2x^2, 0] \subset \gamma(-\epsilon, \epsilon) - R_+,$$

which always holds (for any $\delta > 0$ and $\gamma > 0$).

(iii) For any $u \neq u_0$ and $\varepsilon > 0$ we can find $\delta > 0$ such that $\forall x \in (x_0 - \delta, x_0 + \delta)$,

$$G(x, u) - G(x_0, u) \equiv [0, (x - x_0)^2|u|] - (x - x_0)^2 \subset (-\varepsilon, \varepsilon) - R_+.$$

(This property is even stronger than the R_+-s.l.s.c. with $G(x_0, u)$.) Note that δ may be taken as $\sqrt{\dfrac{\varepsilon}{|u|}}$.

Now we check a property stronger than the $-R$-s.l.s.c. with $g_0 = 0$ of $G(., u_0)$. For any $\varepsilon > 0$, one can find $\delta > 0$ such that, $\forall x \in (x_0 - \delta, x_0 + \delta)$,

$$G(x, u_0) - g_0 \equiv [-(x - x_0)^2, 0] \subset (-\epsilon, \epsilon) + R_+.$$

So one may choose $\delta = \sqrt{\varepsilon}$.

Let us observe that Theorems 1 and 2 contain as special cases the corresponding results in [5, 9] for the problem with $p_x(x_0, u_0)X = W$. Necessary optimality conditions for (P) and $(\tilde{P})$ when $p_x(x_0, u_0)X$ has a finite codimention need further considerations and are the aim of our forthcoming research.

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